

# Dyck Paths and Up-Down Walks

Gabriella Baracchini

18.204

May 12, 2016

## Abstract

This paper analyzes Dyck Paths and Alpha and Beta Up-Down Walks, a subset of Dyck Paths. It will show the natural implications of Catalan Numbers in counting unique paths. Using Catalan Number knowledge and combinatorics we will count the exact number of unique Up-Down walks of length  $l$  as a function of  $l$ . A closed form equation for the length of these paths is a natural extension of this problem and can be found in the appendices.

## 1 Introduction

For the purpose of this paper, a path is simply a series of steps up or down, and right or left. As in life, there are multiple different paths to get from Point A to Point B. In this paper we will use mathematics to enumerate unique paths of equal length.

The main focus of the paper is Dyck Paths and subsets of them. A Dyck Path is a series of up and down steps. The path will begin and end on the same level; and as the path moves from left to right it will rise and fall, never dipping below the height it began on. You can see, in Figure 1, that paths with these limitations can begin to look like mountain ranges.

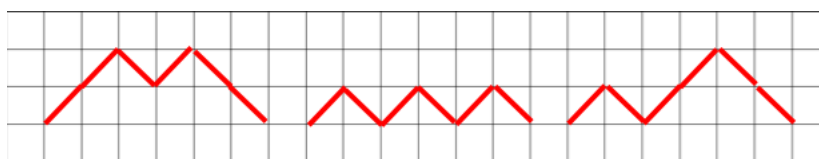


Figure 1: Illustration of Dyck Paths as mountain ranges.

Figure 1 shows that every step, regardless of the direction, is the same length. A more mathematical definition of a Dyck Path is given Definition 1.1.

**Definition 1.1.** Dyck Path A Dyck Path is a series of equal length steps that form a staircase walk from  $(0, 0)$  to  $(n, n)$  that will lie strictly below, or touching, the diagonal  $x = y$ .

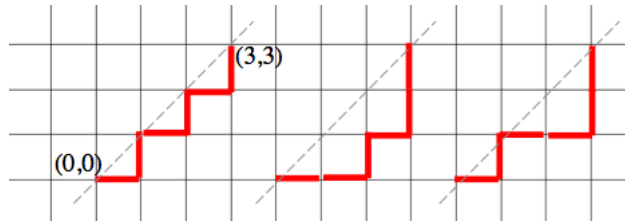


Figure 2: Graphing of various 6-step dyck paths with  $x = y$  line.

In this paper, we will first examine the problem of counting unique Dyck Paths of equal length. The solution to this problem is a series of natural numbers called Catalan Numbers. Next we focus on a more specific subset of Dyck Paths we call Up-Down walks. In this paper we define two types of Up-Down walks: Alpha and Beta. We then use combinatorics to enumerate the number of unique paths of length  $l$ .

## 2 Enumerating Dyck Paths

Now we consider the problem of counting all the unique Dyck Paths of step length  $2n$ . We approach this problem by thinking of the steps as 1s and  $-1$ s that create a sequence. Every "up step" in the path is a 1 in the sequence, and every "down step" is a  $-1$ . Because the path can never cross the  $x = y$  line when graphed, there can never be more  $-1$ s than 1s. Thus the number of unique Dyck Paths becomes the number of sequences with nonnegative partial sums that can be formed from  $n$  1s and  $n - 1$ s. [1]

It is clear, using this sequence methodology and combinatorics, the number of unique Dyck Paths length  $2n$  is  $\frac{1}{n+1} \binom{2n}{n}$ . This sequence of numbers is known as Catalan Numbers. As seen in Figure 3, paths of length  $2n$  correspond to the  $n^{th}$  Catalan Number,  $C_n$ .

**Definition 2.1.** Catalan Numbers Catalan Numbers are a sequence of natural numbers that occur in various counting problems.

The formula for the  $n^{th}$  Catalan Number,  $C_n$ , is follows.

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{2n!}{n!(n+1)!} = \prod_{k=2}^n \frac{n+k}{k}, n \geq 0$$

$C_n$	0	1	2	3	4	5
$n$	1	1	2	5	14	42

$n = 0$ :	*	1 way
$n = 1$ :	$\wedge$	1 way
$n = 2$ :	$\wedge\wedge, \wedge$	2 ways
$n = 3$ :	$\wedge\wedge\wedge, \wedge\wedge, \wedge\wedge, \wedge\wedge, \wedge\wedge$	5 ways

Figure 3: Catalan Numbers correspond to the number of unique Dyck Paths.  
[3]

### 3 Up-Down Walks

Up-down walks are simply a subset of dyck paths. An up-down walk, is defined as a series of steps which starts with steps that do not decrease the value, reaches a peak or plateau in the middle, and ends with steps that do not increase the value. This can be analogized to the path someone takes up and then down a ladder, starting and ending on the floor. It follows that the walk will always start and end at the same height value, and either increase in value in the middle or stay flat throughout. In the walk, the last step up must occur before the first step down. An up-down walk will always start and end at the same height value, and either increase in value in the middle or stay flat throughout. These walks can be represented graphically or described using vector notations.

In this paper we consider two types of up-down walks: Alpha and Beta.

- *Alpha Walks* are composed of only three specific steps. All steps in the walk have width 1 and change in height by -1, 0, or 1. The individual steps can be described in vector notation as  $(1, -1)$ ,  $(1, 0)$ , and  $(1, 1)$ . The graphs in Figure 4 represent different Alpha Walks of length 4.

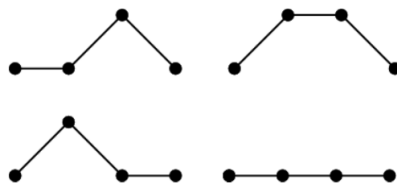


Figure 4: Illustration of Alpha Up-Down Walks of length 4.

- *Beta Walks* are also composed of only three specific steps. The steps that change the height, by -1 or 1, have a width of  $\frac{1}{2}$ . The steps that do not change the height

have width 1. These individual steps can be described in vector notation as  $(\frac{1}{2}, -1)$ ,  $(\frac{1}{2}, 1)$ , and  $(1, 0)$ . Figure 5 illustrates different Beta Walks of length 2.

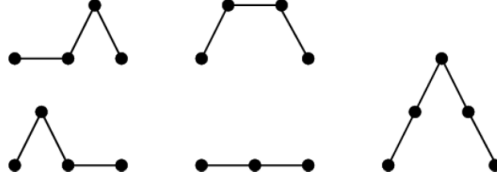


Figure 5: Illustration of various Beta Up-Down Walks of length 2.

As seen in Figure 5, Beta Walks of length 2 can include a varying number of steps.

For each Alpha and Beta Up-Down Walk with more than 1 step there are multiple unique path or walks. This paper derives formulas,  $F_A(l)$  and  $F_B(l)$ , for Alpha and Beta Up-Down Walks respectively, that give the number of unique walks of length  $l$  as a function of  $l$ . These formulas are derived by using the summations of binomial coefficients that count various combinations of step placements.

## 4 Deriving the Number of Unique Up-Down Walks

To derive formulas for the total number of unique up-down walks with length  $l$  we pay attention to the fact that each walk ends at the same height value it began on. To end at the original height value, the number of steps with height 1 is equal to the number of steps with height  $-1$ . We define these types of steps as *height changing* steps. The total number of height changing steps is always even.

### 4.1 Alpha Up-Down Walks

The quantity of unique Alpha Walks is relatively easier to derive because every step in Alpha Up-Down Walks has width 1.

*Claim.*  $F_A(l) = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{l}{2i}$ .

*Proof.* It is trivial to see, by the definition of an Alpha Up-Down Walk, an Alpha Walk length  $l$  has exactly  $l$  steps in the the walk. Additionally, we note that there must be the same number of up-steps as down-steps because all up-down walks begin and end at the same height. It follows that in every path, there is an even number of height changing steps. We can think of height changing steps coming in pairs (one up-step for every down-step).

If  $l$  is even, the maximum pairs of height changing steps the walk can contain is  $l/2$ , when every step is height changing. It follows that an Alpha Walk of length  $l$  can contain zero to  $\lfloor l/2 \rfloor$  pairs of height changing steps.

There are multiple unique Alpha Walks with  $l > 1$  due to various placements of height changing steps. We can count all the possible placements of the height changing steps with binomial coefficients.

For example, an Alpha Walk of length 4, thus 4 steps, with 2 height changing steps (one pair) has  $\binom{4}{2} = 6$  unique orderings of height changing steps. The 6 unique orderings are shown in Figure 5.

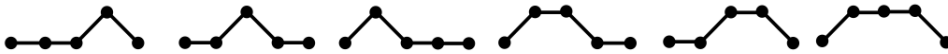


Figure 6: Illustration of all six unique Alpha Walks of length 4 with 2 height changing steps.

We keep in mind that for any ordering of height changing steps, we know that the first half in the walk are up-steps and the last half are down-steps.

To count the number of unique walks of length  $l$ , we have to count all the ways 0, 2, 4, ..., or  $l$  height changing steps can be placed within  $l$  steps. We do this by summing all of the binomial coefficients,  $\binom{l}{2i}$ , from  $i = 0$  to  $\lfloor l/2 \rfloor$ . The variable,  $i$ , is the number of height changing pairs and  $2i$  is the number of height changing steps.

We therefore have counted every possible ordering of height changing and flat steps for every possible number of pairs of height changing steps that can fit in the Alpha Walk.

We note that because the binomial coefficient is only counting the placement of height changing steps, and not the placement of specifically 1 or -1 steps, the formula will not count paths that do not meet the up-down path specification.

□

Appendix A.1 is an extension of this proof using a generating function, to turn this summation into a closed-form formula for the number of unique Alpha Up-Down Walks.

## 4.2 Beta Up-Down Walks

Finding the formula for the number of unique Beta Up-Down Walks is much trickier because not all walks of length  $l$  will have the same number of steps, due to the varying

step widths. To account for this we have to slightly alter our function for Alpha Walks as seen below.

*Claim.*  $F_B(l) = \sum_{i=0}^l \binom{l+i}{2i}$ .

*Proof.* First we note that a Beta Walk of length  $l$  can have a various number of steps. For example, a Beta Up-Down Walks of length 2 can have 2, 3 or 4 steps depending on whether they are height changing steps or not. In fact, all walks of length  $l$  can have  $l$  (all steps have 0 height change) to  $2l$  (all steps are height changing) amount of steps.

It is important to note that for each number of steps, there is a specific number of height changing steps in the walk.<sup>1</sup> For a walk of length  $l$  with  $l + i$  steps, there is  $2i$  height changing steps in the walk. For example, a walk with length 4, but 7 steps, will always have exactly 6 height changing steps ( $l = 4, l + i = 7, i = 3, 2i = 6$ ).

To count the number of unique placements of height changing steps in the Beta Walk we choose the number of height changing steps ( $2i$ ) from the number of total steps ( $l + i$ ), giving us the binomial coefficient  $\binom{l+i}{2i}$ .

To account for the varying number of steps we sum this binomial coefficient over every possible number of steps for a walk of length  $l$ ,  $l$  to  $2l$ .

Therefore,  $F_B(l) = \sum_{i=0}^l \binom{l+i}{2i}$  will give us the total number of unique beta up-down walks of length  $l$ , by adding together, all the possible placements of height changing steps, over all the possible number of steps for a Beta Walk of length  $l$ .

□

Appendix A.2 is an extension of this proof using generating functions and Taylor series, to turn this summation into a closed-form formula for the number of unique Beta Up-Down Walks.

## 5 Final Remarks

The problem of enumerating Dyck Paths is one representation of a problem that occurs frequently throughout mathematics. Dyck Paths, and thus Catalan Numbers, are seen in counting, chess, coding, etc. [2] Thoroughly understanding Dyck Paths and Catalan Numbers is an important and interesting part of mathematics.

---

<sup>1</sup>This is different from alpha up-down walks where a walk with a certain number of steps could have various numbers of height changing steps.

## 6 Acknowledgements

I would like to thank Professor Alexander Postnikov of the MIT Department of Mathematics for his time and help throughout the course. I would also like to thank my classmates for their continuous feedback and help cultivating my mathematical presenting and writing skills.

## A Generating Functions Appendix

Generating functions are a useful tool that will easily take our binomial coefficient summation to a closed-form function that tells us how many unique walks there are for Alpha and Beta Walks length  $l$ .

With the combinatorial formulas for the number of unique Alpha and Beta walks of length  $l$  from the Alpha and Beta walk proofs, we can create the generating functions seen in the subsections below.

### A.1 Alpha Walk Generating Function Manipulation

We use generating functions and the Binomial Theorem to get closed-form function for the number of unique Alpha Up-Down Walks,  $F_A(l)$ .

*Claim.*  $F_A(l) = 2^{l-1}$ .

*Proof.* We begin with our combinatorial formula from Section 4.1.

$$F_A(l) = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{l}{2i}$$

We will assume in  $F_A(l)$  that  $x = 1$  to create the following generating function

$$H_A(x) = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{l}{2i} x^{2i}$$

Note that we strategically choose  $2i$  as the exponent for  $x$  to make it easier for us to apply the Binomial Theorem to find a closed-form function.

To transform  $H_A$  to a closed-form function we will use the Binomial Theorem.

**Theorem 1.** Binomial Theorem  $\sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$ , where  $n$  is a natural number.

We will let  $2i = k$ . Note that this will change the summation limits. The summation is now taken from  $k = 0$  to  $k = l$  due to the change in variables.

$$H_A(x) = \sum_{k=0}^l \binom{l}{k} x^k$$

Because  $k = 2i$ ,  $H_A(l)$  only holds for *even* values of  $k$ . We use the Binomial Theorem and expand the formula out so that it only holds true for even values of  $k$  as seen below.

$$H_A(x) = \sum_{k=0}^l \binom{l}{k} x^k = \frac{1}{2}((1+x)^l + (1-x)^l)$$

We can resubstitute  $x = 1$  into the equation to get

$$F_A(l) = \frac{1}{2}((1+1)^l + (1-1)^l) = \frac{1}{2}(2^l + 0) = 2^{l-1}$$

Therefore, we can see our closed-form function for the number of unique Alpha Up-Down Walks is

$$F_A(l) = 2^{l-1}.$$

□

## A.2 Beta Walk Generating Function Manipulation

We transform our combinatorial formula  $F_B(l)$  into a closed-form formula for the number of unique Beta Up-Down Walks.

*Claim.*  $F_B(l) = \frac{1+\sqrt{5}}{2\sqrt{5}} \left[ \frac{3+\sqrt{5}}{2} \right]^l + \frac{-1+\sqrt{5}}{2\sqrt{5}} \left[ \frac{3-\sqrt{5}}{2} \right]^l$

*Proof.* To begin start with our combinatorial formula  $F_B(l) = \sum_{i=0}^l \binom{l+i}{2i}$ , and transform it into the standard generating function  $H_B(z)$  below.

$$H_B(z) = \sum_{l=0}^{\infty} \sum_{i=0}^l \binom{l+i}{2i} z^l$$

We would like to make  $H_B(z)$  a close-form formula for the number of beta walks, length  $l$ .

We will do that by first manipulating  $H_B(z)$ , a single variable generating function, into  $G(x, y)$ , the general two variable generating function format, seen below.



$$G(x, y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{a+b}{a} x^a y^b = \sum_{n=0}^{\infty} (x+y)^n = \frac{1}{1-x-y} \quad (1)$$

By defining  $a$  and  $b$  in  $G(x, y)$  to match variables in an single variable generating function, we can transform  $H_B(z)$  into a two variable generating function.

We'd like to start by turning  $\binom{a+b}{a}$  into  $\binom{l+i}{2i}$ . We let  $a = 2i$  and  $b = l - i$ . It follows that  $x^a y^b$  turns into  $x^{2i} y^{l-i}$ . We would like to get  $x^{2i} y^{l-i} = z^l$ . We can do this if we let  $x = z^{1/2}$  and  $y = z$ . For simplicity we square both equalities and get  $z = x^2 = y$ . Thus we hope that  $H_B(z) = H_B(x^2)$  is related to  $G(x, x^2)$ .

To check ourselves, we will consider the special case described above. We will start with the general two variable generating function and make its coefficients equal to the combinatorial formula for the number of unique Beta Walks. We start with the general two variable generating function below

$$G(x, y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{a+b}{a} x^a y^b$$

We let  $y = x^2$ ,

$$H_B(z) = H_B(x^2) = G(x, x^2) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{a+b}{a} x^{a+2b}$$

Remember that  $z = x^2$ , we want to compute the coefficient for  $z^l$  and thus we compute the coefficient for  $x^{2l}$ . This mean  $a + 2b = 2l$ . Using this relationship we obtain,

$$\begin{aligned} \text{The coefficient of } x^{2l} \text{ in } G_B(x, x^2) &= \sum_{b=0}^l \binom{2(l-b)+b}{2(l-b)} \\ &= \sum_{b=0}^l \binom{2l-b}{2l-2b} \end{aligned}$$

$$\text{Letting } b = l - i, \text{ the coefficient becomes } \sum_{i=0}^l \binom{l+i}{2i}$$

We can see that the manipulation works and therefore, we have  $G_B(x, x^2) = H_B(x^2) + Q_B(x)$  where  $Q_B(x)$  consists of only odd powers of  $x$ . However, we are only looking for the even power of  $x$ .

We can apply Equation 1 to our two variable generating function to get

$$G_B(x, x^2) = \frac{1}{1 - x - x^2}$$

To get only the even terms of the function we subtract the odd terms as seen below.

$$H_B(x^2) = \frac{1}{2} \left( \frac{1}{1 - x - x^2} + \frac{1}{1 + x - x^2} \right) = \frac{1}{2} \left( \frac{2 - 2x^2}{1 - 3x^2 + x^4} \right) = \frac{1 - x^2}{1 - 3x^2 + x^4}$$

For this function to be useful to us we replace  $x^2$  with  $x$ ,

$$H_B(x) = \frac{1 - x}{1 - 3x + x^2}.$$

In Lemma 2 we use partial fractions to decompose this function and then Taylor series to eventually derive the closed form function  $F_B(l)$  below.

$$F_B(l) = \frac{1 + \sqrt{5}}{2\sqrt{5}} \left[ \frac{3 + \sqrt{5}}{2} \right]^l + \frac{-1 + \sqrt{5}}{2\sqrt{5}} \left[ \frac{3 - \sqrt{5}}{2} \right]^l.$$

□

*Lemma 2.* We will expand  $H_B(x)$  using partial fractions and then transform it to a closed-form function using Taylor Series.

$$H_B(x) = \frac{1 - x}{1 - 3x + x^2} = \frac{a}{1 - \phi_+ x} + \frac{b}{1 - \phi_- x},$$

for some  $a$  and  $b$  and where  $\phi_+ = \frac{3 + \sqrt{5}}{2}$  and  $\phi_- = \frac{3 - \sqrt{5}}{2}$  are the inverse roots of the denominator of  $H_B(x)$ .

Elementary algebra gives

$$a = \frac{1 + \sqrt{5}}{2\sqrt{5}}, b = \frac{-1 + \sqrt{5}}{2\sqrt{5}}$$

Now, we need to remember the Taylor series for  $\frac{1}{1 - \alpha x}$ . This is

$$\frac{1}{1 - \alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \dots$$

Thus when we expand each of the fractions for the expression for  $H_B(x)$  using Taylor series.

$$H_B(x) = a \left( 1 + \left( \frac{3 + \sqrt{5}}{2} \right) x + \left( \frac{3 + \sqrt{5}}{2} \right)^2 x^2 + \left( \frac{3 + \sqrt{5}}{2} \right)^3 x^3 + \dots \right)$$

$$+b\left(1 + \left(\frac{3 - \sqrt{5}}{2}\right)x + \left(\frac{3 - \sqrt{5}}{2}\right)^2 x^2 + \left(\frac{3 - \sqrt{5}}{2}\right)^3 x^3 + \dots\right)$$

Substituting in the value we found for  $a$  and  $b$ , we get

$$\begin{aligned} H_B(x) &= \frac{1 + \sqrt{5}}{2\sqrt{5}} \left(1 + \left(\frac{3 + \sqrt{5}}{2}\right)x + \left(\frac{3 + \sqrt{5}}{2}\right)^2 x^2 + \left(\frac{3 + \sqrt{5}}{2}\right)^3 x^3 + \dots\right) \\ &+ \frac{-1 + \sqrt{5}}{2\sqrt{5}} \left(1 + \left(\frac{3 - \sqrt{5}}{2}\right)x + \left(\frac{3 - \sqrt{5}}{2}\right)^2 x^2 + \left(\frac{3 - \sqrt{5}}{2}\right)^3 x^3 + \dots\right) \end{aligned}$$

It can be seen from the above Taylor expansion of the generating function  $H_B(x)$ , that the closed form function  $F_B(l)$  that it correlates to is

$$F_B(l) = \frac{1 + \sqrt{5}}{2\sqrt{5}} \left[\frac{3 + \sqrt{5}}{2}\right]^l + \frac{-1 + \sqrt{5}}{2\sqrt{5}} \left[\frac{3 - \sqrt{5}}{2}\right]^l.$$

## References

- [1] "Dyck Path." *Wolfram Alpha*. Wolfram Mathworld, n.d. Web. 2 May 2016.
- [2] Hilton, Peter, and Jean Hedersen. "Catalan Numbers, Their Generalization, and Their Uses." *The Mathematical Intelligencer* 13.2 (2009): 64-75. Web. 2 May 2016.
- [3] Jiang, Xiaotong. *Applications of Catalan Numbers*. Rep. Sweet Briar College, 12 July 2012. Web. 2 May 2016.