18.212 lec 9.1

Eulerian triangle

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 4 & & 1 & \\
1 & & 11 & & 11 & & 1 \\
 & & & \vdots & & &
\end{array}
$$

$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k)$

$A(1,0) = 1$

$\Updownarrow$ Can check

$$f_n(x) = \frac{\sum_{k=0}^{n-1} A(n,k)\, x^k}{(1-x)^{n+1}}$$

$$f_n = \frac{d}{dx}(x f_{n-1})$$

$\tilde{A}(n,k) := \#\{w \in S_n \mid des(w) = k\}$, then $\tilde{A}(n,k) = A(n,k)$.

Proof: Perms $w \in S_n$ are obtained from perms $u \in S_{n-1}$ by inserting the new entry "n":

$$u = \underline{\quad}\, u_1 \,\underline{\quad}\, u_2 \,\underline{\quad} \cdots \underline{\quad}\, u_{n-1} \,\underline{\quad}$$

"n"

2 cases: (I) If we insert "n" into an ascent slot of u, then $des(w) = des(u)+1$

(II) If we insert "n" into a descent slot of u, then $des(w) = des(u)$

Note: We say $u_0 = -\infty$ $u_n = \infty$ and $u_i \,\underline{\quad}\, u_{i+1}$ is an ascent slot if $u_{i+1} > u_i$.

Assume that $des(w) = k$, then u obtained from removing n has two scenarios:

I: $des(u) = k-1$, and u has $n-k$ descent slots.

II: $des(u) = k$, and u has $(k+1)$ ascent slots

Bijection $w \leftrightarrow u$ gives $\tilde{A}(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k)$

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Stirling Numbers

1^{st} kind:

signless $\quad c(n,k) = \#\{w \in S_n \mid cyc(w) = k\} = \#\{w \in S_n \mid rec(w) = k\}$

signed $\quad s(n,k) = (-1)^{n-k} c(n,k)$

2^{nd} kind:

$S(n,k) = \#$ set partitions of $[n]$ with k blocks.

Def: A set partition $\pi = (B_1 | B_2 | \dots | B_k)$ of set $[n] = \{1, \dots, n\}$ s.t

- $[n] = B_1 \sqcup B_2 \sqcup B_3 \sqcup \dots \sqcup B_k$
- $\forall i \in [k]: \ B_i \neq \emptyset$

where the order of block doesn't matter.

Ex: $S(4,2) = 4 + 3 = 7$

(123|4) (124|3)
(134|2) (234|1)

(12|34) (13|24)
(14|23)

1^{st} Stirling triangle

Reccurance: $c(n,k) = c(n-1,k-1) + \underline{(n-1)}c(n-1,k)$

$$\begin{array}{ccc} & c(0,0) & \\ c(1,0) & & c(1,1) \\ c(2,0) & c(2,1) & c(2,2) \\ & \vdots & \end{array} = \begin{array}{cccc} 1 & & & \\ 0 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 2 & 3 & 1 \\ & \vdots & & \end{array}$$

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2nd Stirling triangle

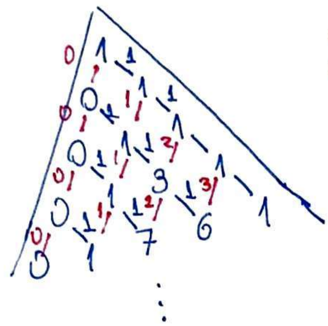
$S(0,0)$

$S(1,0)\ S(1,1)$

$S(2,0)\ S(2,1)\ S(2,2)$

$\vdots$

$=$



$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

Pf of 1st number reccurance:

$u = (\ \dots\ )(\ \dots\ )\cdots(\ \dots\ ) \in S_{n-1}$ (cycle notation)

"n" Get w by inserting n into u.

2 cases:

(i) n is a fixed point of w

(ii) We insert "n" into on of the cycles of u : $(n-1)$ ways

Ex: $(1\ \ 3\ \ 4)(2\ \ 5)$

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Pf of 2nd number reccurance:

Same insertion into block notation.

2 cases:

(i) "n" forms a new block

(ii) We insert "n" into one of the k blocks.

Denote $(x)_n := x(x-1)\dots(x-n+1) = \prod_{i=0}^{n-1}(x-i)$ called the folly factorial a.k.a. Pochhomer symbol.

Thm: (1) $\sum_{k=0}^{n} s(n,k)\,x^k = (x)_n$.

(2) $\sum_{k=0}^{n} S(n,k)\,(x)_k = x^n$.

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Ex: $n=3$

(1) $\sum_{k=0}^{n} s(n,k)\, x^k = -0\, x^0 + 2x^1 - 3x^2 + x^3 = x(x-1)(x-2) = (x)_n$

(2) $\sum_{k=0}^{n} S(n,k)\,(x)_k = 0 \cdot 1 + 1x + 3x(x-1) + 1x(x-1)(x-2) = x^3 = x^n$

2 linear basis of $\mathbb{R}[x]$:

$1, x, x^2, \dots$

$1, x, x(x-1), \dots$

Cor: Fix $N \in \mathbb{N}$. Consider matrices $[s(n,k)]_{1 \le k, n \le N}$, $[S(n,k)]^{-1}_{1 \le k, n \le N}$

have $[s(n,k)]_{1 \le k, n \le N} = \left([S(n,k)]_{1 \le k, n \le N}\right)^{-1}$

Pf: They represent change of basis matrices between $\{x^i \mid i \in [N] \cup \{0\}\}$ and $\{(x)_i \mid i \in [N] \cup \{0\}\}$ for space of degree at most N polynomials.