18.212 lec 3.1

Thm 1: $C_n := \#\left\{\begin{matrix}\text{Dyck paths}\\ \text{w/ } 2n \text{ steps}\end{matrix}\right\} = \frac{1}{n+1}\binom{2n}{n}$

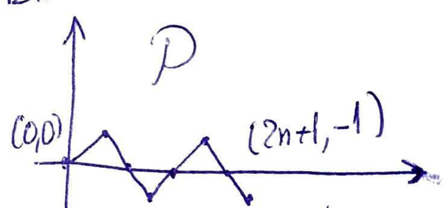
Proof #1: (Reflection method): $C_n = \binom{2n}{n} - \binom{2n}{n-1}$

A "subtraction-free" proof?

Q: Can we subdivide all $\binom{2n}{n}$ lattice paths w/ n up & n down steps into a disjoint union of blocks of size $n+1$ s.t. each block contains exactly one Dyck path?

A slight modification...

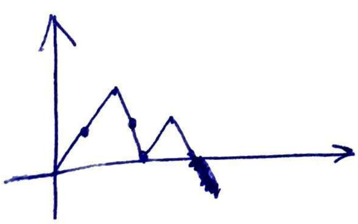
Ex $n=3$



$$\frac{1}{n+1}\binom{2n}{n} = \frac{2n!}{n!\,(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n}$$

$\binom{2n+1}{n} = \#$ all lattice paths w/ n up steps and $n+1$ down steps.

An *augmented Dyck path* is a Dyck path w/ an extra down step at the end.



Pf #2: (Cyclic shifts)

$C: P \mapsto P'$ obtained by P from moving its last step to the begining of P.

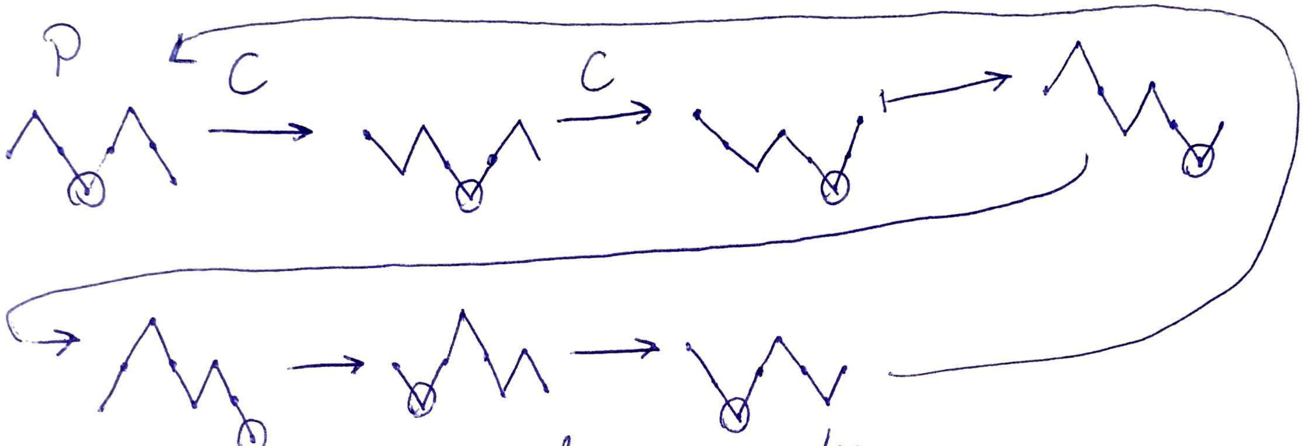
18.212 lec 3.2

Ex $n=3$

C: 

$(1,-1,-1,1,1,-1,-1) \longmapsto (-1,1,-1,-1,1,1,-1)$

This gives an action of the cyclic group $\mathbb{Z}/(2n+1)\mathbb{Z}$ of order $(2n+1)$ on lattice paths.



an orbit of the cyclic group action.

Claim 1: Each orbit contains exactly $2n+1$ paths (all $2n+1$ paths are distinct)

Claim 2: Each orbit contains exactly 1 augmented path.

Let $m(P)$ be the position of the first minimum of P.

Observations: (1) $m(P) \in \{1, 2, \dots, 2n+1\}$

(2) P is an augmented Dyck path iff $m(P) = 2n+1$

Claim 3: As we apply cyclic shifts $P \xrightarrow{C} P' \to P'' \to \dots \to P$, $m(P)$ changes cyclically. $i \to i+1 \to \dots \to 2n+1 \to 1 \to 2 \to \dots \to i$

Claim 3 $\Rightarrow$ Claims 1 & 2.

12.212 lec 3.3

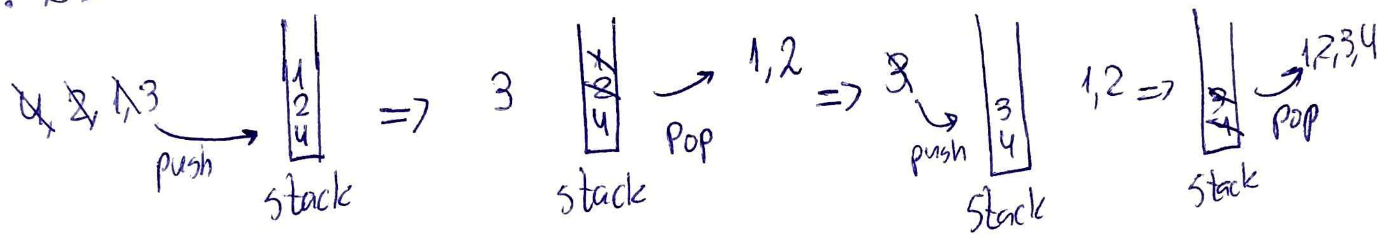
Sorting of permutations

$w = w_1, w_2, \dots, w_n$ is a permutation of $1, 2, \dots, n$.

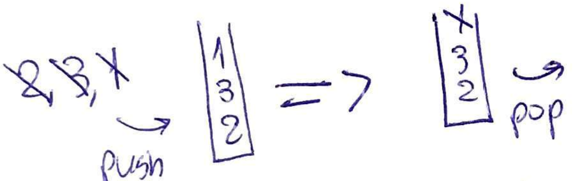
The symmetric group S_n is the set of all perms of $1, \dots, n$.

Stack sorting

Stack : LIFO (last-in-first-out) data structure

Ex:


4, 2, 1, 3 is stack-sortable

Ex


stuck! => 2, 3, 1 is not stack sortable

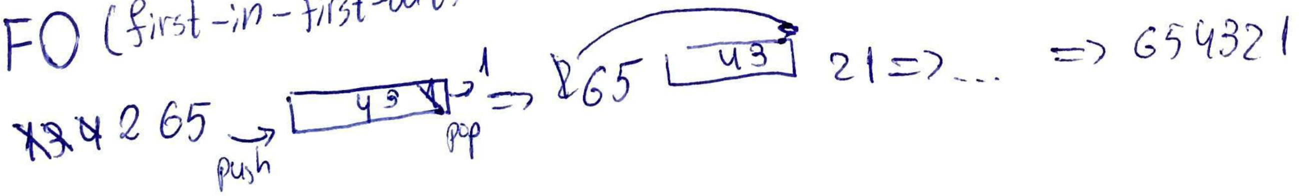
Theorem: # Stack-sortable perms in $S_n = C_n$

Theorem: Stack sortable perms are exactly 231-avoiding perms

(i.e. $\nexists\ i < j < k$ s.t. $w_j > w_i > w_k$)

Que sortable perms

FIFO (first-in-first-out)

Ex:


18.212 lec 3.4

321-avoiding perms : $\nexists$ $i<j<k$ s.t $w_i > w_j > w_k$.

Thm: # Queue-sortable perms (are exactly 321-avoiding perms) $= C_n$