Analysis and Applications of Burnside's Lemma

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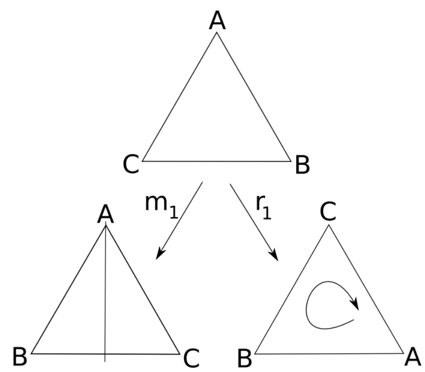
Abstract

Burnside's Lemma, also referred to as Cauchy-Frobenius Theorem, is a result of group theory that is used to count distinct objects with respect to symmetry. It provides a formula to count the number of objects, where two objects that are symmetric by rotation or reflection are not categorized as distinct. The proof involves discussions of group theory, orbits, configurations, and configuration generating functions. The theorem was further generalized with the discovery of the Polya Enumeration Theorem, which expands the theorem to include all number of orbits on a set. This theorem not only enumerates the number of distinct objects, but also the configurations of each object and its frequency. The result from Polya enumeration theorem has been extensively used, in particular in the enumeration of chemical isomer compounds. This paper will explore chemical compound enumeration along with another interesting application within music theory.

1 Introduction

To find the number of colorings of a fixed six-sided cube using n colors, we know that the number of colorings would be n^6 because each side has the option of each color. However, many of the colorings would be the same if we were allowed to rotate the cube. To solve coloring problems for asymmetric objects where we do not account for the cube rotating, we can just use binomial and multinomial theorems which involve basic combinatorics. However, now if we take into account that the cube can be rotated, our objective is to count the number of distinct colorings that cannot be flipped or rotated to match one another.

The following diagram is an example of transformations done on a triangle with corners labeled A, B, and C. The triangle has been transformed through reflection and rotation, and all three diagrams represent one distinct coloring.



In order to count the number of distinct colorings of an object, we can apply Burnside's Lemma and Polya's Enumeration Theorem; while Burnside's Lemma will enumerate the number of distinct objects, Polya's Enumeration Theorem will provide us additional information about the characteristics and configurations of these distinct objects, as well as how many exist. For the example of the colorings of a cube, Burnside's Lemma will tell us how many distinct colorings exist, while Polya's theorem will provide details on each configuration of colors within each coloring.

Polya's theorem also has numerous applications which we will explore as well. We will look into the application of Polya's theorem in chemical isomer enumeration; while many compounds have the same chemical formula, their structure in 3D space is different based on the arrangement of the molecules. These different structures, called isomers, can be determined easily using Polya's theorem, and is widely used in chemistry since its discovery in 1937 by Cayley. For music theory, Polya's theorem is used in the enumeration of distinct chords without accounting for chords that are transpositions or inversions. Chord enumeration can be very useful in music composition in order to determine the possible chords and tonal combinations that are unique.

2 Basics of Group Theory

As Burnside's Lemma is a result of group theory, we will first provide some basic definitions and notations involved in group theory that will be relevant in this paper.

Definition 1: Group A group is a mathematical object that consists of a set of elements and an operation that satisfies certain properties. The properties that must be satisfied are four axioms called the group axioms: closure, associativity, identity, and invertibility.

Definition 2: Orbit The orbit of an element x in X is the set of elements in X that can be moved by the elements in G. In other words, it consists of all the possible results of transforming an element x. The orbit x is denoted by $G.x$ where:

$$G.x = \{g.x | g \in G\}$$

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The orbits partition the set X. As each orbit represents all the possible results of transforming an element x, the set of orbits represents the distinct objects within the set. The set of orbits over all elements of X, not counting repetition, can be represented as X/G. Thus, the number of orbits, which we are solving for using Burnside's Lemma, is the cardinality X/G.

Definition 3: Fixed Point For an element $g \in G$, a fixed point of X is an element $x \in X$ such that $g.x = x$. In other words, x is unchanged by the group's operation.

After defining a fixed point, we can further define the elements fixed by a specific $g \in G$. Represented as $|X^g|$, these elements constitute the subset of X that is fixed by g, and gives the set of configurations in X that are unchanged after a given transformation.

Definition 4: Stabilizer A stabilizer, represented as $|G_x|$, is the set of elements in G that fix x.

3 Burnside's Lemma

For a finite group G that acts on set X, let X/G be the set of orbits of X. Then, Burnside's Lemma states that

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

In Definition 3, we defined $|X^g|$ above to be the subset of X that is fixed by g. This also means the the number of orbits is equal to the average number of fixed points of G.

Proof of Burnside's Lemma

The first step is to represent the sum $\sum_{g \in G} |X^g|$ as a sum over all the elements of X. The result is the equality that

$$\begin{aligned}\sum_{g\in G}|X^g| &= \sum_{g\in G}|\{x\in X : g.x = x\}| \\ &= |\{(g,x) : g\in G, x\in X, g.x = x\}| \\ &= \sum_{x\in X}|\{g\in G : g.x = x\}| \\ \sum_{g\in G}|X^g| &= \sum_{x\in X}|G_x| \qquad (1)\end{aligned}$$

We defined $|G_x|$ above as the stabilizer of x. Intuitively, this equality makes sense because we are representing the sum of all elements but just fixing on different variables; the left side is summing over all values of g, while the right side is summing over all values of x.

Next, we will use the Orbit-Stabilizer Theorem, which will define the relationship between orbits and stabilizers within our group. We will prove the theorem below.

Theorem 1: Orbit-Stabilizer Theorem Let G be a finite group of permutations of a set X. Then, the orbit-stabilizer theorem gives that

$$|G| = |G_x| * |G.x|$$

Proof For a fixed $x \in X$, $G.x$ be the orbit of x, and G_x is the stabilizer of x, as defined above. Let L_x be the set of left cosets of G_x. This means that the function $f_x : G.x \to L_x$, given that $gx \mapsto gG_x$, is a bijection.[6]

If y $\in G.x$ so $y = g_1x = g_2x$ for $g_1, g_2 \in G$, then this means that

$$g_2^{-1}g_1x = g_2^{-1}g_2x$$

$$g_2^{-1}g_1x = x$$

Thus, this results in the following:

$$g_2^{-1}g_1 \in G_x$$

$$g_1G_x = g_2G_x$$

This results in f being well-defined. We know that f is surjective because by definition, the function f is a function of $G.x$ to the left coset, it follow that f is surjective.

We also know that if $gG_x = g'G_x$, then for some value of $h \in G_x$, $g = g'h$. This means that $gx = (g'h)x = g'(hx) = g'x$. Thus, the function f is injective.

We have thus shown the function f is a bijection, which proves our theorem. $\square$

$|G.x|$ represents the size of the orbit for x, so each x in the orbit contributes $\frac{1}{|G.x|}$ to the total sum for that orbit. Thus, if the sum of all the x in a given orbit always equals to 1, then the total sum of $\sum_{x\in X} \frac{1}{|G.x|}$ will be equal to the total number of orbits, $|X/G|$.

Now, we can substitute in the Orbit-Stabilizer Theorem to our expression of the total number of orbits:

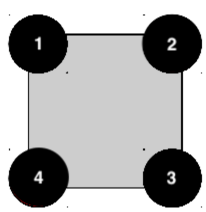
$$\begin{aligned} \# of orbits &= \sum_{x\in X} \frac{1}{|G.x|} \\ &= \sum_{x\in X} \frac{|G_x|}{|G|} \\ &= \frac{1}{|G|} \sum_{x\in X} |G_x| \end{aligned}$$

Finally, we can use Equation 1 proven above to conclude that:

$$\frac{1}{|G|} \sum_{g\in G} |X_g|$$

Example: Colorings of Points on a Square

Let us now look at an example of enumerating the different possible coloring of points on a square with n colors using Burnside's Lemma. The points of the square are labeled 1-4 in clockwise order.



First, we want to define all possible transformations $g \in G$. This represents all the possible rotations and reflections that we can make on the square. The four transformations are listed below:

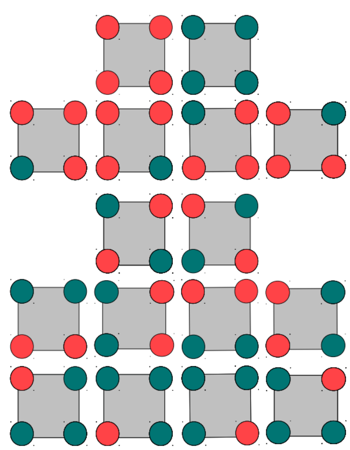
1. *Rotate 0 (Identity Transformation)*
   This transformation does not change the colorings at all, which means that the number of fixed points is the total number of possible colorings: n^4

2. *Rotate 90*
   This transformation results in all points affecting one another, so the only fixed points would be colorings of all the same color: n

3. *Rotate 180*
   This swaps two pairs of vertices that are across from each other, which means only the vertices within the pairs need to be the same color: n^2

4. *Rotate 270*
   Similar to the 90 degree rotation, all points swap with one another, so the only fixed points are colorings of all the same color: n

The figure below displays all the possible colorings and transformations of the square using two colors. Besides the first row which contains two orbits, each of the following rows represents one orbit.



We can now use Burnside's Lemma to sum up all the configurations fixed on a tranformation g, which results in the following number of distinct colorings of a square using n colors:

$$\frac{1}{|G|}\sum_{g\in G}|X^g|$$

$$\frac{1}{4}(n^4+n+n^2+n)$$

4 Polya Enumeration Theorem

While Burnside's Lemma provides the number of distinct colorings, it does not include information on the type of configuration. Polya's Enumeration Theorem weights the colors in one or more ways, so there could be any number of colors given the set of colors has a generating function with finite coefficients.[2]

Definition 5: Cycle Index Let G be a group whose elements are permutations of X, where $|X| = k$. We will define a polynomial with k variables $x_1, x_2 ... x_k$ with non-negative coefficients to create a product $x_1^{b_1}, x_2^{b_2}, ... x_m^{b_k}$ such that the polynomial is

$$P_G(x_1, x_2, ... x_k) = \frac{1}{|G|} \sum_{g \in G} x_1^{b_1}, x_2^{b_2}, ..., x_m^{b_k}$$

This formula is very similar to Burnside's Lemma, but now we specify how many of each cycle there are as we differentiate between the cycles of different length.

Theorem 2: Polya's Enumeration Formula Let X be a set of elements and G be a group of permutations of X that lead to equivalent colorings of X. We will use the colors $w_1, w_2 ... w_m$ and this can be expressed by the following generating function, where k is the largest cycle length.

$$P_G\left(\sum_{j=1}^{m} w(j), \sum_{j=1}^{m} w(j)^2, ... \sum_{j=1}^{m} w(j)^k \right)$$

Now, let us do a simple example using a necklace and apply Polya Enumeration Theorem to list the possible configurations of the necklace.

Example 1: If we have a necklace with three beads and n colors, then our cycle index can be written as

$$P_G = \frac{1}{3}(x_1^3 + 2x_3)$$

Now, let us see what the color configurations if we had two colors of red, R, and blue, B.

$$\begin{aligned} P_G((R+B), (R^2+B^2), (R^3+B^3)) &= \frac{1}{3}((R+B)^3 + 2 * (R^3+B^3)) \\ &= \frac{1}{3}(3A^3 + 3A^2B + 3AB^2 + 3B^3) \\ &= A^3 + A^2B + AB^2 + B^3 \end{aligned}$$

The final equation lists out all the possible distinct colorings of the necklace and the number of each color that exist for each of them.

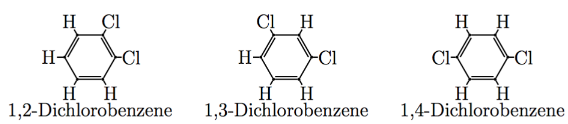
5 Applications of Polya Enumeration Theorem

Polya Enumeration Theorem has a variety of applications that have been adapted to many different fields of study. The most widely known application is within chemical isomer enumeration.

Chemical Isomer Enumeration

In chemistry, a chemical formula can represent more than one molecule due to varying arrangements of the molecules in space. The molecules that have the same chemical formula but different chemical structures are called isomers, and we can use Polya Enumeration Theorem to enumerate these types of isomers.

Let's look at an example of a chemical formula with various isomers. The molecule dichlorobenzene $C_6H_4Cl_2$ has the following isomers:[3]



We can use Polya Enumeration Theorem to determine the number of ways of attaching two chlorine atoms and four hydrogen atoms onto the carbon ring. To do this, we assign numbers 1-6 for the six carbons of the hexagon, and Y be the set with a chlorine and hydrogen atom with weights Cl and H respectively. Then, we get the configuration generating function:

$$\begin{aligned} F(H, Cl) &= Z_{C_6}(H + Cl, H^2 + Cl^2, ..., H^6 + Cl^6) \\ &= \frac{1}{6}((H + Cl)^6 + (H^2 + Cl^2)^3 + 2(H^3 + Cl^3)^2 + 2(H^6 + Cl^6) \\ &= H^6 + H^5Cl + 3H^4Cl^2 + 4H^3Cl^3 + 3H^2Cl^4 + HCl^5 + Cl^6 \end{aligned}$$

The coefficients in the generating function represent how many isomers exist for that chemical formula, so in this case the coefficient of H^4Cl^2 means that there are three isomers of dichlorobenzene which are shown in the diagram above.[3]

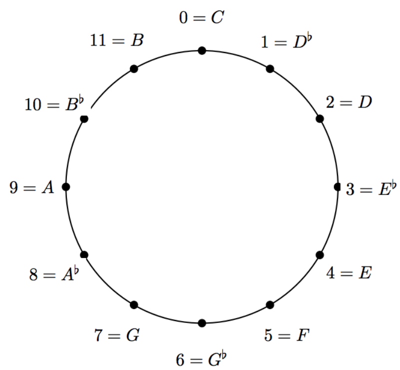
Music Theory Enumeration

We can also apply this theorem to music theory in the enumeration of distinct chords. This is useful to identify patterns within chord sequences, in addition to counting the number of patterns in intervals and rhythms. It can also be helpful in transposing music if transpositions can be easily identified.

In Western music, tones are equivalent if they differ by an octave. There are a total of 12 tones, which are

$$C, D\flat, D, E\flat, E, F, G\flat, G, A\flat, A, B\flat, B$$

The tones are typically arranged in a circle to show that they cycle through an octave. The tones are shown on the following page, represented as numbers from 0 to 12.[4]



We will look more closely at triads, which are a subset of tones made with three distinct tones. We will consider two triples to be equivalent if one can be translated to another. For example, the two chords [C,E,G] = [0,4,7] and [F,A,C] = [5,9,0] would be considered equivalent because the [F,A,C] chord is a shift of 5 tones above each note in the first chord. Thus, if the spaces between the triads are equal then the chord has simply been translated.

In addition, triples are equivalent if the tones are permuted with the same set of notes. For example, [C,E,G] = [0,4,7] is the same as the chord [G,C,E] = [7,4,0] because they consist of the same notes, just in a different ordering. This is called an inversion.

To solve this, we can use Burnside's Lemma. Let X be the set of all distinct triples of elements from Z. If we consider all the ways to select three tones from a set of 12 tones regardless of ordering, we get $|X| = \binom{12}{3}$. Because it does not account for ordering, this already eliminates the problem of different chord permutations and does not double count these. Thus, the only other transformation we need to account for is translation of the chords. We can represent this as

$$\{(a, b, c), k\} = \{a + k, b + k, c + k\}$$

We need to determine the number of distinct orbits to determine the number of different triads we can make. To do this, we identify the elements of our group:

1. *k = 0 (Identity Transformation)*
   All of these are fixed points because nothing is changed after the identity transformation, so $|X| = \binom{12}{3}$

2. *k = 4*
   This will lead to a fixed point if and only if the following equation holds true

   $$4\{a, b, c\} = \{a + 4, b + 4, c + 4\} = \{a, b, c\}$$

   This results in four triples, so $|X^4| = 4$

3. *k = 8*
   Using the same reasoning as above, we also have $|X^8| = 4$

These are the only possible fixed points that exist because in any d-tone system where we are selecting n-tuples of tones, the only value k that we can have fixed points is when kn is a multiple of d.[4] In this situation, we have a 12-tone system and we are selecting 3 tuples of tones. So, this means that the only possible fixed points is when 3k is a multiple of 12, so when k = 0,4,8.

Using Burnside's Lemma, we calculate that the number of distinct orbits, which represent the number of distinct triads, is equal to

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}|X^g| &= \frac{1}{12}(|X^0 + X^4 + X^8) \\ &= \frac{\binom{12}{3} + 4 + 4}{12} \\ &= \frac{228}{12} \\ &= 19\end{aligned}$$

This represents the number of distinct triads for triples, but similar methods can be used to enumerate the different n-tuples tones as well.

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