An Introduction to Chromatic Polynomials

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Abstract

This paper will provide an introduction to chromatic polynomials. We will first define chromatic polynomials and related terms, and then derive important properties. Once the basics have been established, we will explore applications and theorems related to chromatic polynomials, and introduce the idea of chromatic polynomials associated with hypergraphs and chromatic polynomials associated with fractional graph colouring. To conclude the paper, we will discuss some unsolved graph theory problems related to chromatic polynomials.

1 Introduction

Chromatic polynomials were first defined in 1912 by George David Birkhoff in an attempt to solve the long-standing four colour problem. First, it is necessary to notice that the number of ways a map can be coloured using k colours has a polynomial dependence on k, which we will prove in a rigorous way later in the paper. By viewing maps as loopless planar graphs and defining mathematical properties of colourings on these graphs, Birkhoff hoped to prove that any two-dimensional map can be coloured with just four colours so that no neighbouring bodies are assigned the same colour. Although he was unsuccessful in this attempt, chromatic polynomials became an important object in algebraic graph theory and continue to be a subject of great interest today.

Birkhoff's definition is limited in that it only defines chromatic polynomials for planar graphs. The concept of chromatic polynomials was later extended in by Hassler Whiteney 1932 to graphs which cannot be embedded into the plane.

Today, the chromatic polynomial has been studied in many novel forms. We are now able to define properties of graphs with more interesting geometries, and on the frontier there has even been study of generalizations of the problems as are seen in hypergraphs or fractional colourings.

1.1 Proper and Improper Colourings

A graph colouring or vertex colouring is defined as the assignment of a colour or unique label to each vertex. A proper colouring is any colouring such that no adjacent vertices share the same colour. Any colouring which does not satisfy this requirement will be called an improper colouring. We say that a graph G is n-colourable if there exists a proper colouring of G with using a set of colours with cardinality n.

Notice that by this definition of a proper colouring, we may colour each vertex with a unique colour. We can even choose a colour set with cardinality n larger than that of the vertex set, and it will be still considered to be n-colourable. However, it is often more interesting to restrict $n=$ —S— to be smaller than the cardinality of the vertex set. The smallest number of colours needed in order to properly colour a graph is called the chromatic number of G, represented by $\chi(G)$ [2].

More formally, define $V(G)$ as the vertex set of any graph G, and $E(G)$ as the edge family of graph G. We will define a vertex colouring as some function $\sigma : V(G) \rightarrow \{1, 2, 3, \cdots\}$, where integers represent unique colours. Any proper colouring of G is such that for any $e_{ij} \in E(G)$, $\sigma(i) \neq \sigma(j) fori, j \in V(G)$.

2 Properties

2.1 Basic Properties

Theorem 1. *The chromatic function of a simple graph is a polynomial.*

Proof. Before we discuss properties of chromatic polynomials, we must prove that they are indeed polynomials. Construct colourings of G by partitioning $V(G)$ into independent sets and assigning a unique colour to each independent set. As a review, an independent set is a set of vertices in which no two members are adjacent. Given a colour set of k members, there will be k ways to choose the colour for the first set, $(k-1)$ ways to choose the colour for the second set, $(k-2)$ ways to choose the colour for the third set, and so on. There are then $k(k-1)(k-2)\cdots$ possible ways to colour the vertices of graph G such that they result in a proper colouring, and it is clear that this is some polynomial. □

Corollary 1. *The degree of a chromatic polynomial on n colours is at most n*

Proof. There is only one possible partition of $V(G)$ which has n parts, the partition in which all vertices are separated. The polynomial for this partition is one of degree n. The polynomials for all other partitions of $V(g)$ are all of degree $< n$. We know that the sum of such polynomials yields one of degree n. □

2.2 Deletion Contraction

There are many graphs for which it is quite difficult to determine the chromatic polynomial purely by inspection. The Deletion-Contraction property can be very useful in breaking down the problem into parts which are much easier to solve. Define $G - e$ as graph G with edge e deleted. Define G/e as the graph

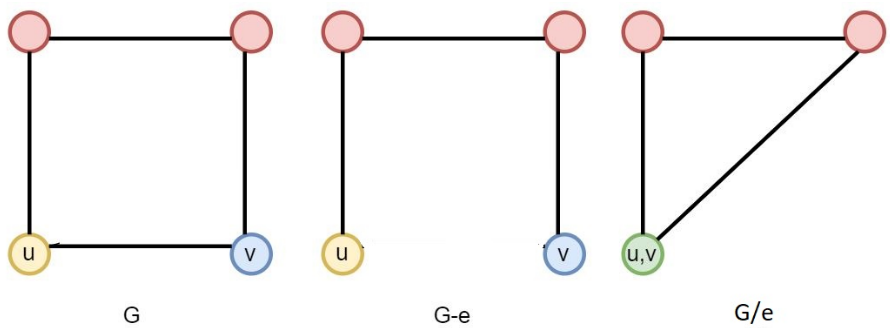


Figure 1: $G - e$ is simply constructed by deleting edge e. Visually, G/e_{uv} is the result of shrinking u and v towards the middle, and then deleting the two incident edges which would form loops.

produced by contracting edge e i.e. simultaneously deleting edge e and merging the two vertices which it originally connected.

Theorem 2. $P(G, k) = P(G - e, k) - P(G/e, k)$

Proof. We will demonstrate that this is true using combinatorial arguments. The number of colourings of $G - e_{vw}$ where vertices v and w are the same colour is the same with or without the contraction. This is easy to see, because the contraction would simply join the two vertices into one vertex of the same colour that both vertices shared prior to the contraction. The number of colourings of $G - e_{vw}$ where v and w are different colours is the same with or without the deletion. In other words, if v and w are different colours, we can choose to join or separate them without affecting whether or not the given colouring is proper. These two cases represent the two possibilities for colourings of G, and so we have that $P(G, k) + P(G/e, n) = P(G - e, k)$. The proof is concluded by subtracting $P(G/e, k)$ from both sides [6]. □

2.3 Bounding the Chromatic Number

Theorem 3. *For graph G with maximum degree D, the maximum value for χ is D unless G is complete graph or an odd cycle, in which case the chromatic number is $D + 1$.*

Proof. This statement is known as Brooks' theorem, and colourings which use the number of colours given by the theorem are called Brooks' colourings. A simplified proof using greedy colouring and depth first search has been shown by Lovasz.This proof can be found in [1], which I suggest as supplemental reading. □

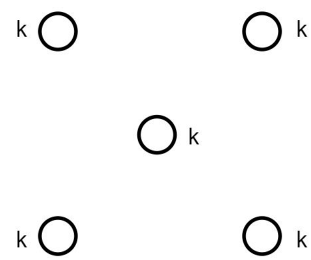
Theorem 4. *Any graph with clique size k requires at least k colours for a proper colouring.*

Proof. A clique is defined as a subset of vertices such that every pair of vertices in the subset is adjacent. Another way to think of this is that the induced subgraph is a complete one. Each vertex within the clique is adjacent to $k-1$ other vertices which are all in turn adjacent to the other $k-1$ vertices in the clique excluding themselves. By the definition of a proper colouring, the colours assigned to each of the vertices in such a clique, including that of the original vertex, must all be distinct. The claim follows.

Notice that the reverse statement is not true. In fact, by Mycielski construction it is possible to produce graphs which retain the property of being triangle-free while reaching arbitrarily high chromatic numbers [3].

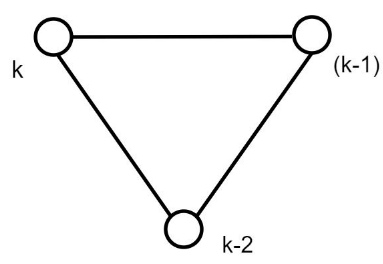
□

3 Examples



Claim 1. *The chromatic polynomial for an empty graph on n nodes is k^n*

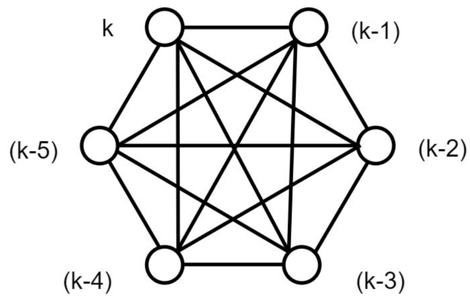
Proof. Because no vertex is adjacent to any other vertex in the graph, we may choose any arbitrary colour within our colour set to assign to any vertex in the graph. Multiplying the k options of colour for each of the n nodes, we have that $P(G,k) = k^n$ □



Claim 2. *The chromatic polynomial for a triangle graph is $(k)(k-1)(k-2)$*

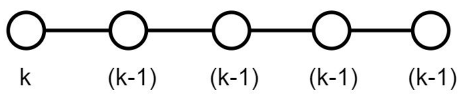
Proof. We can choose any of the k colours for the first vertex we colour. For the second vertex we colour, we have $(k-1)$ choices, because we cannot choose the same colour as that of the first vertex as all 3 vertices are adjacent. Similarly, we will have $(k-2)$ choices of colour for the last vertex we colour. This yields a polynomial $k(k-1)(k-2)$.

□



Claim 3. *More generally, the chromatic polynomial for a complete graph on n nodes is $(k)(k-1)(k-2)\cdots(k-n+1)$*

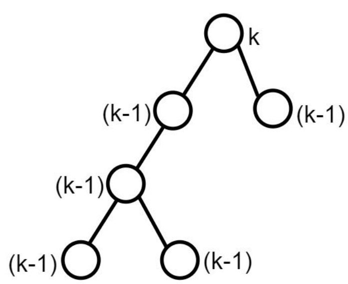
Proof. The argument for this is identical to that which we showed for the triangle graph, but terminates later when we reach the nth vertex. If we systemically assign colours as we did for the triangle graph, the number of colours we will have available for the last vertex will be $(k-n+1)$, because we started at $(k-0)$ and we count down for $(n-1)$ more terms. □



Claim 4. *The chromatic polynomial for a path graph on n vertices is $k(k-1)^{(n-1)}$.*

Proof. Let us begin colouring the graph from the leftmost node. There are k choices of colour for the first vertex, and $(k-1)$ choices of colour for the second vertex because it is adjacent to the first vertex. For the third vertex, we also have $(k-1)$ choices of colour because we cannot use the 1 colour which we used for the second vertex (to which the third vertex is adjacent). Similarly, we will have $(k-1)$ choices of colour for all of the remaining vertices.

□



Claim 5. *The chromatic polynomial for any tree is also* $k(k-1)^{(n-1)}$.

Proof. We will do this proof by induction on n. For our base case, choose $n = 1$. The claim clearly holds as we have k choices of colour for the 1 vertex. Assume that the claim holds for a graph on n vertices. Select any leaf node and detach it from the tree. The chromatic polynomial for the resulting tree will be $k(k-1)^{(n-2)}$. Now, replace the leaf node. We can assign any colour to this node except for that of its neighbour, and thus have $(k-1)$ choices of colour. $k(k-1)^{(n-2)}(k-1) = k(k-1)^{(n-1)}$, so the claim holds by induction on n. □

4 Special Cases

4.1 Uncoloured Option

Let us consider what the equivalent to a chromatic polynomial would be given the option to not colour a particular vertex. This is distinct from adding another colour, because it doesn't affect the possible colouring of adjacent vertices. For any graph, to remove uncoloured vertices and their incident edges to get a subgraph. You can produce any induced subgraph by this method, so there is no undercounting or overcounting.

The polynomial associated with improper colourings of a graph would then be the sum of the chromatic polynomials of the subgraphs induced on each subset of vertices, including the zero-vertices subgraph, assigning to it the chromatic polynomial 1.

4.2 Hypergraphs

A hypergraph is a generalization of a graph in which an edge may connect any number of vertices. The vertex set is defined similarly to any other graph, but because each hyperedge may connect many vertices, the hyperedge set may have members which are sets of size greater than 2. In other words, $E(H)$ is a subset of $P(V) - \emptyset$ where $P(V)$ is the power graph, the set of all possible subsets of V including the empty set and V itself. From this, it is easy to see that if the cardinality of every hyperedge in G is equal to 2, then G is a graph. Notice that we cannot have a hyperedge with 0 members, but we can have a hyperedge which circles the entire graph and contains all elements of V.

A weak proper colouring of a hypergraph is such that every hyperedge connects at least two vertices of distinct colours. In other words, no hyperedge of cardinality at least 2 can be monochromatic. The name suggests that these conditions are weaker than those of a strong proper colouring wherein vertices contained within the same hyperedge must all be assigned distinct colours. However, strong proper colourings are often ignored in research involving chromatic polynomials for hypergraphs, because they can be constructed by simply analyzing a chromatically equivalent simple graph [7].

Given a hypergraph $H = (V, E)$, construct a simple graph G on V in which vertices u, v are adjacent if and only if they are contained in some edge of H.

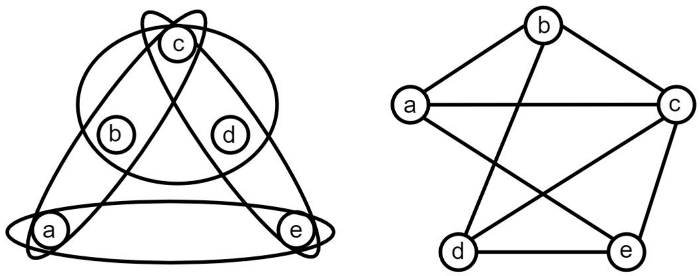


Figure 2: Hypergraph H and its associated simple graph G.

In other words, for every hyperedge e in H, form a clique in G. As we showed before , each vertex in a clique of a simple graph must be a different colour, which is also true in this case for vertices contained within a hyperedge by the definition of a strong proper colouring. From here, it is clear that the proper colourings for the simple graph G and the strong proper colourings of H are the same. More interesting properties of colourings of hypergraphs which do not apply to simple graphs can be found in [7].

4.3 Fractional Colourings

Recently, a new branch of graph theory, fractional graph theory, has emerged. While graph theory often focuses on integer-valued concepts, fractional graph theory focuses on how these concepts can be extended to non-integral values. This field includes the topics of fractional arboricity, fractional matching, fractional colouring, and more.

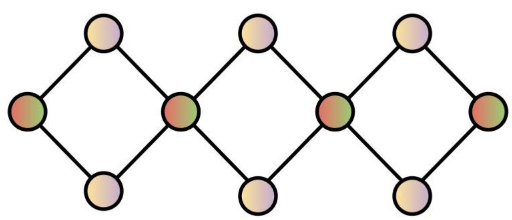


Figure 3: 4:2 colouring of a simple graph G

As this is an introductory paper, we will only define fractional colourings, and suggest additional readings. The fractional colouring of a graph can be viewed as the linear programming relaxation of a traditional colouring as defined in Section 1. This is to say that vertices may be assigned multiple colours with different weights, as opposed to each taking 1 colour. The subtlety arises from the fact that none the colours assigned to one vertex may be assigned to adjacent vertices. Otherwise, each vertex might be split into many monochromatic vertices for a simpler problem.

A x-fold coloring of G is a colouring such that s colours are assigned to each vertex, and is proper if the sets of adjacent vertices are disjoint. An $k : x$-coloring is a x-fold coloring out of k available colors [5].

5 Unsolved Problems

What makes a polynomial chromatic? In other words, given some polynomial, how can we determine whether or not there exists some graph for which it is the chromatic polynomial? The polynomial

$$k^4 - 3k^3 + 3k$$

seems to fit the general form of a chromatic polynomial, yet there is no graph which has this as its chromatic polynomial. So while we have necessary conditions that we can check a given polynomial against, there is still no set of sufficient conditions.

In the examples, we saw that two unique graphs might share the same chromatic polynomial. What is a necessary and sufficient condition for two graphs to have the same chromatic polynomial?

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