18.217 lec 7.1

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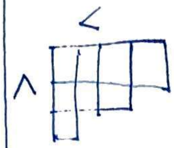
Recall: Standard Young Tableau (SYT) of shape $\lambda \vdash n$



fill with $1, \ldots, n$
Increasing along rows & columns

$f_\lambda = \#\{\text{SYT of shape } \lambda\}$

Robinson–Schensted Correspondance

Thm: There is a bijection

$$S_n \longleftrightarrow \left\{ (P,Q) \,\middle|\, \begin{array}{l} P, Q \text{ are SYT} \\ \text{of same shape } \lambda \vdash n \end{array} \right\}$$

Note: This generalizes to semin-standard tableaus called Robinson–Schensted–knuth correspondance (RSK).

To start, define an insertion algorithm on SYT:

Def: Shensted insertion $T \leftarrow x$

Ex:

2	4	8	11
5	4		
7			

$\leftarrow 6$

1) If x is bigger than all numbers in the column place it at the end
2) If x is smaller than some of then numbers, bumb out the smallest number greater than it. x'.
3) Continue by inserting our x' to the next row...

Check: This still results with a semi-standard tableau.

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RSK algorithm: $w = (w_1, w_2, \dots, w_n) \longmapsto (P, Q)$

Start w/ $P = Q = \emptyset$

Shensted insert $w_1, \dots, w_n$ in order into P. On insert i: write down i in the box of Q which gets added to P.

Ex: $w = 624351$

$$P: \emptyset \to \begin{array}{|c|}\hline 6\\\hline\end{array} \to \begin{array}{|c|}\hline 2\\\hline 6\\\hline\end{array} \to \begin{array}{|c|c|}\hline 2&4\\\hline 6\\\cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 2&5\\\hline 4\\\cline{1-1} 6\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 2&3&5\\\hline 4\\\cline{1-1} 6\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 6\\\cline{1-1}\end{array} = P$$

$$Q: \emptyset \to \begin{array}{|c|}\hline 1\\\hline\end{array} \to \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 6\\\cline{1-1}\end{array} = Q$$

Inverse RSK

$$P = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&5\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline\end{array} \quad \emptyset$$

$$Q = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \emptyset$$

must have inserted 4 on the last move

must have ended with last box the 5 is $\Rightarrow$ this could only resulted from 2 kicking 5 to second row $\Rightarrow$ we must have inserted 2

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Cor: $\sum_{\lambda\vdash n} (f_\lambda)^2 = n! \leftarrow |S_n|$

f_λ: tableaux of shape λ

Thm: For $w \in S_n$, if $w \mapsto (P,Q)$ then $w^{-1} \mapsto (Q,P)$

Cor: $\sum_{\lambda\vdash n} f_\lambda = \#\{w \in S_n \mid w^{-1} = w\}$

~~(Part of) Green's~~

Def: The shape λ of P & Q is the <u>Schensted shape</u> of w.

E.g. (from earlier) Schensted shape 31524 is $(3,2)$

Def: Given permutation $w = w_1 \dots w_n$, an increasing sequence is $i_1 < \dots < i_k$ s.t. $w_{i_1} < \dots < w_{i_k}$.

Eg 3 <u>1</u> 5 <u>2</u> <u>4</u> has increasing sequence 124

Def: " " <u>decreasing</u> " " ~~inversions~~ $w_{i_1} > w_{i_2} > \dots > w_{i_k}$.

<u>(Part of) Green's thm:</u> Let λ be the Schensted shape of w, then

- λ_1 (the size of first row) = max length of increasing sequence in w.
- λ_1' (size of first column) = max length of decreasing sequence in w.

Remark: w is 321 avoiding $\Longleftrightarrow$ No decreasing sequence of length ≥ 2 $\Longleftrightarrow$ At most 2 rows

bij: $\{w \in S_n \mid 321 \text{ avoiding}\} \longleftrightarrow \{SYT \text{ of shape } n \times 2\}$

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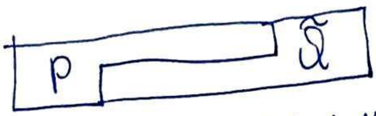
$\{w \in S_n \mid 321 \text{ avoiding}\} \xleftrightarrow{\text{bij}}$ SYT of shape $n \times 2$

(P, Q)



↖ Q rotated 180° replace i with $2n-i+1$.

Eg:

$$31524 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 9 \\ \hline 3 & 5 & 6 & 8 & 10 \\ \hline\end{array}$$