18.211 Optional Problem Set 7

All of these problems are bonus problems. You can turn in **by email** to `zhenkun@mit.edu` (cc'd to `apost@math.mit.edu`) any number of these problems before Wednesday, December 11, 2019. December 11 is the last day of classes, so no solutions can be accepted after that date.

Alternatively, if you don't want to scan your solutions, you can get in touch with Zhenkun Li and hand him your handwritten solutions (also before December 11).

You should try to solve some (or all) of these problems if you want to get an $A+$ for the course.

Problem 1. Let k and n be two positive integers such that $k \leq n/2$. Construct a bijection $f : \binom{[n]}{k} \to \binom{[n]}{n-k}$ between the set $\binom{[n]}{k}$ of k-element subsets of $[n]$ and the set $\binom{[n]}{n-k}$ of $(n-k)$-element subsets of $[n]$ such that, for any $I \in \binom{[n]}{k}$, we have $f(I) \supseteq I$.

Problem 2. Consider a random walk on an infinite set of vertices labelled by integers $i \in \mathbb{Z}$ such that we can go from i to $i+1$ with probability $1/2$ or to $i-1$ with the probability $1/2$. Calculate the probability $p(n)$ that a walk with $2n$ steps that starts and ends at 0 reaches its maximal value exactly once. For example, the walk $0 \to (-1) \to 0 \to 1 \to 2 \to 1 \to 0$ reaches its maximum value once, but the walk $0 \to 1 \to 2 \to 1 \to 2 \to 1 \to 0$ reaches its maximum value twice. Give a combinatorial proof.

Problem 3. In class, we defined the *tree inversion polynomial* as $I_n(x) := \sum_T x^{\mathrm{inv}(T)}$, where the sum is over all spanning trees T of the complete graph K_{n+1} and $\mathrm{inv}(T)$ is the number of inversions of T. Prove that

$$I_n(x) = \sum_f x^{\binom{n}{2} - \sum_{i=1}^n f(i)},$$

where the sum is over all parking functions $f : [n] \to [n]$.

Problem 4. Find an explicit expression for the number of spanning trees T of the complete graph K_{2n} such that T contains a perfect matching. (For example, for $n = 2$, there are 12 such trees T among all 4^2 spanning trees of K_4.)

Problem 5. Find an explicit formula for the number of spanning trees T of the complete graph K_{2n} such that each vertex v of T is either a leaf, i.e., $\deg_T(v) = 1$, or is incident to exactly 3 edges, i.e., $\deg_T(v) = 3$.

(For example, for $n = 2$, there are 4 such trees among all 4^2 spanning trees of K_4.)

Problem 6. Show the the diameter of a simple undirected graph G is strictly less than the number of distinct eigenvalues of the adjacency matrix A of G.

Problem 7. Let G be a simple undirected graph. Since the adjacency matrix A of G is symmetric, all eigenvalues of A are real. Let $\alpha_{\max}$ be the maximal eigenvalue of A. Let d_{ave} be the average degree of G, and let $d_{\max}$ be the maximal degree of G. Show that $d_{\text{ave}} \le \alpha_{\max} \le d_{\max}$.

Problem 8. Let L be an $n \times n$ matrix with all zero row and column sums. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n = 0$ be the eigenvalues of L. Recall that the (i, j)-th *cofactor* of L equals $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained from L be removing the ith row and jth column. Show that all cofactors of L are equal to each other and to the number $\frac{1}{n}\lambda_1 \lambda_2 \cdots \lambda_{n-1}$.

Problem 9. Let k and n be two positive integers. A *k-colored permutation* of size n is a permutation $w \in S_n$ where each cycle of w (including each fixed point) is colored in one of k colors. Find the number of k-colored permutations of size n. (For example, the number of 3-colored permutations of size 2 is $3^2 + 3 = 12$.)

Problem 10. Find a bijection between the set A_n of all set partitions of $[n]$ and the set B_n of set partitions π of $[n+1]$ such that there is no pair of consecutive integers i and $i+1$ that belong to the same block of π.