18.212 lec 6.1

OH: Mon 2:30 or 3 pm $\Rightarrow$ Fri 3pm office hour.

Tue 1 or 4 pm

Thur 1 or 4 pm

Fri 3 pm

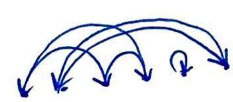
Permutations: $w \in S_n$

- 1-line notation : $w = w_1, \dots, w_n$
- 2-line notation : $\begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$
- graphical notation :

- cycle notation : $(\dots)(\dots) \dots (\dots)$
- a bijection $w : [n] \to [n]$, $i \to w_i$
- Permutation matrices $A_w = \begin{array}{cc} i & \\ \begin{bmatrix} & 0 & & \\ & 0 & & \\ & \vdots & & \\ \cdots & 1 & \cdots & \\ & 0 & & \\ & 0 & & \\ & \vdots & & \end{bmatrix} & w_i \end{array}$

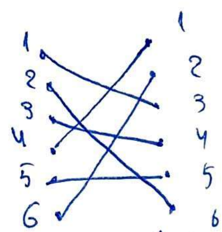
Ex

- $w = 3,6,4,1,5,2$
- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix}$



a fixed point

- $w = (1,3,4)(2,6)(5)$



- $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Multiply permutations:

$u, w \in S_n$ uw is a composition map or product of permutation matrices.

$uw(i) \to u(w(i))$ and $A_{uw} = A_u A_w$

Adjacent transpositions $s_i = (i, i+1)$

Ex: $n=3$ $\quad s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\quad s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$s_1 \cdot s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

18.212 lec 6.2

Statistics on permutations

$\sigma : S_n \to \mathbb{Z}_{\geq 0}$

generating function $f_{\sigma,n}(x) = \sum_{w \in S_n} x^{\sigma(w)}$

Two statistics σ, η are called equidistributed if they have $f_{\sigma_n}(x) = f_{\eta,n}(x)$

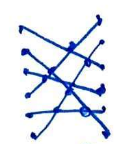
- # inversion := $inv(w) := \{(i,j) \mid$ ~~such that~~ $1 \leq i < j \leq n, \ w_i > w_j\}$
- # descents := $des(w) = \{ i \in [n-1] \mid w_i > w_{i+1}\}$
- # cycles := $cyc(w)$ = # all cycles in w including fixed points
- <u>Major index</u> $maj(w) := \sum_{i : w_i > w_{i+1}} i$
- # <u>records</u> $rec(w) = \#\, i$ s.t $w_i > w_j \ \forall j < i.$

(also know as left to right maxima)

- # exceedences $exc(w) = \#\{i \mid w_i > i\}$
- # anti exceedences $aexc(w) = \#\{i \mid w_i < i\}$
- # weak exceedences $wexc(w) = \#\{i \mid w_i \geq i\}$

Example for $w = 3,6,4,1,5,2$:

$inv(w)$ = # crossing in bijection graph notation $= 9$



$des(w) = 3$ $\qquad exc(w) = 3$ $\qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \hat{3} & \hat{6} & \hat{4} & 1 & 5 & 2 \end{pmatrix}$

$cyc(w) = 3$

$maj(w) = 2 + 3 + 5 = 10$

$rec(w) = 2 \quad (\underline{3}\,\underline{6}\,4\,1\,5\,2)$

18.212 lec 6.3

Eulerian statistics $\sim$ des

Mahonian statistics $\sim$ maj

Stirling statistics $\sim$ cyc

Theorem: The following statistics are equidistributed:

(1) des $\sim$ exc $\sim$ aexc $\sim$ (wexc -1) $\in$ Eulerian

(2) inv $\sim$ maj

(3) cyc $\sim$ rec

Proof of cyc $\sim$ rec:

- $w = (134)(26)(5) = (26)(341)(5)$

$w = (a_1\ )(a_2\ )\dots(a_k\ )$ s.t. (1) a_i's are maximal elements in their cycles

(2) $a_1 < \dots < a_k$

Ex: $w = (413)(5)(62) \to \tilde{w} = 413562$ (erase brackets and think of it as a one line notation)

Claims (1) $cyc(w) = rec(\tilde{w})$

(2) $w \longmapsto \tilde{w}$ is bijective