18.211 PROBLEM SET 4 (due Monday, October 28, 2019)

Problem 1. Let A_n, $n \geq 0$, be the sequence given by the recurrence relation $A_n = A_{n-1} + 12A_{n-2}$, for $n \geq 2$, and $A_0 = 0$, $A_1 = 7$.

(a) Find the ordinary generating function for the sequence A_n.

(b) Find the exponential generating function for the sequence A_n.

(c) Find an explicit closed formula for A_n.

Problem 2. Consider the sequence of polynomials $H_n(x)$, $n \geq 0$, which are recursively defined by $H_0(x) = 1$, $H_1(x) = x$, and

$$H_{n+1}(x) = x\, H_n(x) + n\, H_{n-1}(x), \text{ for } n \geq 1.$$

(These polynomials are related to the *Hermite polynomials.*) Find an explicit nonrecursive formula for $H_n(x)$ involving a single summation of certain closed expression.

Problem 3. Consider the sequence of polynomials $T_n(x)$, $n \geq 0$, given by $T_0(x) = 1$, $T_1(x) = x$, and

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x), \text{ for } n \geq 1.$$

(a) Find a closed expression for the ordinary generating function $f(x, y) := \sum_{n\geq 0} T_n(x)\, y^n$.

(b) Find a closed expression for the exponential generating function $g(x, y) := \sum_{n\geq 0} T_n(x)\, \frac{y^n}{n!}$.

(c) Show that $T_n(\cos(\theta)) = \cos(n\theta)$.

Problem 4. Consider the sequence of numbers A_n, $n \geq 0$, whose exponential generating function equals the tangent function:

$$\sum_{n\geq 0} A_n \frac{x^n}{n!} = \tan(x).$$

(These numbers are known under many different names: the tangent numbers, the zigzag numbers, the numbers of alternating permutations. They are closely related to the Bernoulli numbers.)

(a) Show that $f(x) = \tan(x)$ satisfies the differential equation

$$\frac{df}{dx} = f(x)^2 + 1.$$

(b) Show that $A_{2n} = 0$, for all n; and the numbers A_{2n-1} are given by the recurrence relation:

$$A_{2n-1} = \sum_{k=1}^{n-1} \binom{2n-2}{2k-1} A_{2k-1}\, A_{2(n-k)-1}, \text{ for } n \geq 2; \text{ and } A_1 = 1.$$

Problem 5. For two integers $0 \le k \le n$, find an explicit formula for the number of Dyck paths from $(0, 0)$ to $(2n, 0)$ that start with k (or more) "up" steps.

For example, for $n = 3$ and $k = 2$, there are 3 such paths: $(UUDDUD)$, $(UUDUDD)$, $(UUUDDD)$. (Here "U" denotes an "up" step, and "D" denotes a "down" step.)

Hint: You can use the reflection principle.