18.211 Problem Set 2 (due Friday, October 4, 2019)

Problem 1. Find the minimal number of adjacent transpositions needed to transform the permutation $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ into the permutation $6, 7, 8, 9, 10, 1, 2, 3, 4, 5$.

Problem 2. Fix two nonnegative integers m and n. Find a closed expression for the sum of binomial coefficients

$$\sum_{a=0}^{m} \sum_{b=0}^{n} \binom{a+b}{a}.$$

Problem 3. For an integer $n \geq 3$, prove the following identity for q-binomial coefficients:

$$\begin{bmatrix} n+3 \\ n \end{bmatrix}_q = \sum_{k=0}^{3} q^{k^2} \begin{bmatrix} 3 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Problem 4. Prove that the number of compositions of n with all parts greater than or equal to 2 equals the Fibonacci number F_{n-1}.

Problem 5. Prove that the number of compositions of n with all odd parts equals the Fibonacci number F_n.

Problem 6. Fix two integers $n \geq 0$ and $k \geq 2$. Show that the following 3 numbers are equal:

(a) The number of compositions of n with parts equal to 1 or k.
(b) The number of compositions of $n+k$ with all parts greater than or equal to k.
(c) The number of compositions of $n+1$ with all parts congruent to 1 modulo k.

Problem 7. Use the formula $x^n = \sum_{k=0}^{n} S(n,k)\,(x)_k$ to find explicit expressions for the Stirling numbers of the second kind $S(n,1)$, $S(n,2)$, $S(n,3)$ and $S(n,4)$ as linear combinations of $1^n, 2^n, 3^n$, and 4^n.

Problem 8. Let $p(n)$ denote the number of integer partions of n. Show that $p(n) - p(n-1)$ equals the number of integer partitions of n with all parts greater than or equal to 2.

Problem 9. Fix two integers $n \geq 0$ and $k \geq 1$. Show that the following 3 numbers are equal:

(a) The number of integer partitons of n with at most k parts.
(b) The number of integer partitions of $n+k$ with exactly k parts.
(c) The number of integer partitions of n with all parts less than or equal to k.

Problem 10. Show that the number of integer partitions of n with odd parts equals the number of integer partitions of n with all distinct parts.