

18.01A Practice Final Exam 120 minutes

$$\int_0^{\infty} e^{-x^2/2} dx = \sqrt{\frac{\pi}{2}}$$

Poisson random variable: $P(k) = e^{-m} \frac{m^k}{k!}$, $k = 0, 1, 2, \dots$, mean m .

Exponential density function: $f(x) = \frac{1}{m} e^{-x/m}$, mean m

Normal density function: (mean m , standard deviation σ): $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-m)/\sigma)^2/2}$

Standard normal density function $\phi(z)$ is the above $f(z)$, with $m = 0$, $\sigma = 1$.

Table of values for $\Phi(Z)$, $Z \geq 0$: the distribution function for $\phi(z)$

z :	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	
$\Phi(z)$:	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159	
z :	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	
$\Phi(z)$:	.8413	.8643	.8849	.9032	.9192	.9332	.9452	.9554	.9641	.9713	
z :	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$\Phi(z)$:	.9772	.9821	.9861	.9893	.9918	.9938	.9953	.9965	.9974	.9981	.9987

1. (15) a) Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\sin 2x}$.

b) Let $f(x) = \frac{x}{\ln x}$, $g(x) = \int_2^x \frac{dx}{\ln x}$. Show $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

2. (15) Evaluate the integral $\int_0^1 x\sqrt{1-x^4} dx$, by making the change of variable $x^2 = \sin u$.

3. (10) Evaluate $\int x \sin x dx$.

4. (10) Write as a sum of simpler fractions: $\frac{3x^2 + 1}{(x^2 + 1)(x - 1)}$.

5. (10) If $F(x) = \int_0^x \sqrt{1+u^4} du$, express the value of $\int_2^6 \sqrt{16+t^4} dt$ in terms of values of $F(x)$.

6. (15) A circular pond has the shape of the bottom half of an ellipsoid, formed by rotating the ellipse $\left(\frac{x}{10}\right)^2 + y^2 = 1$ about the y -axis (*not* the x -axis).

It is filled with algae, whose density $\delta = 1 + y$ (in gms/cubic meter) varies with y (it's highest near the surface $y = 0$, where there is most sunlight, and 0 at the bottom).

Find the total mass of algae in the pond.
(Alternatively, for 10 points find the volume of the pond, by setting up and evaluating a definite integral.)

7. (10) After a practice shooting session at an MIT PhysEd class in pistol, examination of a circular target of radius 100 cm showed the density $\delta(r)$ of bullet holes (holes/cm²) depended only on the distance r from the center.

If this density was given by $\delta = \frac{1}{1+r^2}$, how many shots were fired at the target? Give an “exact” answer, then estimate this number to the nearest multiple of 25.

8. (10) For which values of k does $\int_1^{\infty} \frac{dx}{x^k \sqrt{x+1}}$ converge? (Indicate reason.)

9. (20) Tell whether each of the following converges (absolutely or conditionally?) or diverges, indicating reason.

a) $\sum_1^{\infty} \frac{n^2}{\sqrt{n^5+1}}$ b) $\sum_1^{\infty} \frac{(-1)^n}{\sqrt{n}}$

10. (10) Find the radius of convergence of $\sum_1^{\infty} \frac{x^n}{2^{2n} \sqrt{n}}$, showing work.

11. (10) Find the first three non-zero terms in the Taylor series around 0 for $\frac{\sin t}{1+t}$. (Strong suggestion: Use algebraic methods.)

12. (10) Find the sum $f(x) = \sum_0^{\infty} \frac{x^{2n+1}}{2n+1}$ by first differentiating the series.

13. (15:5,10) A continuous probability density on the interval $0 \leq x \leq 1$ is given by $f(x) = Ax^2(1-x)$, where A is a constant.

a) Show that $A = 12$.

b) The mean of $f(x)$ is .6 ; find its standard deviation σ .
(If you can't do this, for 6 points show its mean is .6 .)

14. (15) John Barrymore playing Hamlet used to forget his lines twice a night. On any given night, was he more likely to forget more often than that or less often than that? (Calculate both probabilities, and use $e^2 = 7.4$).

15. (10) A 911 operator finds that the average waiting time between calls is 20 minutes, but needs to be away for 5 minutes. What's the probability an incoming call will be missed? (Estimate to two significant figures, showing work; no calculators.)

16. (15: 5,10) Assume aphids (tiny harmless bugs) are distributed on broccoli with a normal probability density, having mean 200 and standard deviation 20 (aphids/pound).

a) What fraction of a truckload of broccoli would contain over 240 aphids/pound? (Show work or reasoning.)

b) Under pressure from the Farm Lobby, the Dept. of Agriculture agrees to allow 99% of the trucked broccoli to be sold. How many aphids/pound should it announce as the legal upper limit for the aphid count per pound? (If necessary, make a rough interpolation mentally in the table.)