

The Single Jeep and Jeep Caravan Problems

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Abstract

The Jeep problem was introduced in 1947 and has since spawned numerous variations with real-life applications. The goal of the original problem was to find the correct procedure for a single Jeep to reach a certain distance using the minimum amount of fuel. The second part of this paper introduces the idea of multiple Jeeps, called a Jeep Convoy[4] by CG Phipps in 1947. The goal of the Jeep Convoy problem is to advance a single jeep of the caravan as far as possible. Both Fine and Phipps found solutions relating the Jeep problem to the Harmonic series, and proved that given unlimited fuel and time a Jeep could cross any size desert. This paper will cover the original problem introduced by NJ Fine[1] and will also look at three variations of the Jeep Convoy problem.

Key words: Jeep Problem, Caravan Problem, fuel capacity, fuel cache.

1 Introduction

The Jeep problem was introduced by NJ Fine in 1947[1] and has since been studied and written about every few years. The problem was first introduced as a single Jeep traveling across a desert. The goal of the Jeep is to cross the desert, which means traveling a given distance d . The Jeep starts from a base camp where they have unlimited access to fuel, but the Jeep can only carry a limited amount of fuel. Additionally, the Jeep is allowed to transport and leave fuel in fuel caches, also known as stations, along its route in order to refuel while away from the base camp. The goal of the problem is to escape the desert using the minimum amount of fuel possible. There are two variations of this problem, the One-Way Journey and the Round-Trip journey.

Soon after NJ Fine published his initial results, CG Phipps generalized the problem into a caravan made of m jeeps[4]. The goal of the caravan was to advance a single jeep of its company as far as possible into the desert. In this problem the jeeps of the caravan are still constrained by a fuel capacity, but may create fuel caches and transfer fuel among Jeeps. There are three common variations of this problem: the result where no jeeps return to the base camp, the result where all jeeps return to the base camp, and the result where all jeeps but one return to the base camp.

The purpose of this paper is to present the solutions to the Single-Jeep Problems and Caravan Problems. All results are from the source papers listed in the bibliography section of this paper. First I will introduce the original Single-Jeep one-way problem created by Fine, then look at a solution published

by David Gale[2] in 1970. The Single-Jeep round-trip variation will follow, then the Caravan problem will be presented in Part 3. This section of the paper will present the optimal strategy for each variation of the Caravan problem. Finally, the paper will conclude by describing some other important variations of the Jeep Problem along with mentioning some real-world applications of the problem.

2 Single Jeep Problems

The goal of this problem is to determine the minimum amount of fuel necessary to reach a location L that is a distance d away from our starting point. As stated in the introduction, the jeep's driver may store fuel from the jeep in fuel caches along the way, allowing her to later refuel the jeep without returning to her base.

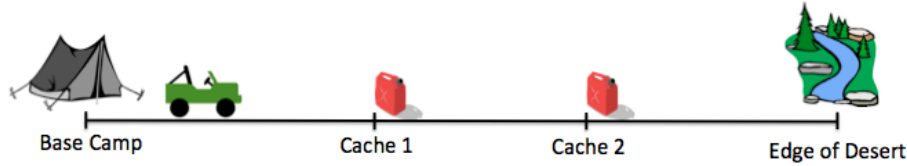


Figure 1: A visualization of the single jeep problem.

To formalize the problem, let's have the Jeep travel along the x -axis starting at $x = 0$. So there is an unlimited amount of gas at $x = 0$ but the Jeep has a limited fuel capacity. To make things simpler let us measure fuel by distance rather than gallons. So we define the jeep's capacity to equal c , where c is the number of miles the jeep could travel given a full tank of fuel. The Jeep is trying to cross a desert that is d miles across, so the Jeep must travel from $x = 0$ to $x = d$.

- d : distance from the Jeep's base camp to the edge of the desert
- c : distance the Jeep can travel on a full tank of gas
- $x = 0$: the Jeep's base camp
- $x = d$: the edge of the desert



Figure 2: Set $x=0$ at the Base Camp and $x=d$ at the edge of the desert.

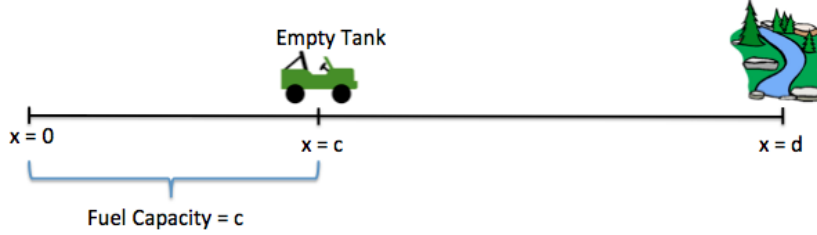


Figure 3: The jeep can travel a distance c on one full tank of gas.

If d is less than or equal to c the Jeep simply drives across the desert, no fuel cache's are necessary. If d is greater than c then it is necessary to place fuel cache's along the x -axis so that the Jeep can refuel at some location while crossing the desert.

2.1 One-way Journey

When NJ Fine first introduced the Jeep problem he proved a solution that shows the minimum fuel necessary to cross a given distance and also solves for the procedure. That is, Fine's solution defined where the fuel cache's should be placed and how the Jeep should travel in order to use the minimum amount of fuel.

In May of 1970 David Gale[2] proved the same solution using Banach's Formula for total path distance. Gale's solution is somewhat easier to understand and so it is summarized in this section. For those interested in NJ Fine's work, see his paper titled "The Jeep Problem"[1].

While the original problem asks to minimize fuel requirements to travel a distance d , we will solve the equivalent problem of maximizing distance given a certain amount of fuel.

We will use Banach's formula for the path length of a curve:

$$\text{Path Length} = \int_0^d n(x)dx \quad (1)$$

In Banach's formula $n(x)$ is the number of points on the curve that map to x . We will define $n(x)$ for the Jeep Problem to be equal to the number of times during the journey that the Jeep is at point x . So each point x_i in the interval $[0,d]$ has an associated $n(x_i)$.

Gale's Solution Using Banach's Formula[2]

Given an amount of fuel f , we must find the maximum distance $d(f)$ the Jeep can reach. Let us define a sequence of points $x_0, x_1, \dots, x_f$ on the x axis such that x_k is at the point where k units of fuel will be consumed to the right of that point. For example, $x_f = 0$ because all of the fuel available will be used to the right of $x = 0$.



Figure 4: A visualization of Gale's method of dividing the interval into smaller intervals.

Gale proves the following Lemma in his paper:

$$\text{If } x < x_k \text{ then } n(x) \geq 2k + 1. \quad (2)$$

The Lemma states that when x is to the left of x_k , the point at x must be visited at least $2k + 1$ times. Gale also proves that there must be exactly one unit of length between x_k and x_{k+1} , which we combine with Banach's Formula and the above Lemma:

$$\begin{aligned} 1 &= \text{Path Length between } x_k \text{ and } x_{k+1} \\ 1 &= \int_{x_{k+1}}^{x_k} n(x) dx \geq (2k + 1)(x_k - x_{k+1}) \end{aligned} \quad (3)$$

From here we sum from 0 to $f - 1$ to solve for an upper bound of $d(f)$:

$$d(f) \leq 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2f - 1} \quad (4)$$

The upper bound is in fact the odd harmonic series, and since we know the odd harmonic series diverges we know that the Jeep can cross any desert or reach any point.

2.2 Round-trip Journey

Once we've solved the One-Way Jeep problem the Round-Trip problem is trivial. Looking at the Lemma (2) from above, we change it to reflect a round trip by adding 2 instead of one:

$$\text{If } x < x_k \text{ then } n(x) \geq 2k + 2. \quad (5)$$

This leads to a solution for $d(f)$ equal to the harmonic series instead of the odd harmonic series:

$$d(f) \leq 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{f} \quad (6)$$

Comparing the one-way and round-trip results we see that for short distances a round trip journey can require much less fuel than two one-way journeys. As distances increase however, the advantage to round-trip's decreases.

3 Jeep Caravan Problems

Let us now look at the similar problem where we have a caravan of jeeps traversing the desert. In "The Jeep Problem: A More General Solution", author CJ Phipps[4] looks at the problem of a convoy, or caravan, of jeep's working together to explore as far as possible into the surrounding desert. Each jeep is identical, starts with a full tank of fuel, and has the same fuel capacity. In this problem the caravan may create fuel caches and may also transfer fuel from jeep to jeep. The goal of the caravan is to advance a single Jeep as far away from the base camp as possible.

There are three variations of this problem that we will cover. The first is the case where no jeeps return to the base camp, the second is the case where all jeeps must return, and the third is the case where all jeeps but the single farthest-traveling jeep must return to the base camp.

3.1 No Jeeps Return

In this case we transfer fuel from some jeep and then abandon it along the way. Eventually all jeeps will have been abandoned in the desert.

Let us say our caravan consists of m identical jeeps, each with fuel capacity c . Our strategy involves traveling a certain distance, transferring fuel from one jeep, and abandoning that jeep as the rest of the caravan continues forward. The best strategy to pick the stations where we abandon jeeps takes into account the number of active jeeps and the fuel capacity.

On the first leg of the journey the caravan should travel a distance equal to c divided by m . This is the distance where the total amount of fuel the caravan has consumed is equal to one tank of fuel. When you empty one jeep and transfer the fuel to the other $m-1$ jeeps they will all have full tanks, and you abandon the empty jeep.

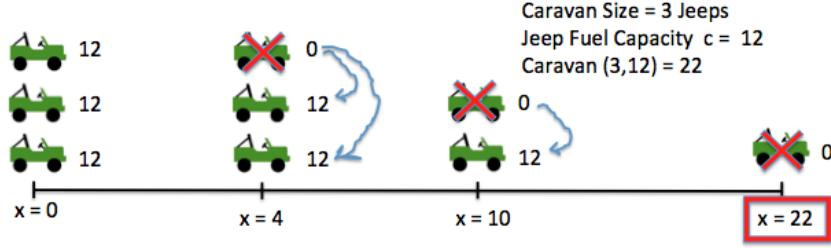
If the caravan stopped before this distance c/m then the other $m-1$ jeeps will have full tanks but there will be some surplus fuel. At this point the caravan is at the same point as the optimum strategy except they are closer to their home base. If the caravan stopped after the distance $x = c/m$ then the jeep is wasting fuel by continuing with the caravan. The maximum distance possible for the caravan will be smaller than if they followed the optimal strategy.

Since we now know that this is the optimal strategy we apply it to the remaining legs of the journey, abandoning a jeep every c/m miles where m decreases by 1 every leg. When there is only one jeep remaining it simply drives until it runs out of fuel.

$$\begin{aligned}
 Caravan(m, c) &= c + \frac{c}{2} + \frac{c}{3} + \dots + \frac{c}{m} \\
 Caravan(m, c) &= c * (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}) \\
 Caravan(m, c) &= c * \sum_{n=1}^m \frac{1}{n}
 \end{aligned} \tag{7}$$

The above equation represents the maximum possible distance a jeep with fuel capacity c from a caravan of m jeeps can reach. For an example involving three jeeps see Table 1.

Table 1: No Jeeps Return



3.2 All Jeeps Return

In this case we must leave each jeep with enough fuel to return to base camp. The final Jeep, the one that journey's farthest into the desert, must also return to base camp after their exploration.

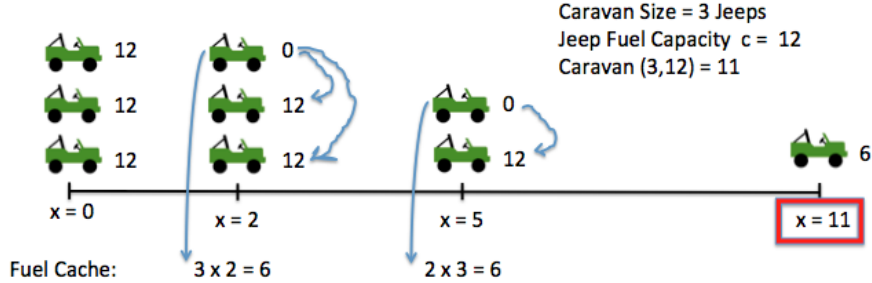
This problem is similar to the previous problem where no jeeps return. We solve for each leg length by dividing the fuel capacity by the number of active jeeps, but since we must leave fuel for the return journey the leg's must be half as long. Each time the caravan stops to leave behind a jeep they also leave enough fuel for every jeep in the caravan to return to the previous station. The jeep that stays at that station saves enough fuel to return to the previous station then transfers it's remaining fuel to the active jeeps in the caravan.

Since the distances are simply divided by 2, equation 1 from above can be adapted to solve the "All jeeps return" variation.

$$\begin{aligned}
 Caravan(m, c) &= c + \frac{1}{2} * \frac{c}{2} + \frac{1}{2} * \frac{c}{3} + \dots + \frac{1}{2} * \frac{c}{m} \\
 Caravan(m, c) &= c * \left(\frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{3} + \dots + \frac{1}{2} * \frac{1}{m} \right) \\
 Caravan(m, c) &= c * \sum_{n=1}^m \frac{1}{2 * n}
 \end{aligned} \tag{8}$$

For an example involving three jeeps see Table 2.

Table 2: All Jeeps Must Return



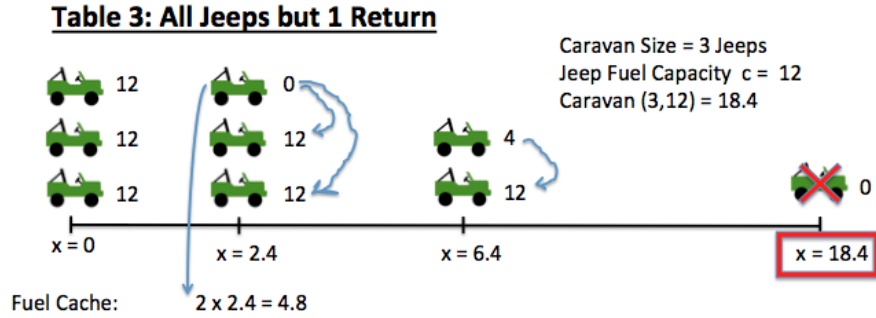
3.3 All Jeeps but One Must Return

In this case all jeeps except for the farthest jeep must return to base camp. The final jeep will be abandoned when it runs out of fuel while exploring farther out into the desert.

This case is more tricky and requires us to adapt our previous argument. Our distance for each leg will depend on fuel capacity and number of jeeps like before, but instead of dividing c by m or $2m$, we must divide by $2*m - 1$. The factor of 2 is because the jeeps must return, but since one jeep never returns we can subtract one from $2*m$.

$$\begin{aligned}
 Caravan(m, c) &= c + \frac{c}{2 * 1 - 1} + \frac{c}{2 * 3 - 1} + \dots + \frac{c}{2 * m - 1} \\
 Caravan(m, c) &= c * \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2 * m - 1} \right) \\
 Caravan(m, c) &= c * \sum_{n=1}^m \frac{1}{2 * n - 1}
 \end{aligned} \tag{9}$$

For an example involving three jeeps see Table 3.



4 Conclusion and Other Significant Variations

Although the Jeep Problem is over 60 years old, its many variations are still studied today. Since this type of optimization problem can be directly related to real-world situations new variations are introduced every few years. Below are some of the more interesting variations that have been solved along with their corresponding publications.

One common Jeep Problem variation was introduced by Dewdney[6] in 1987. He introduced the "Desert Fox puzzle," which has the same basic constraints and goals as the Jeep Problem but allows each Jeep to also carry one fuel drum of limited capacity. Additionally, fuel cache's can only be supplied from fuel drums, fuel cannot be transferred from the Jeep's primary fuel supply. Another variation also allows fuel drums but constrains their use by saying once a fuel drum is opened it can't be transported. This requires the remaining fuel from the opened drum to be used as a fuel cache or added to the primary fuel supplies of the Jeeps.

The two "types" of Jeeps, those that carry drums and those that don't, are referred to as either "Fine Jeeps" named for the original Jeep Problem author NJ Fine, or "Dewdney Jeeps" for AK Dewdney who introduced the most common variant of Fine's original problem.

Another variation involves Jeeps or airplanes with varying fuel efficiencies and capacities[3]. The military originally was very interested in the Jeep optimization problem because it directly related to the problem of transporting airplane fleets in the Pacific Theater. One interesting paper that discusses vehicles with different fuel efficiencies include "The Range of a Fleet of Aircraft" published by JN Franklin in 1960.

One last variation of interest was introduced by Herb Bailey[5] in 2009. Bailey considered the convoy problem in both a friendly desert, with conditions similar to Phipps's convoy problem, and a hostile desert, where Jeep's cannot leave fuel caches unattended or wait alone in the desert.

References

- [1] N.J. Fine, *The Jeep Problem*, Mathematical Association of America, Washington, D.C., 1947.
- [2] David Gale, *The Jeep Once More or Jeeper by the Dozen*, Mathematical Association of America, U.C. Berkeley, 1970.
- [3] J.N. Franklin, *The Range of a Fleet of Aircraft*, Society for Industrial and Applied Mathematics, U.S.A., 1960.
- [4] C.G. Phipps, *The Jeep Problem: A More General Solution*, Mathematical Association of America, University of Florida, 1947.
- [5] Herb Bailey, *Jeeps Penetrating a Hostile Desert*, Mathematical Association of America, U.S.A., 2009.
- [6] AK Dewdney, *Computer recreations*, Scientific American, U.S.A., 1987.