Ramsey Numbers

Christos Nestor Chachamis

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Abstract

In this paper we introduce Ramsey numbers and present some related results. In particular we compute the values for some easy cases and examine upper and lower bounds for the rest of the numbers. Using the bounds derived, we computed the values for some other, not so easy, numbers.

1 Introduction

Frank Ramsey introduced the theory that bears his name in 1930. The main subject of the theory are complete graphs whose subgraphs can have some regular properties. Most commonly, we look for *monochromatic* complete subgraphs, that is, subgraphs in which all of the edges have the same color [Ram30]. In this paper we only examine graphs in which two colors are used: red and blue. There are similar (but less tight) results about graphs with more than two colors.

Some of the result shown in this paper are trivial, while others are harder to come up. For sections 2 and 3 I found the work in [Gou10] useful, while for section 4 I mostly used the work in [GG55].

For the rest of the paper we use the notation K_n for a complete graph with n vertices. We denote by $R(s,t)$ the least number of vertices that a graph must have so that in any red-blue coloring, there exists either a red K_s or a blue K_t. These numbers are called *Ramsey numbers*.

2 Preliminary results

Computing exact values for Ramsey numbers is a rather hard task. A huge amount of computational power is needed to generate all colorings of graphs and check the conditions that should be satisfied by the subgraphs. In this section we present a limited amount of Ramsey numbers whose exact value is known and easy to calculate.

2.1 Trivial values

Trivial are called the Ramsey numbers for which either $s = 2$ or $t = 2$, that is, there exists either a complete graph of friends or a pair of people that do not know each other.

Theorem 1. *R(n,2) = n.*

Proof. First, we consider a complete $(n-1)$-gon in which every edge is colored blue. In this case, there is neither a red edge, nor a complete blue n-gon, so $R(n, 2) > n - 1$.

Next, we consider any graph with n vertices. If any edge is colored red, then we have found the red pair of vertices. Otherwise, all edges are blue, so we have found the blue n-gon. This means that in any graph of n vertices there is either a blue K_n or a red K_2, so $R(n, 2) \leq n$.

Combining the above two results we get that $R(n, 2) = n$. □

By symmetry of $R(s, t)$ and $R(t, s)$, we also get that $R(2, n) = n$.

2.2 A classic result

The easiest non-trivial case is the number $R(3, 3)$. It states that in a party of that many people, there are either 3 that know each other, or 3 that do not know each other. The problem of determining this value has appeared in the early days of mathematical competitions like Putnam.

Theorem 2. *R(3,3)=6.*

Proof. First, we claim that $R(3, 3) > 5$. To show that this is true, we consider the pentagon shown in Figure 1. There is no monochromatic triangle, hence our claim is true.

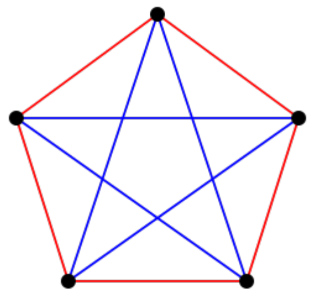

Figure 1: Pentagon without a red or blue triangle

Next, we claim that $R(3,3) \leq 6$. Consider an arbitrary coloring of the edges of a complete graph with 6 vertices. Take one of the vertices and call it X. There are 5 edges adjacent to X. Since there exist just two colors, at least 3 of those edges will be colored by the same color (say blue) □

3 Asymptotics

Even though it is tough to compute the exact values due to computational restrictions, there are many bounds that were proven mathematically. We present some of them in this section.

3.1 A naive upper bound

The following theorem is easy to prove:

Theorem 3. *If* $s > 2$ *and* $t > 2$*, then* $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Proof. Assume on the contrary that $R(s,t) > R(s-1,t)+R(s,t-1)$ for some values of s,t. Let $n = R(s-1,t) + R(s,t-1)$ and consider a complete graph of n vertices and a reb-blue coloring such that there is no red K_s or blue K_t. Pick a random vertex v. Let N_R be the set of vertices which are connected to v with a red edge and N_B be the set of vertices which are connected to v with a blue edge. It holds that $|N_R| + |N_B| = n - 1$.

By assumptions for the graph, there should be no blue K_t in N_R. Also, if there exists a red K_{s-1} in N_R, then the set $N_R \cup \{v\}$ has a red K_s, contradiction. Thus $|N_R| \leq R(s-1,t) - 1$. Using the same argument, we get

$|N_B| \leq R(s, t-1) - 1$, so

$$n - 1 = |N_R| + |N_B| \leq R(s-1, t) + R(s, t-1) - 2 = n - 2,$$

contradiction, and we showed that $R(s, t) \leq R(s-1, t) + R(s, t-1)$. $\square$

Since $R(n, 2) = R(2, n) = \binom{n}{1}$, we get the following result using induction:

Corollary 1. $R(s, t) \leq \binom{s+t-2}{s-1}$.

A particularly interesting case is the diagonal Ramsey numbers, that is, those of the form $R(k, k)$. Using corollary 1 we get that $R(k, k) \leq \binom{2k-2}{k-1}$. This bound has complexity $O(\frac{4^{k-1}}{\sqrt{k-1}})$, so it grows exponentially fast. However, even for small k's this bound is not very tight.

If $R(s-1, t)$ and $R(s, t-1)$ are both even, then we have the following theorem:

Theorem 4. $R(s, t) \leq R(s-1, t) + R(s, t-1) - 1$

Proof. Suppose $R(s-1, t) = 2p$ and $R(s, t-1) = 2q$. Take a graph of $2p + 2q - 1$ vertices and a vertex A. There are $2p + 2q - 2$ edges ending at A. Then, consider the following cases:

1. $2p$ or more edges end at A
2. $2q$ or more edges end at A
3. $2p - 1$ red edges end at A and $2q - 1$ blue edges end at A

For first case, consider the set T_1 of the vertices at the farther ends of the $2p$ or more segments. Since the numbers of vertices in T_1 is greater than or equal to $R(s-1, t)$, there is either a red K_{s-1} or a blue K_t. However, if there is a red K_{s-1}, then the set $T_1 \cup \{A\}$ is a red K_s. Thus, the theorem holds in this case.

The same argument shows that the theorem holds for second case as well.

The third case cannot hold for every vertex A of the graph. Indeed, if it did, there would be $(2p + 2q - 1)(2p - 1)$ red endpoints, which is an odd number. However, every edge has two endpoints, so this number should be even. This means that there exists at least one vertex for which either case 1 or case 2 holds. Since theorem was shown for these two cases, it holds for the third case, too. $\square$

In section 3.2 we present a lower bound for diagonal Ramsey numbers.

3.2 A lower bound using probabilistic method

The following theorem is due to Erdos [Erd47].

Theorem 5. *Let $k, n \in \mathbb{N}$ be such that $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Then $R(k,k) > n$.*

Proof. In order to show that $R(k,k) > n$, it is sufficient to show that there exists a colouring of the edges of K_n that contains no monochromatic K_k. Consider an edge colouring of K_n in which colours are assigned rabdomly. Let each edge be coloured independently, and such that for all edges e it is

$$\mathbb{P}(\text{edge e is red}) = \mathbb{P}(\text{edge e is blue}) = \frac{1}{2}.$$

There are $\binom{n}{k}$ copies of K_k in K_n. Let A_i be the event that the i^{th} K_k is monochromatic. Then:

$$\mathbb{P}(A_i) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}},$$

where the leading 2 is because there are two colours from which to choose. Then:

$$\mathbb{P}(\exists \text{ a monochromatic } K_k) = \mathbb{P}(\cup_i A_i) = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

However, $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ by the assumption of the theorm, so

$$\mathbb{P}(\exists \text{ a colouring with no monochromatic } K_k) > 0.$$

Hence, there exists a colouring with no monochromatic K_k. □

We can use the result proved above to show another useful bound:

Corollary 2. *For $k \geq 3$, $R(k,k) > 2^{\frac{k}{2}}$.*

Proof. Given $k \geq 3$, define $n := \lfloor 2^{\frac{k}{2}} \rfloor$. Then

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{1-\frac{k(k-1)}{2}} \leq \frac{\left(2^{\frac{k}{2}}\right)^k}{k!} \cdot 2^{1-\frac{k^2}{2}+\frac{k}{2}} = \frac{2^{1+\frac{k}{2}}}{k!}.$$

However, $\frac{2^{1+\frac{k}{2}}}{k!} < 1$ if $k \geq 3$, so Theorem 5 applies. □

Corollary 2 is particularly interesting because it provides an insight into how diagonal Ramsey numbers grow. Specifically, it shows that they grow exponentially in k.

3.3 A special case: no red triangles

When we deal with the case of $R(3,t)$, one would expect that computations would be easier, since we can have a fairly limited number of red edges. As it turns out, we still don't know the exact numbers for most values of t (we only know for $t \leq 9$). However, this cases allows for stricter bounds.

It was shown very early [GY68] that the upper bound of $R(3,t)$ is $O(t^2 \log\log t/\log t)$. Erdos had also shown the lower bound $R(3,t) = \Omega(t^2/(\log t)^2)$ [Erd61]. Later, both of these bounds were improved. Ajtai, Komlós and Szemerédi [AKS80] showed that $R(3,t) = O(t^2/\log t)$, while Kim [Kim95] showed that $R(3,t) = \Omega(t^2/\log t)$.

The above results show that $R(3,t) = \Theta(t^2/\log t)$, which is much better than the exponential bounds found in previous subsections.

4 Obtaining other Ramsey numbers

In this section we compute the values $R(3,4)$, $R(3,5)$ and $R(4,4)$. The theorems and proofs that follow were first shown in [GG55].

Theorem 6. *$R(3,4) = 9$ and $R(3,5) = 14$.*

Proof. From Theorem 4 it follows that

$$R(3,4) \leq R(2,4) + R(3,3) - 1 = 4 + 6 - 1 = 9.$$

Then, we claim that $R(3,5) > 13$. Indeed, we consider a K_{13} in which we number vertices with numbers 0-12 and color the edges such that an edge is red if and only if the difference of the numbers of the two adjacent vertices is 1, 5, 8 or 12 (assume all operations happen modulo 13). The red edges as shown in Figure 2. Then, the graph contains no red triangle and no blue K_5. It is easy to see that there is no red triangle.

We can also show that there is no blue K_5. Assume on the contary that a blue K_5 exists. By symmetry, assume that a vertex of the k_5 is the 0. Then, the other vertices must be in the "clusters" $2, 3, 4$, or $6, 7$, or $9, 10, 11$. By pidgeon hall principle, at least two are in the same cluster. Since the edge between them is not blue, they are in a cluster of three total numbers. Without loss of generality assume they are 2 and 4. Then the others can only be 6 and 11. But these two differ by 5, contadiction.

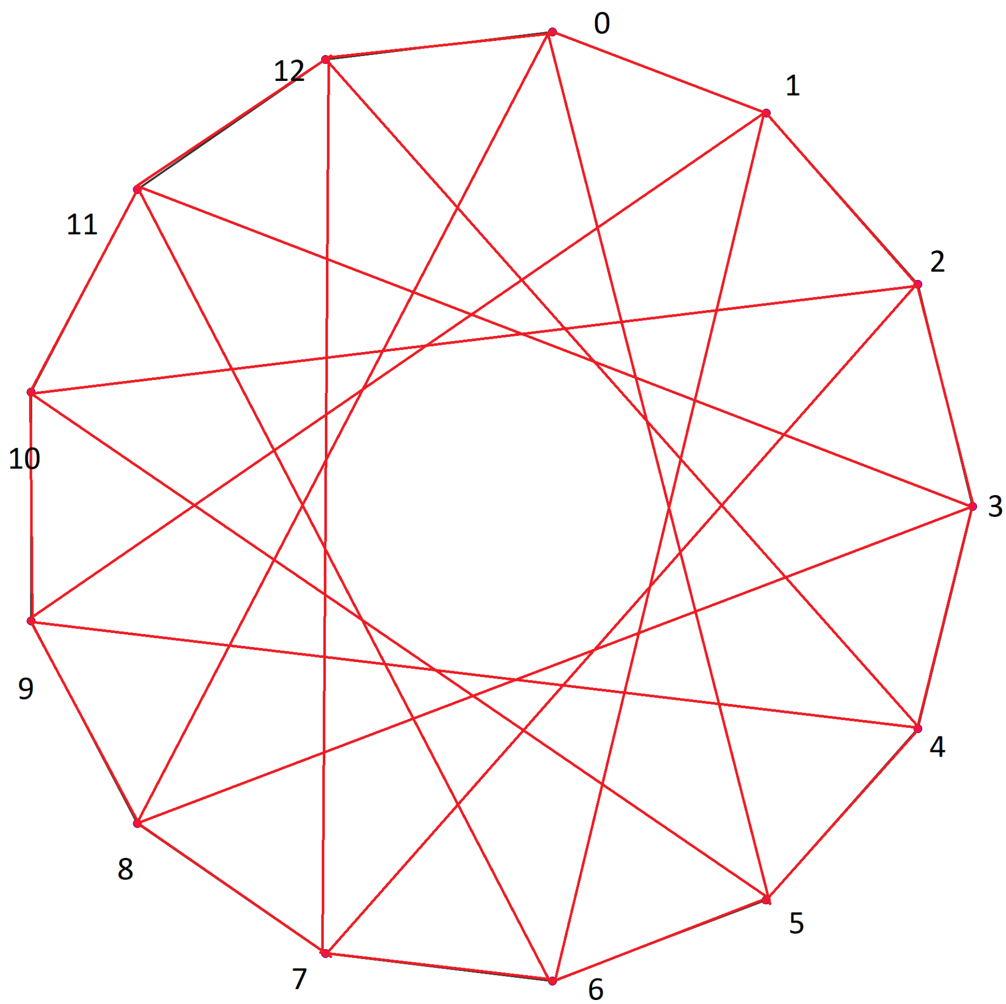

Figure 2: The red edges of the tridecagon. No red triangle exists

Thus $R(3,5) > 13 \Rightarrow R(3,5) \geq 14$. However,

$$R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$$

This means that we must have $R(3,4) = 9$ and $R(3,5) = 14$. □

Theorem 7. $R(4,4) = 18$.

Proof. First, using Theorem 3 we get that

$$R(4,4) \leq 2R(3,4) = 18.$$

It is enough to show that $R(4,4) > 17$. We consider a K_{17} in which we number vertices with numbers 0-16 and color the edges such that an edge is if and only if the difference of the numbers of the two adjacent vertices is 1, 2, 4, 8, 9, 13, 15, 16 (all operations are modulo 17). By symmetry, it is enough to show that vertex 0 cannot be in a red K_4 or a blue K_4.

Vertex 0 is connected by red edges with the vertices 1, 2, 4, 8, 9, 13, 15 and 16. Assume there is a red K_4. If 1 is in that, the remaining vertices must be in the set $\{2, 9, 16\}$, but no two of them are connected with red vertices. Similarly, for 2, the set of remaining vertices should be in $\{1, 4, 15\}$, for 4, the set of remaining vertices should be in $\{2, 8, 13\}$, and for 8, the set of remaining vertices should be in $\{4, 9, 16\}$. No red edges are contained in these sets. The rest are symmetric. Thus there can be no red K_4 that contains 0.

What about a blue K_4? Vertex 0 is connected by red edges with the vertices 3, 5, 6, 7, 10, 11, 12 and 14. Assume there is a blue K_4. If 3 is in that, the remaining vertices must be in the set $\{6, 10, 14\}$, but no two of them are connected with blue vertices. Similarly, for 5, the set of remaining vertices should be in $\{10, 11, 12\}$, for 6, the set of remaining vertices should be in $\{3, 11, 12\}$, and for 7, the set of remaining values should be in $\{10, 12, 14\}$. No blue edges are contained in these sets. The rest are symmetric. Thus there can be no blue K_4 that contains 0.

Hence $R(4,4) > 17 \Rightarrow R(4,4) = 18$ □

4.1 Computation based research and future work

There are five more numbers which are known: $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$ and $R(4,5) = 25$. They were discovered in [GY68],

[GR82] and [MR95]. For all of those numbers the researchers need not only a great amount of ingenuity, but also a great amount of computational power. The methods and bounds presented before are no longer very useful, as one needs to check a large number of cases and graphs.

After investigating the basic theory behind Ramsey Theory, it becomes obvious that it is a field that currently goes beyond the theoretical graph theory techniques. As numbers grow on an exponential scale, so does the number of cases to be checked or enumerated, and researchers should either increase their computational powers, or search for answers on techniques from different fields.

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