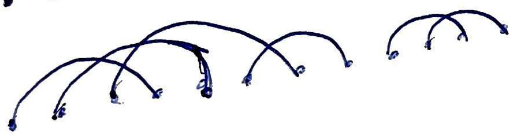
18.212 lec 2.1 Catalan Numbers (Cont'd)

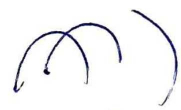
$C_n = \frac{1}{n+1}\binom{2n}{n} = \#$ of ...

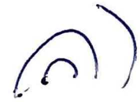
(A) Dyck paths w/ 2n steps
(B) Triangulations of (n+2)-gon
(C) Paranthesizations of (n+1) letters
(D) Plane binary trees w/ n vertices
(E) Complete plane binary trees w/ n+1 leaves
(F) Plane trees w/ n+1 vertices
(G) non-crossing perfect matchings w/ 2n vertices
(H) non-nesting perfect matchings w/ 2n vertices

Ex n=6

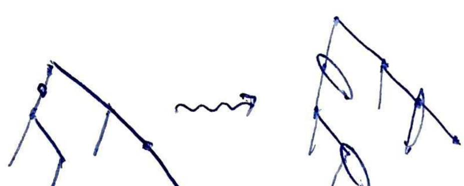
G: 

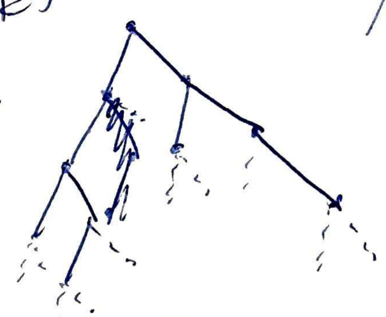
H: 

forbid 

forbid 

Bijection (D) $\to$ (E)

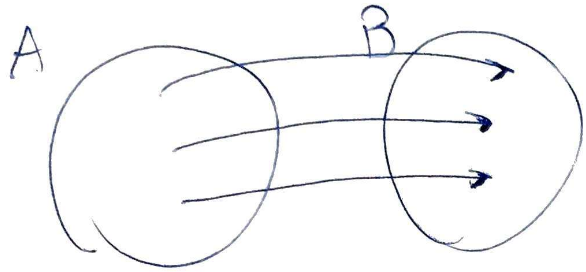


 $\mapsto$ 

n vertices

n+1 leaves

18.212 lec 2.1



Two maps

$f: A \to B$ and $g: B \to A$

$g \circ f = id = f \circ g$ bijection

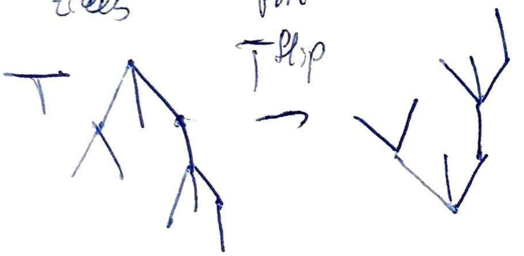
aka one-to-one correspondence

(F) $\longleftrightarrow$ (A)

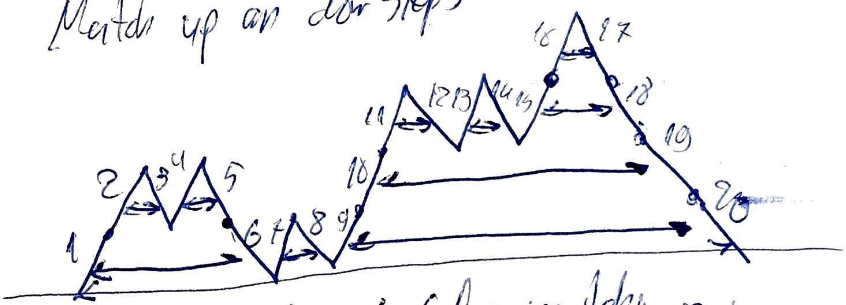
plane trees — Dyck paths

T

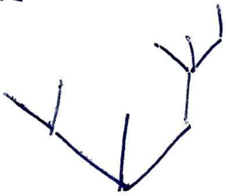
T flip $\rightarrow$



Match up an down steps

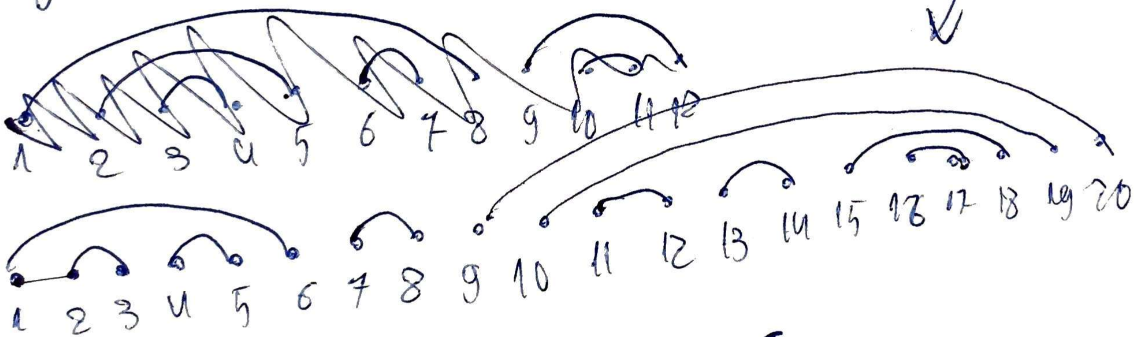


↓ Glue matching pairs

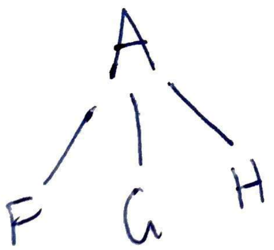


(A) $\longleftrightarrow$ (G)

bij



B – D – E – F

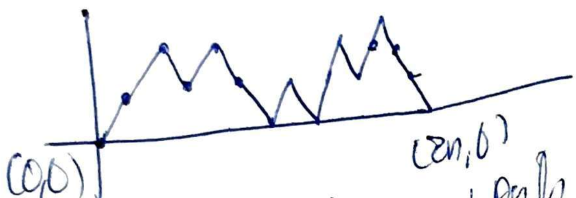


18.212 lec 2.3

Thm 1 $C_n \stackrel{def}{:=} \#\left\{\begin{matrix}\text{Dyck paths}\\ \text{w/ } 2n \text{ steps}\end{matrix}\right\} \stackrel{?}{=} \frac{1}{n+1}\binom{2n}{n}$

Pf #1: Reflection Method

$\#\{\text{all lattice paths from } (0,0) \text{ to } (2n,0)\} = \binom{2n}{n}$



$C_n = \binom{2n}{n} - \#\{\text{"bad" paths}\}$

$\{\text{"bad" paths}\} = \{\text{paths intersecting line } y=-1\}$

$X :=$ the first intersection with $y=-1$

$P \longleftrightarrow P'$ obtained from P by reflecting this over X portion amongus the line $y=-1$.

this map $P \longmapsto P'$ is a bijection between all bad paths and all lattice paths from $(0,0)$ and $(2n,-2)$

$|P'| = \binom{2n}{n-1} = \frac{n}{n+1}\binom{2n}{n} \Longrightarrow \#\text{Dyck paths} = \frac{1}{n+1}\binom{2n}{n}$