18.211 Problem Set 1 (due Wednesday, September 18, 2019)

Problem 1. 15 students solved 100 homework problems in total. (If two students solved the same problem, we count it as two solutions.) Prove that there are two students who solved exactly the same number of problems.

Problem 2. Pick any 10 points inside a square with sides of length 3 inches. Prove that we can always find two of these points so that the distance between them is less than 1.5 inch.

Problem 3. 2019 people came to a party. Show that we can always find 2 of them who were born on the same day of the week, and whose birthdays differ by at most 1 day.

Problem 4. A *halfking* is a new kind of chess piece that can attack directly to the right, left, up, or down. In other words, a halfking moves like a king but only vertically or horizontally, and not diagonally.

(a) Find the maximal number M of halfkings one can place on the 8×8 chessboard so that none of them can attack another halfking. Use the pigeon-hole principle to argue that one cannot place more than M non-attacking halfkings.

(b) Find the number of ways to place M non-attacking halfkings on the 8×8 chessboard, where M is the number from part (a).

Problem 5. Pick any collection of n lines a plane. These lines subdivide the plane into regions. We say that two regions are *adjacent* if they share a common edge. (But two regions that share only a vertex are not considered adjacent.) Prove by induction that we can color the regions in two colors so that any pair of adjacent regions have different colors.

Problem 6. Prove that, for a positive integer n, we have

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Problem 7. Let us consider two infinite sequences of integer numbers $F(0), F(1), F(2), \cdots = 0, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers); and $A(0), A(1), A(2), \cdots = 1, 1, 3, 8, 21, 55, \ldots$; which are given by the following recurrence relations:

$$F(0) = 0,\ F(1) = 1, \quad \text{and } F(n) = F(n-1) + F(n-2),\ \text{ for } n \geq 2;$$

$$A(0) = 1, \ \text{ and } A(n) = A(n-1) + 2\ A(n-2) + \cdots + n\, A(0),\ \text{ for } n \geq 1.$$

Prove that $A(n) = F(2n)$, for $n \geq 1$.

Problem 8. Prove that a positive integer is divisible by 3 if and only if its sum of digits is divisible by 3.

Problem 9. Find the number of ways to place 8 non-attacking rooks on the 8×8 chessboard so that all rooks are placed on black squares.

Problem 10. Find the number of permutations $w_1, w_2, \ldots, w_{10}$ of numbers $1, 2, \ldots, 10$ such that $w_i \geq i - 2$, for all $i = 1, \ldots, 10$.