18.212 lec. 10.1

Stirling #s : $s(n,k)$ (I kind) $S(n,k)$ (II kind)

Thm: (1) $\sum_{k=0}^{n} s(n,k)\, x^k = (x)_n$

(2) $\sum_{k=0}^{n} S(n,k)\,(x)_k = x^n$

Proof of (2): If the identity holds for x being a positive integers, then these polynomials should indeed be equal.

Assume $x \in \mathbb{Z}_{\geq 0} \Rightarrow (x)_k = k!\binom{x}{k}$

RHS: $x^n = \#$ functions $f: \{1,2,\dots,n\} \to \{1,\dots,x\}$

LHS: $\sum_{k=0}^{n} \underbrace{k!\,S(n,k)}\binom{x}{k}$

k element subsets

ordered partitions of n with k blocks (nonempty)
$\pi = (B_1 | B_2 | \dots | B_k)$

$\Rightarrow k!\,S(n,k) = \#$ surjective maps $f: [n] \to \{i_1,\dots,i_k\}$

$\binom{x}{k} = \#$ k element subsets $\{i_1,\dots,i_k\}$ of $\{1,2,\dots,x\}$

$\Rightarrow k!\,S(n,k)\binom{x}{k} = \#$ functions $f: \{1,2,\dots,n\} \to \{1,\dots,x\}$ s.t. $|\operatorname{Im} f| = k$
i.e. f takes on exactly k different values.

$\Rightarrow \sum_{k=0}^{n} k!\,S(n,k)\binom{x}{k} = \#$ functions $f: \{1,2,\dots,n\} \to \{1,\dots,k\} = x^n$ $\square$

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Sperner's thm (1928): let $I_1, I_2, \ldots, I_N$ be different subsets of $[n]$ s.t $\forall i \neq j$: $\underbrace{I_i \not\subseteq I_j \text{ and } I_j \not\subseteq I_i}_{\text{i.e they are incomparable}}$. Then $N \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Note: By considering all $\lfloor \frac{n}{2} \rfloor$ sized subsets, we get that the bound is sharp.

Posets (Partialy ordered sets)

Def: P is a set with a bineary relation $\le$, i.e. $\forall\, a, b$: $a \le b$ is either true or not.

Axioms: (1) $a \le a \quad \forall a \in P$ (reflexivity)

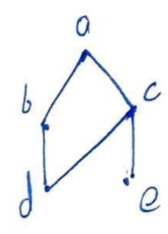
(2) $a \le b$ and $b \le a \Rightarrow a = b$

(3) $a \le b,\ b \le c \Rightarrow a \le c$ (transitivity)

Notation: • "$a < b$" denotes $\begin{cases} a \le b \\ a \neq b \end{cases}$ ($a \le b$ & $a \neq b$)

• "$a <\!\cdot\, b$" denotes that b coveres a, or a is covered by b, i.e: $a < b$ and $\nexists\, c \in P$ s.t $a < c < b$

Hasse diagram of P: a directed graph, with verticies corresponding to P's elements, and arrows corresponding to $a <\!\cdot\, b$ relations.

Ex: 

$\longleftrightarrow$ $b <\!\cdot\, a$, $c <\!\cdot\, a$, $d <\!\cdot\, b$, $d <\!\cdot\, c$, $e <\!\cdot\, c$

When P has a unique maximal element, i.e its not covered by anything, we denote it by $\hat{1}$. Viceversa for the minimal element, denoted by $\hat{0}$.

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Def: $a, b \in P$ are <u>incomparible</u> if $a \not\leq b$ & $b \not\leq a$

Def: A <u>chain</u> $C \subset P$ is a subset, s.t any two elements are comparible.

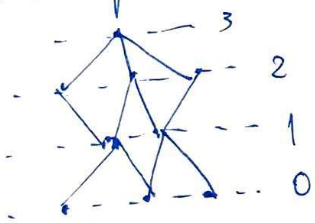
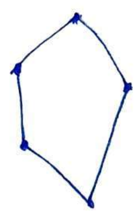
Def: A <u>situated chain</u> is a chain of form $C = \{c_1 \lessdot c_2 \lessdot \dots \lessdot c_k\}$

Def: An <u>anti-chain</u> $A \subset P$ is a subset s.t. any two elements are incomparable.

Def: A poset P is <u>ranked</u> if $\exists$ function $\rho: P \to \mathbb{Z}$ s.t.
(rank function)

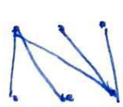
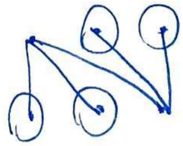
$\forall a, b \in P: \quad a \lessdot b \Rightarrow \rho(a) + 1 = \rho(b)$

For a finite poset P, we assume that $\rho: P \to \{0, 1, \dots, l\}$ is a surjective function.

Ex:  is ranked  is not ranked

For ranked poset P, let $P_i = \{a \in P \mid \rho(a) = i\}$ and let $r_i = |P_i|$ ← rank numbers

Def: A finite ranked poset is
- <u>rank symmetric</u> if $\forall i = 0, \dots, l$ we have $r_i = r_{l-i}$
- <u>unimodel</u> $\exists s \in \{0, \dots, l\}$ s.t $r_0 \leq r_1 \leq \dots \leq r_s \geq r_{s+1} \geq \dots \geq r_l$
- <u>spernard</u>: If a maximal size of chain an anti-chain $= \max(r_0, r_1, \dots, r_l)$

Ex:  is ranked, unimodel but not spernard! 

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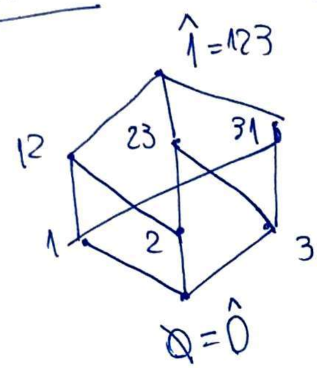
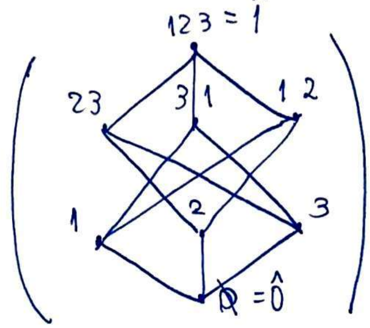
Boolean lattice: B_n the poset of all subsets $I \subseteq [n]$ ordered by inclusion

Ex: B_3:





Thm: B_n is rank-symmetric, unimodel & Sperner.