Infinite chess

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1 Abstract

We present a summary of known results about infinite chess, and propose a concrete idea for how to extend the largest known game value for infinite chess form ω^4 to ω^5. Currently, it is known that whether white has checkmate in n or fewer moves is decideable for finite positions. For finite positions, the highest known game value is $n\omega$ for all n and for inifinite positions the highest known game value is ω^4.

2 Introduction

Infinite chess is chess on an infinite board. There can be any number of pieces on the board, and exactly one of each king. Pawns never promote, and there is no 3 fold repetition or 50 move rule.

The main question we ask is "can white win in n moves or fewer?". Because some positions have a winning strategy for white, but not in any finite number of moves, we use ordinal numbers to describe positions with transfinite values. We assign ordinal numbers to positions that white has a winning strategy as follows:

- We assign 0 to a position in which the black king is checkmated
- If it is white to move, we assign the position the minimum of all the possible positions white can move to.
- If it is black to move, we assign the position the supremum of all the possible positions that black can move to.

The ordinal number assigned to a position is its ıvalue. These recursive rules assign a value to every winning position for white; a position that isn't winning for white is not assigned any value.

In section 3 we talk about finite positions, and show how to construct a position with value $n\omega$ for all n. We also give a summary of the proof of the decideability of mate-in-n for finite positions. In section 4 we talk about infinite positions, and describe how to construct a position with value ω^4. In section 5 we propose a modification to the position from section 4 that could potentially produce a position with value ω^5.

3 Finite positions

Theorem 3.1. *There exists a finite position with value $n\omega$ for all n. [2]*

Proof. Consider this position with a black rook, white king, and a trapped black king about to be checkmated in 2 moves. This position has value ω. It is black to move. Black moves their rook as far away as they like from the white king. White then moves their knight to threaten mate in 1. Black then begins a series of harassing checks with the rook on the white king. In order to get out of check, the white king must chase after the black rook one space at a time. This gives black n moves of delay where n is how far the black rook moved on the first turn. When the white king gets there, the black rook can't keep checking without getting captured, so black loses their rook, and then on the next turn white can checkmate black.

Because black on their first turn could choose a position with value n for any finite n, the initial position has value ω since ω is the supremum of the set of natural numbers.

We can extend this position to $n\omega$ for any n by starting with white threatening mate in n instead of mate in 1. In this position, black starts by moving their rook as far away as they like. Then white makes one move of progress on their mate in n sequence. Then a series of harassing checks on the white king happen. When the white king gets to the rook, the rook again moves as far away as it likes. This sequence repeats n times, at which point white can checkmate. Since there are n times in which black decides how many moves it will take until the next such time, the position has value $n\omega$. $\square$

Theorem 3.2. *The mate in n problem is decideable for finite positions. [1]*

Proof. The outline of the proof is as follows:

- We model positions as strings, and show that they form a regular language.
- Then we show that several predicates on positions such as BlackInCheck(p) and LegalMove(p, q) are also regular.
- We then note that the MateInN predicate can be expressed as a series of quantifiers applied to these primitive predicates. Since the truth of arbitrary quantified regular predicates on regular languages is decideable, MateInN is decideable.

First, we model positions as a regular language. Let A be the set of pieces in the position. A position will be a sequence of tuples (x, y, c) for each piece i in A. x and y are the coordinates of piece i, and c is a bit that is 1 if piece i has been captured, and 0 otherwise. A position additionally has a single bit indicating which player's turn it is.

The only condition on a position to be considered valid is no two pieces are on the same space. Since a finite automaton can check if a string is in the form of a sequence of tuples, and since it can also make a finite number of equality checks to make sure that no two pieces, this ecoding of positions forms a regular language.

Now we show that several predicates are also regular languages. First we have Attack(i, x, y, p), which says that piece $i \in A$ can attack square (x, y) in position p. We do case analysis on the differnt pieces i. If i is a knight, pawn, or king, there are only finitely many spaces it can legally move to, so Attack(i, x, y, p) is satisfied if (x, y) is in this set, so the predicate is regular for these pieces. If i is a bishop, then Attack is true if two conditions hold: (1) the bishop is on the same diagonal as the space (x, y), and (2) there is no piece in the way. If the bishop is at (a, b), (1) can be expressed as the linear relation $x + y = a + b$ or $x - y = a - b$. Since linear relations can be recongized by finite automata, (1) is a regular language. For (2), a piece i located at (a', b') is in the way of the bishop moving toward the top right if $a' + x = b' + y$ and $x < a' < a$, and a similar linear relation holds for a piece in the way of a bishop moving in a different direction. Since there is a fixed finite set of pieces, there is a fixed finite list of linear relations, so this is also a regular language. If i is a rook, then Attack is essentially the same as for a bishop, with slightly simpler linear relations of $x = a$ or $y = b$. If i is a queen, then Attack is just the disjunction of Attack for a bishop and Attack for a rook.

The next predicate is BlackInCheck(p). This is true if an piece is attacking the black king. Formally, for a piece set $A = \{i_1, i_2, ...i_n\}$, if the black king is located at (x, y), this becomes BlackInCheck(p)=Attack(i_1, x, y, p)$\vee$ Attack(i_2, x, y, p) $\vee \ldots \vee$ Attack(i_n, x, y, p). That is, BlackInCheck is the disjuction of the Attack predicates for each piece. Since this is a first order formula of regular predicates, it is also a regular language.

The next predicate is LegalMove(p, q). This is true if there is a single legal move from position p that results in position q. For this to be true, the following must hold:

1. Exactly one piece is in a different position
2. In position p, that piece was attacking the space it moved to in q.
3. If there was a piece in the space it moved to, that piece was of the opposite color and is now marked as captured.
4. The piece that moved is of the color of the player whose turn it was in postiion p, and it is now the other player's turn in q.
5. The player that just moved is not in check in q.

(1) is easy becasue the piece set is a fixed finite set, so we can just enumerate all combinations of exactly one piece moing.
(2) is easy because it is just the Attack predicate from before.
(3) is easy since the set of pieces is finite and we can just check the color and capture bits for all of them.
(4) is just checking two color bits.
(5) is just the BlackInCheck predicate.
Thus, LegalMove is a regular language.

Now we get the predicate BlackInCheckmate, which is defined as BlackInCheck(p) $\wedge \neg\exists q$ LegalMove(p,q). That is, black is in check and has no legal move. This is regular since it is a first order formula of regular predicates.

We define the predicate MateIn(n) recursively as follows:
MateIn(0) is BlackIn Checkmate
MateIn(n, p) is $\exists q \forall r$ LegalMove(p, q) $\wedge$ LegalMove(q, r) $\wedge$ MateIn($n-1$)
For every n, we can expand out the recursive definition to get MateIn(n) in the form of a first order predicate with $2n$ alternating quantifiers. Since this is a first order predicate, MateIn(n) is a regular language for all n. $\square$

4 Infinite positions

Theorem 4.3. *There exists a position with value ω^4. [3]*

We will give an overview of the position, and describe the main line and why the position has value ω^4. We won't prove that no player has a deviating strategy that is better; for a full proof see [3].

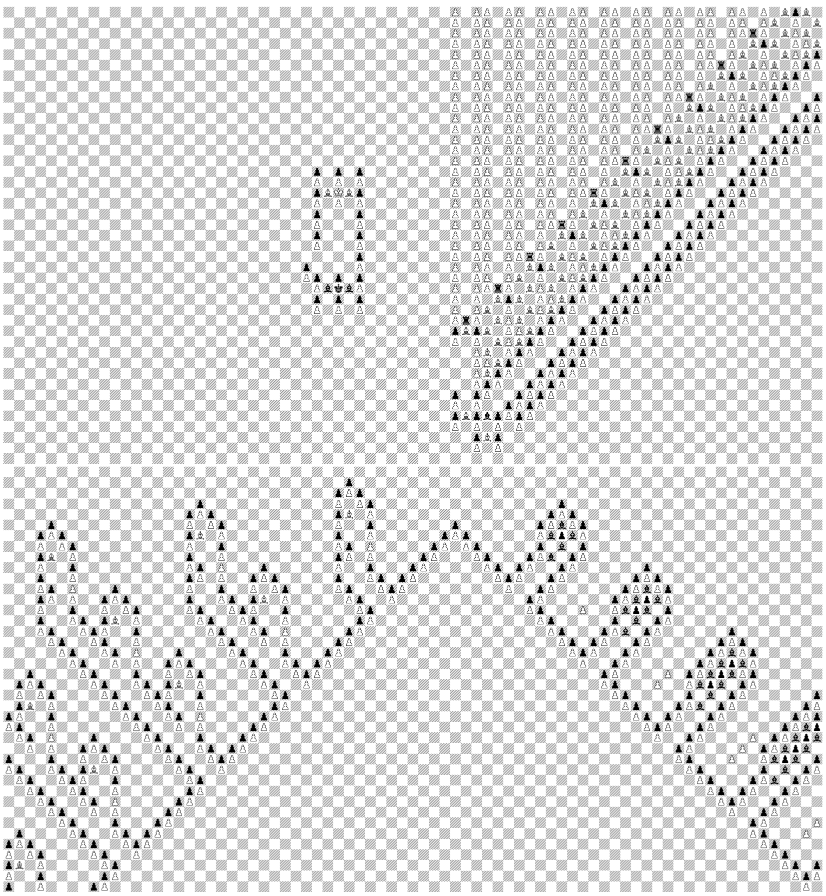

Figure 1: A position in infinite chess with value ω^4

The positions has 2 major components: The rook towers on the right, and the bishop cannons on the bottom. In the left corner we also have the "throne room" where the two kings are trapped, threatening to be checkmated if a bishop can get in range.

In the upper left we have the rook towers. Each rook tower consists of a black rook which can move arbitrarily far up its file, and infinite columns of white pawns on each side of it. At the bottom of each rook tower is a series of trapped white bishops, which are blocking a pawn that when released can threaten the next rook tower. The final rook tower, instead of just releasing a white pawn to activate the next rook tower, released a white bishop that will quickly go checkmate the black king. Below all of the rook towers is a black bishop that can move arbitrarily far to the upper right.

The main line starts with black moving this bishop arbitrarily far to the upper right, effectively choosing how many rook towers will come into play. White then captures the bishop with one of the pawns. This pawn then marches foward, eventually breaking a small hole in the pawn wall on the top of this bishop tunnel. This hole allows whtie to get a single pawn walking up to threaten the black rook in one of the rook towers. Once the pawn captures the single black pawn next to the rook, black's rook becomes under attack and needs to move. If it doesn't, then white pawn will capture it, creating a hole that whtie can use to untangle his bishops and release pawn to activate the next rook tower. Black thus moves his rook up arbitrarily far. White then captures the rook and spends n moves shuffling his pawns foward, where n is the number of spaces black moved his rook up. At this point, white can then untangle his bishops and activate the next rook tower.

Since black can choose any number of rook towers to activate, and for each rook tower, can choose any number of moves for white to spend before activating the next rook tower, the rook towers alone give the position a value of ω^2.

In the lower half of the board we have the bishop cannons. To the right are the cannons that release the bishops, and to the left are the gateways that the bishops use to attempt to exit and checkmate the white king.

Between each of white's moves on the rook towers main line, black chooses a bishop cannon to activate with b bishops in it by moving the front bishop of that cannon out to attack the entrance to a gateway with n exits. White does not need to immediately respond to this, and can make a move on the main line. Black then moves that bishop to one of the n exits on the gateway. White is forced to respond by moving his bishop in the gateway out one space. If white does not, then black could escape by capturing the white pawn and then would immediately go checkmate white. By white moving their bishop into position, if black tries to break free with their bishop, the white bishop will chase after and checkmate black a turn sooner. Black proceeds to then move their bishop to a different exit, forcing another white reply, and keeps doing this until there are no more exits left.

At this point, black releases another bishop from the bishop cannon. Unlike the first bishop, which white did not have to immediately respond to, white must immediately respond here. If white does not, black will push the pawn that was freed up by moving the bishop to break a hole in the pawn wall surrounding the bishop cannon. This hole would then allow a black bishop to escape the cannon and promptly go checkmate the white king. White's response is to push a guard pawn up one space, such that if black tried to break a hole with their pawn, white can just recaputure with the guard pawn keeping the pawn wall intact.

Thus, white must respond to every bishop released, and for each bishop released black can choose how many responses white must make, black can force a delay of ω^2 for each move that white can make in the main line in the rook towers.

Thus, the total value of this position is $\omega^2\omega^2 = \omega^4$.

Proof. $\square$

5 A position with value ω^5

Conjecture 5.4. *There exists a position with value ω^5.*

Here we propose a modification to the ω^4 position by [3] that has value ω^5. We describe the main line, and give a brief argument for why players don't have any useful deviations from the main line.

We add to each bishop cannon a special pawn that is released when the first bishop fires. This pawn can march down a corridor and release a single black dark squared bishop into the bottom right quadrant of the board. Unlike before, this quadrant is enclosed, so black cannot use this bishop to immediately checkmate the white king. Insetad, in order to break out black needs to attack a key white guard pawn with more pieces than white has defenders.

There is actually a family of inifinitely many such guard pawns, each of which has room for only finitely many black pieces simultaneously attacking it. The guard pawn is keeping black from releasing his queen into the wild, where it can wreak havoc and ultimately let black release mating bishops into the main open area. Each of these gaurd pawns is locked by a black

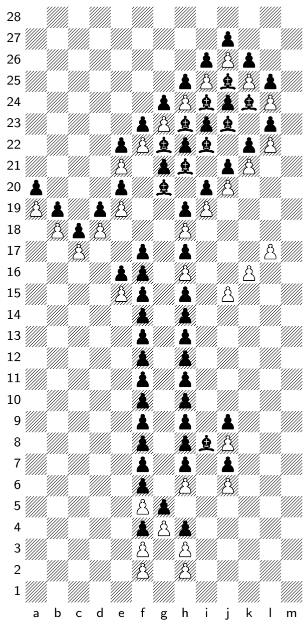

Figure 2: A modified bishop cannon

pawn in the way which black must first spend a move pushing in order to make that guard pawn available to attack. For each n, there is a guard pawn with n spaces available for black bishops to threaten the guard pawn. Here is an example for $n = 3$.

White has a bank of inifinitely many white bishops which can be slowly released to defend the guard pawn. White can spend a move releasing a single defending bishop. These bishops can be lined up on the red diagonal by white.

The main line goes as follows. Black first chooses a white guard pawn to target and spends a move "unlocking" it. White then gets one play in the rook towers. Then black starts firing bishop cannons. Unlike before, the first n bishop cannons now have their first bishop fire with force, since white has to respond by immediately releasing a guard bishop to defend their pawn. If white doesn't, then black will release their own bishop first, break open the white guard pawn, and release the queen that will wreak havoc and ultimately force a checkmate on the white king.

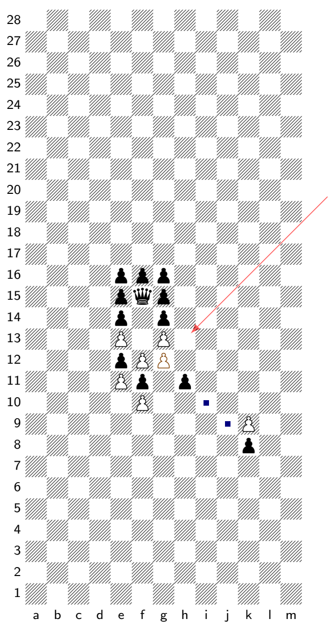

Figure 3: A white guard pawn with space for 3 attacking black bishops

References

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[2] C. D. A. Evans and J. D. Hamkins. "Transfinite game values in infinite chess". In: *ArXiv e-prints* (Feb. 2013). arXiv: `1302.4377 [math.LO]`.

[3] C. D. A. Evans, J. D. Hamkins, and N. L. Perlmutter. "A position in infinite chess with game value ω^4". In: *ArXiv e-prints* (Oct. 2015). arXiv: `1510.08155 [math.LO]`.