The Spectral Characterization of Simply-Laced Dynkin Diagrams

Abstract

Simply-laced Dynkin diagrams appear in the classification of many mathematical objects, such as root systems of simple Lie algebras, representations of Lie groups, Gabriel's classification of quivers of finite type, symmetries of regular polytopes, and chip-firing games. In this paper, we characterize the simply-laced Dynkin diagrams as the simple undirected graphs with eigenvalues less than two and describe connections to a few appearances of Dynkin diagrams.

1 Introduction

For any undirected graph G, the resulting adjacency matrix A is symmetric with nonnegative entries. By the spectral theorem for symmetric matrices, we know that we can diagonalize this matrix with respect to an orthonormal basis, and in particular, the eigenvalues are all real. We can show that the maximal eigenvalue λ_1 is nonnegative and can be computed from the Rayleigh quotient

$$\lambda_1 = \max_{|x|=1} x^T A x$$

In addition, multiplying a vector by this matrix is equivalent to placing values at the vertices of the graph, and then replacing the value of each vertex by the sum of the values of its neighbors, accounting for edge multiplicity. From this interpretation, certain objects, such as the Sponsor Game and Kostant's Game, described in Sections 4 and 5, respectively, naturally lead to the question of which graphs have maximal eigenvalues at most 2. One can also construct graphs from the Cartan matrices for simple simply-laced Lie algebras, and one discovers that the resulting extended Dynkin diagrams have maximal eigenvalue exactly 2. Thus, a classification of which graphs have eigenvalue (less than) 2 provides insight into the classification of these objects.

2 Introduction to Spectral Graph Theory

For an undirected graph G, one can construct a matrix A, called an adjacency matrix, by labeling the vertices with distinct numbers $1, 2, \ldots, |V|$ where V is the set of vertices of G, and we let the matrix entry $A_{ij} = u(i \to j)$ where $u(i \to j)$ is the number of edges from vertex i to vertex j with $1 \leq i, j \leq |V|$. Since our graph is undirected, we have that $A_{ij} = A_{ji}$ so the matrix is symmetric, and by the spectral theorem, A can be diagonalized with real eigenvalues using some orthogonal basis. For this paper, we will also assume that G has no self-loops, so all the terms on the diagonal of A will be zero.

Remark 2.1. One may note that the labeling of the vertices was arbitrary, but this does not make a difference to us since a different labeling is simply conjugation by some permutation matrix and conjugation does not change the eigenvalues.

From this, we can prove the following theorem.

Theorem 2.1. *Let λ_1 be the largest eigenvalue of A. Then*

$$\lambda_1 = \max_{|x|=1} x^T A x$$

The term on the right-hand side is called the Rayleigh quotient.

Proof. By the Spectral Theorem, we know there exists some orthogonal matrix $\gamma \in O(|V|)$ such that $\gamma A \gamma^T = D$ where D is diagonal. Then we have

$$\max_{|x|=1} x^T A x = \max_{|x|=1} x^T \gamma^T D \gamma x = \max_{|y|=1} y^T D y$$

where the last equality follows from the fact that γ is orthogonal and so $|\gamma x| = |x| = 1$. Then if y_i is the i^{th} component of y, we have

$$y^T D y = \sum_{i=1}^{|V|} D_{ii} y_i^2$$

Since we have the constraint $\sum_{i=1}^{|V|} y_i^2 = 1$, we see that the above sum of maximized when $y_i^2 = 1$ for the maximal D_{ii} and $y_i = 0$ for everything else. Since $\max_i D_{ii} = \lambda_1$, we see that λ_1 is the maximum value of the Rayleigh quotient. □

Corollary 2.1.1. *Any x with $|x| = 1$ such that $x^T A x = \lambda_1$ must be an eigenvector of A with eigenvalue λ_1.*

Proof. In the proof of the previous theorem, we show that $y^T D y = \sum_{i=1}^{|V|} D_{ii} y_i^2$. In order to maximize this, we need the only y_i which are nonzero must have $D_{ii} = \lambda_1$. Thus, if x maximizes the Rayleigh quotient, it must be a linear combination of eigenvectors of λ_1, so it also has eigenvalue λ_1. □

For a disconnected graph, by picking the labels appropriately, we find that the adjecency matrix is a block diagonal matrix, so the eigenvalues can be found separately for each block. Because of this, we can work with each of the connected components and piece them back afterwards.

Theorem 2.2. *Let G be connected. Then the eigenspace of λ_1 is 1-dimensional and it is spanned by a vector with only positive entries.*

Proof. Let v be an eigenvector with eigenvalue λ_1, and let $\mathcal{P}$ be the indices of v with positive entries and $\mathcal{N}$ be the indices of v with negative entries. If $\mathcal{N}$ is nonempty, then we then have

$$\begin{aligned} v^T A v &= \sum_{1 \le i,j \le n} A_{ij} v_i v_j \\ &= \sum_{i=1}^{n} A_{ii} v_i^2 + \sum_{\substack{i,j \in \mathcal{P} \\ i<j}} 2A_{ij} v_i v_j + \sum_{\substack{i,j \in \mathcal{N} \\ i<j}} 2A_{ij} v_i v_j + \sum_{\substack{i \in \mathcal{P} \\ j \in \mathcal{N}}} 2A_{ij} v_i v_j \\ &< \sum_{i=1}^{n} A_{ii} v_i^2 + \sum_{\substack{i,j \in \mathcal{P} \\ i<j}} 2A_{ij} v_i v_j + \sum_{\substack{i,j \in \mathcal{N} \\ i<j}} 2A_{ij} v_i v_j + \sum_{\substack{i \in \mathcal{P} \\ j \in \mathcal{N}}} 2A_{ij} v_i (-v_j) \end{aligned}$$

Thus, all the entries of v must be non-negative.

Now suppose any of the entries of v are zero. Then there must be some i, j such that $v_i = 0$, $v_j \neq 0$, and $A_{ij} \neq 0$. Then considering the vector $v^{(\theta)}$ with $v_k^{(\theta)} = \sin\theta$ if $k = i$ and $v_k^{(\theta)} = v_k \cos\theta$ if $k \neq i$, we have

$$(v^{(\theta)})^T A_{ij} v^{(\theta)} = \sum_{k \neq i}^{n} A_{kk} v_k^2 \cos^2\theta + \sum_{k \neq i} 2A_{ik} v_k \cos\theta \sin\theta + \sum_{\substack{k_1, k_2 \neq i \\ k_1 < k_2}} 2A_{k_1 k_2} v_{k_1} v_{k_2} \cos^2\theta + \sin^2\theta$$

Differentiating with respect to θ, we get

$$\frac{\partial}{\partial\theta}(v^{(\theta)})^T A_{ij} v^{(\theta)} = -(\sum_{k\neq i}^{n} A_{kk} v_k^2 \sum_{\substack{k_1,k_2\neq i\\ k_1<k_2}} 2A_{k_1k_2} v_{k_1} v_{k_2}) 2\cos\theta\sin\theta + \sum_{k\neq i} 2A_{ik} v_k (\cos\theta^2 - \sin\theta^2) + 2\sin\theta\cos\theta$$

At $\theta = 0$, we see that this function is increasing since $\sin\theta = 0$ and $A_{ik} v_k > 0$ for some k. Thus, the Rayleigh quotient is larger for some $\theta > 0$ so v must have purely positive entries.

To see that the eigenspace of λ_1 has dimension one, we note that the eigenvectors of A are orthogonal. Since the above argument shows that all eigenvectors of λ_1 can be scaled to have all positive entries, and since vectors with only positive entries cannot be orthogonal, we see that the eigenspace must be one-dimensional. ☐

Lemma 2.1. *Let G be connected and $H \subset G$ be a proper subgraph. If λ_1 is the largest eigenvalue of G and λ_1' is the largest eigenvalue of H, then $\lambda_1' < \lambda_1$.*

Proof. Let v be a vector with norm 1 and positive entries which maximizes $v^T A_H v$ where A_H is the adjacency matrix of H. Then consider the vector v_G which is equal to v on the vertices of G which are on H and zero everywhere else. Then v_G is a vector with norm one as well and $v_G^T A_G v_G \geq v^T A_H v$ where equality happens if and only if $(A_H)_{ij} = (A_G)_{ij}$ for $i, j \in H$. If H does not contain every vertex of G, then v_G has an entry which is zero so from the proof of the previous lemma, we know that there is another vector v' which achieves a higher Rayleigh quotient. Thus, we see that any proper subgraph of G has strictly smaller λ_1. ☐

For a more in-depth discussion of spectral graph theory, one can check Péter Csikvári's lecture notes [2].

3 Graphs with Eigenvalue at most 2

3.1 $\lambda_1 = 2$

To find the graphs with maximum eigenvalue at most 2, it suffices to look at connected graphs since we know that the set of eigenvalues is the disjoint union of the eigenvalues of each connected component. In addition, from Lemma 2.1, we know that any graph with maximal eigenvalue less than 2 cannot contain a graph with maximum eigenvalue 2, so we can start by classifying the graphs with eigenvalue 2.

First, we know that a cycle has maximum eigenvalue 2 since any cycle has an eigenvector consisting of only ones.

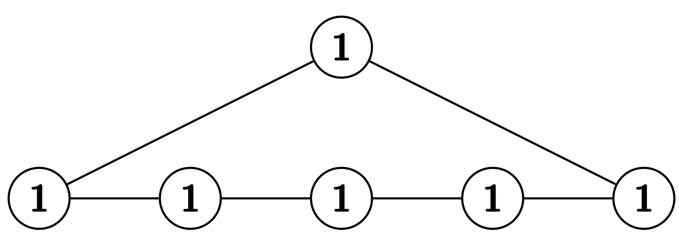

Figure 1: Graph of type $\tilde{A}_n$.

Because of this, the other connected graphs with $\lambda_1 = 2$ must be acyclic, so we just need to look at trees.

We can also show that no node can have degree four or higher unless it is a star of degree four because the following graph has an eigenvector with eigenvalue 2.

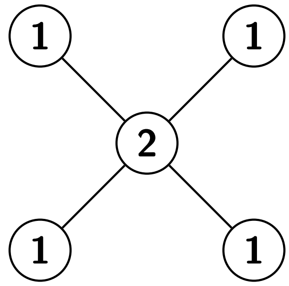

Figure 2: Graph $\tilde{D}_1$.

In fact, we can generalize the above eigenvector to a larger class of graphs, which shows that any tree with two nodes of degree three must be of the following form.

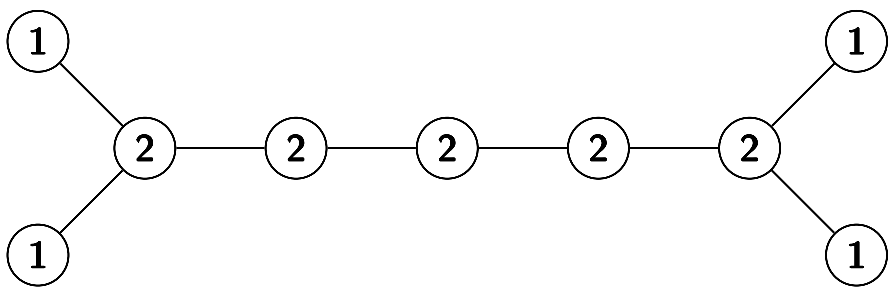

Figure 3: Graph of type $\tilde{D}_n$.

Finally, the three following exceptional graphs also have maximum eigenvalue two.

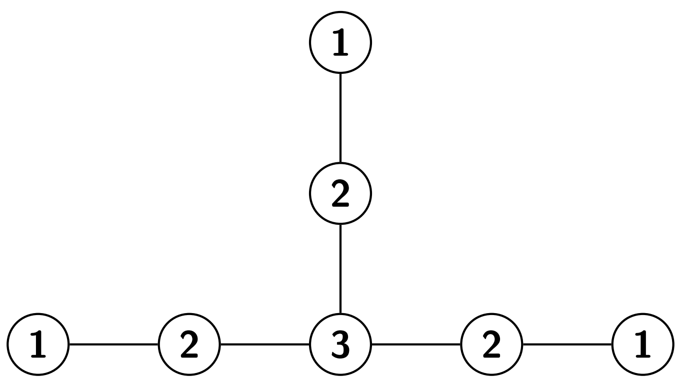

Figure 4: Graph $\tilde{E}_6$.

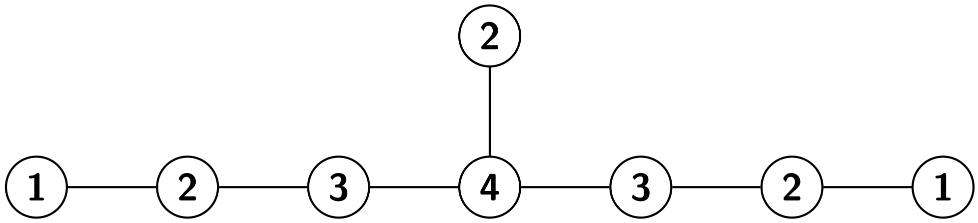

Figure 5: Graph $\tilde{E}_7$.

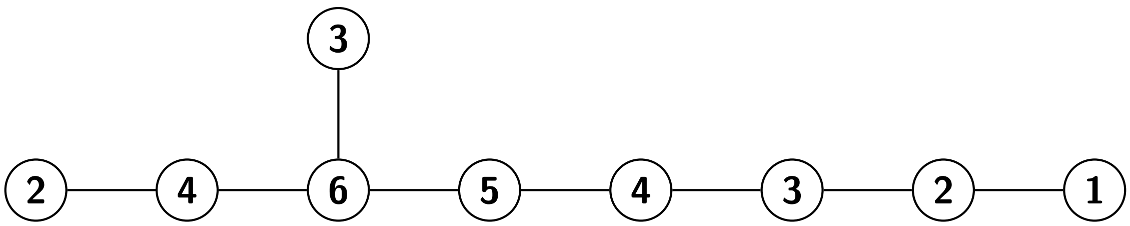

Figure 6: Graph $\tilde{E}_8$.

By performing some casework, one can show that any other tree with a node of degree three not containing any of the above graphs is strictly contained in a graph of type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$, so we have found all of the graphs with maximum eigenvalue two.

3.2 $\lambda_1 < 2$

From Lemma 2.1, we know that the graphs with eigenvalues less than two must not contain any of the above graphs. From Figure 1, we know that these graphs must be acyclic, and from Figures 2 and 3, we know that no node can have degree four and at most one node has degree three. If the tree has no nodes of degree three, then it must be of the following form.

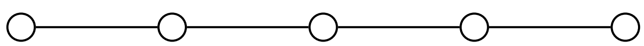

Figure 7: Graph of type A_n.

If the graph has a node of degree 3, and two of the branches going out have length one, then it is a subgraph of some $\tilde{D}_n$ so all such graphs work and are of the following form.

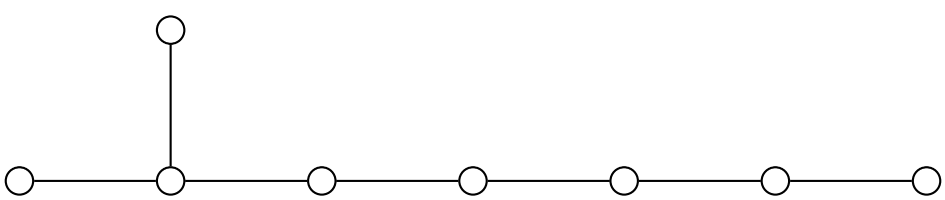

Figure 8: Graph of type D_n.

If the graph has one branch of length two and one branch of length one, the remaining branch can have length up to four before it becomes $\tilde{E}_8$. This gives three more diagrams of the following form.

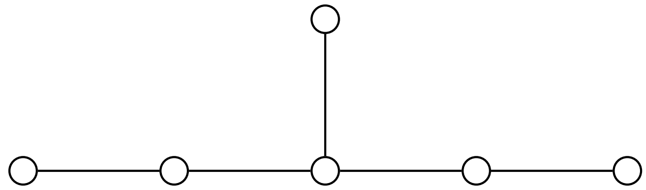

Figure 9: Graph E_6.

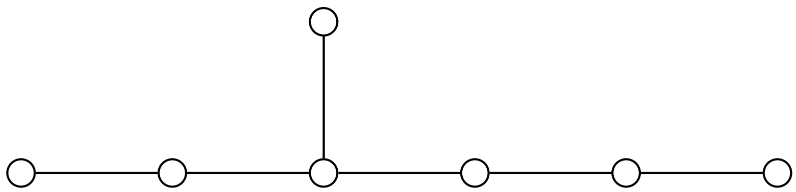

Figure 10: Graph E_7.

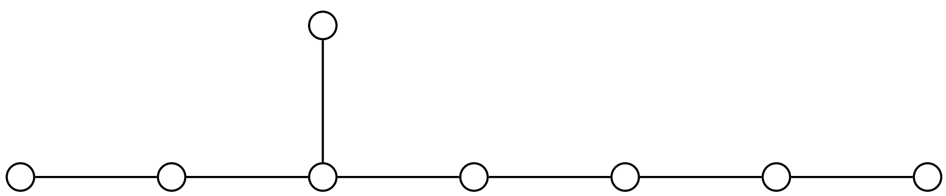

Figure 11: Graph E_8.

By doing some casework, one can see that any graph with a node of degree three not containing any extended Dynkin diagrams must be of the form D_n, E_6, E_7 or E_8 so we have found all the graphs with eigenvalue less than two. Thus, we arrive at the following theorem.

Theorem 3.1 (Main Theorem)**.** *The only graphs with eigenvalues less than two are the finite simply-laced Dynkin diagrams. The only graphs with maximum eigenvalue equal to two are the extended simply-laced Dynkin diagrams.*

4 The Sponsor Game

We now apply the Main Theorem to the following game, which is described in [1]. Suppose we have an undirected, connected graph G with no self-loops, and assume we have labeled the vertices with distinct integers between 1 and n where n is the number of vertices in G. A *configuration* is an assignment of non-negative integers $c_1, \ldots, c_n$ to each vertex of G, where c_i is the value on the i^{th} vertex. A vertex i of G is *unhappy* if

$$c_i < \frac{1}{2} \sum_{(i,j) \in E} c_j$$

where E is the set of edges of G. A vertex i of G is *excited* if

$$c_i > \frac{1}{2} \sum_{(i,j) \in E} c_j$$

The *Sponsor Game* is the following game on configurations:

- Start with a configuration $(c_1, \ldots, c_n)$ such that the c_i are not all zero.
- Pick any vertex of i which is unhappy and add 1 to c_i.
- Stop if there are no unhappy vertices.

The *Excited Sponsor Game* is the following variant:

- Start with the configuration $(c_1, \ldots, c_n)$.
- Pick any vertex of i which is not excited and add 1 to c_i.
- Stop if all vertices are excited.

For this game, we can prove the following theorem.

Theorem 4.1. *The graphs G where the Sponsor Game terminates are the simply-laced Dynkin diagrams and extended simply-laced Dynkin diagrams, and the Sponsor Game will terminate regardless of starting configuration. The graphs G where the Excited Sponsor Game terminates are the simply-laced Dynkin diagrams, and the Sponsor Game will terminate regardless of starting configuration. In addition, in either of these games, the ending configuration does not depend on the moves taken.*

We first define some notation. We denote the vector with all ones by $\mathbf{1}$, and we say $v_1 > v_2$ or $v_1 \geq v_2$ for two vectors v_1, v_2 when each coordinate of v_1 is larger than the corresponding coordinate of v_2. We define $v_1 \geq v_2$ similarly. In addition, $\min(v_1, v_2)$ is the vector where each coordinate is the minimum of the two corresponding coordinates of v_1 and v_2 and $\lceil v \rceil$ is the vector where each coordinate is the ceiling of the corresponding coordinate in v.

Lemma 4.1. *Any graph with a terminating Sponsor Game must have all its eigenvalues at most 2. Any graph with a terminating Excited Sponsor Game must have all its eigenvalues strictly less than 2.*

Proof. We will prove the lemma for only the normal Sponsor Game since the proof for the excited Sponsor Game is similar. Suppose we have a valid configuration v, and let v_1 be an eigenvector for $\lambda_1 > 2$. We know that v_1 will have all positive entries so we can scale it so that $v_1 > v$. Then

$$A(v_1 - v) = Av_1 - Av \geq \lambda v_1 - 2v > \lambda_1(v_1 - v)$$

Then we have

$$\frac{(v_1 - v)^T A(v_1 - v)}{|v_1 - v|^2} > \frac{(v_1 - v)^T \lambda_1 (v_1 - v)}{|v_1 - v|^2} = \lambda_1$$

which contradicts the fact that λ_1 maximizes the Rayleigh quotient. Thus, we must have $\lambda_1 \leq 2$. $\square$

Lemma 4.2. *In both the Sponsor Game and Excited Sponsor Game, if the game terminates, the ending state is unique. In particular, the ending configuration v will be the minimum configuration such that $v \geq c$ and $2v \geq Av$ for the Sponsor Game and $v \geq c$ and $2v > Av$ for the Excited Sponsor Game.*

Proof. Again, the proofs for both games are similar so we prove it only for the Sponsor Game. Suppose $v^{(1)}$ and $v^{(2)}$ are vectors that both satisfy $2v^{(i)} \geq Av^{(i)}$ and $v^{(i)} \geq c$, where c is the starting configuration. Then $v^{(i)} \geq A\min(v^{(1)}, v^{(2)})$ for $i = 1, 2$, so we have $\min(v^{(1)}, v^{(2)}) \geq A\min(v^{(1)}, v^{(2)})$ and $\min(v^{(1)}, v^{(2)}) \geq c$. Thus, there is a minimum vector v such that $2v \geq Av$ and $v \geq c$ if any such vector exists. We now show that the game will terminate with ending configuration v. Since v is the minimum possible terminating configuration, we just need to show that we can never reach any $v' > v$. Suppose there was some series of steps to reach v'. Then there must have been an intermediate configuration c' such that c' and v have the same value at vertex i and we incremented this vertex. We take the first such intermediate configuration, so that all vertices j have values at most v_j. Then

$$2c'_i \geq \sum_{(i,j)\in E} v_j \geq \sum_{(i,j)\in E} c_j$$

Then i is not unhappy so we could not have made a move incrementing vertex i. Thus, our game must terminate at v. □

Proof of Theorem 4.1. We first prove the theorem for the Sponsor Game. We know that every graph with its eigenvalues at most 2 is a simply-laced Dynkin diagram or an extended simply-laced Dynkin diagram. From Lemma 4.2, it suffices to find some vector v with positive integer entries such that $v \geq c$ and $2v \geq Av$. If G is an extended simply-laced Dynkin diagram, our argument in Section 3 show that the eigenvectors with eigenvalue 2 are multiples of a vector with positive integer entries. Then we can scale this vector to find some $v \geq c$ and $2v = Av$. If G is a simply-laced Dynkin diagram, we can take the eigenvector from some extended simply-laced Dynkin diagram $\tilde{G}$ containing G, and scale it to an eigenvector v with positive integer entries such that $v_i \geq c_i$ for $i \in G$. Then we have

$$2v_i = \sum_{(i,j)\in \tilde{E}} v_j \geq \sum_{(i,j)\in E} v_j$$

so the restriction of v to G gives us a good configuration. Thus, the Sponsor Game terminates if and only if G is a simply-laced Dynkin Diagram or an extended simply-laced Dynkin diagram.

For the Excited Sponsor Game, we need to show that every simply-laced Dynkin diagram has some vector v with positive integer entries such that $v \geq c$ and $2v \geq Av$. Take an eigenvector v_1 of A with eigenvalue λ_1 and scale it so that

$$2v_1 - \lambda_1 v_1 \geq \left(\sum_{1\leq i,j\leq n} A_{ij}\right)\mathbf{1}$$

Then we have

$$A\lceil v_1 \rceil \leq A(v_1 + \mathbf{1}) \leq Av_1 + \left(\sum_{1\leq i,j\leq n} A_{ij}\right)\mathbf{1} < 2v_1 \leq 2\lceil v_1 \rceil$$

This shows that $\lceil v_1 \rceil$ gives a good configuration, so the Excited Sponsor Game terminates if and only if G is a simply-laced Dynkin diagram. □

5 Kostant's Game

There's another chip-firing game, described in [1] and [4], which is similar to the normal Sponsor Game above, but instead of adding one to an unhappy vertex v, we replace with with $-v + \sum_{(v,w)\in E} w$. In other words, we replace the value at v with the sum of the values of its neighbors minus itself. From the proofs used in the Sponsor Game, we know that any graph which is not an simply-laced Dynkin diagram automatically will not have a solution. Now it remains to figure out which of these Dynkin diagrams terminate.

First, let's deal with the extended Dynkin diagrams.

Lemma 5.1. *Any stable configuration (one without any playable moves) on an simply-laced extended Dynkin diagram must be an eigenvector with eigenvalue two.*

Proof. Our proof will be very similar to the proofs in the previous section. Let v be our stable configuration, and let v_1 be an eigenvector with eigenvalue two such that $v_1 \geq v$. Then we have

$$Av_1 - Av = 2v_1 - Av \geq 2v_1 - 2v$$

which gives us

$$\frac{(v_1 - v)^T A(v_1 - v)}{|v_1 - v|^2} \geq \frac{2(v_1 - v)^T (v_1 - v)}{|v_1 - v|^2} = 2$$

Since two is the maximum value of our Rayleigh quotient, we see that the Rayleigh quotient of $v_1 - v$ is two. We know from Corollary 2.1.1 that any vector which maximizes the Rayleigh quotient is an eigenvector with eigenvector two, so $v_1 - v$ is an eigenvector with eigenvalue two, and therefore v is an eigenvector with eigenvalue two. □

Now a quick computation shows that any steady configuration on an extended Dynkin diagram can only be reached by itself. Thus, the game only terminates on these graphs if it started on a steady configuration.

For the regular Dynkin diagrams, we first note the following lemma as described in [1], which will be used to describe our graph of configurations.

Lemma 5.2 (Roman Lemma)**.** *Let C be a connected directed graph without self-loops (possibly infinite!). Suppose that for every vertex c and pair of two of its out-going edges, we can find 2 converging paths of the same length using these edges to some other vertex. Then one of the following is true.*

- *C has no vertex with out-degree zero.*
- *C has exactly one vertex c_{end} with out-degree zero and all directed paths from a vertex c eventually reach c_{end} and have the same length.*

A proof of this statement can be found at [3]. For Kostant's game, we can show that if we have two possible moves at a configuration, which alter either node i or node j, we can find paths using both moves of equal length such that we end up at the same configuration. If i and j are not adjacent, we have the paths

$$\begin{tikzcd}
& c \arrow[dl, "i"'] \arrow[dr, "j"] & \\
c' \arrow[dr, "j"'] & & c'' \arrow[dl, "i"] \\
& c_{\text{last}} &
\end{tikzcd}$$

If i and j are adjacent, we have the paths

$$
\begin{tikzcd}
 & c \arrow[dl, "i"'] \arrow[dr, "j"] & \\
c' \arrow[d, "j"] & & c'' \arrow[d, "i"] \\
\cdot \arrow[dr, "i"] & & \cdot \arrow[dl, "j"] \\
 & c_{\text{last}} &
\end{tikzcd}
$$

As a result, the requirements of the Roman Lemma hold on our graph of configurations, so we just need to find a terminating sequence of moves for each configuration.

We do not have the space to fully prove that there is always a terminating sequence of moves, but here is a short sketch. For graphs of the form A_n, one can find a sequence of moves so that the configuration is increasing until a maximum, and then decreasing the rest of the graph. Then after this, one can show that this maximum bounds the values of the nodes of the final configuration. For the graphs D_n, E_6, E_7, and E_8, one can make a sequence of moves so that the node of degree three has the maximum value, the values decrease along the branches as one moves away from this node, and the only unhappy vertex (if there is one) is this node. If this central vertex will gain k from the only available move, then one can show that one can always find a sequence of moves such that each move increases the overall value by at most k. One can also find some group of configurations with some "wiggle room," where the vertices can vary by k in value and they will still by stable configuration. Thus, the sequence of moves must terminate since this group gives an upper bound on what configurations one can reach.

To summarize, we have the following theorem.

Theorem 5.1. *The only graphs for which a nonzero configuration terminates are the simply-laced (extended) Dynkin diagrams. For simply-laced Dynkin diagrams, Kostant's game terminates regardless of the starting configuration, and for simply-laced extended Dynkin diagrams, the game terminates if and only if the starting configuration is an eigenvector with eigenvalue two.*

References

[1] Even Chen. 18.218 Lecture Notes: Taught by Alex Postnikov, 2017.

[2] Péter Csikvári. Lecture Notes on 18.212: Algebraic Combinatorics, 2016. URL: `http://math.mit.edu/~csikvari/schedule_18.212.html`.

[3] Sam Hopkins. Diamond lemma for edge-colored directed graphs. MathOverflow. eprint: `https://mathoverflow.net/q/262511`. URL: `https://mathoverflow.net/q/262511`. URL:https://mathoverflow.net/q/262511 (version: 2017-02-23).

[4] Allen Knutson. Bert Kostant's Find the highest root game, Blog, 2013. URL: `https://plus.google.com/+AllenKnutson/posts/TWsWhCakCQg`.