18.100A Fall 2012: Assignment 5 due MONDAY, Sept. 24

You can collaborate, but must list your collaborators, and write up the solutions independently. Cite by name or number any significant theorems you are using in your arguments. Consulting solutions to problem sets of previous years is not allowed.

Mon: Sections 6.1, 6.3-.4 Other forms of the Completeness Principle:
Nested Intervals, Bolzano-Weierstrass, Cauchy sequences.
Wed: 6.5 (holiday Fri.) Set-speak: sup, inf, max, min; Completeness Principle for sets.

Problem 1. (1) Read Def'n 6.2 of a *cluster point* of a sequence $\{x_n\}$, and prove the forward direction $\Rightarrow$ of the cluster point theorem:

If c is a cluster point of $\{x_n\}$, then $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to c.

You have to construct the subsequence step-by-step, so that its terms occur in the same order that they had in $\{x_n\}$, and so that they converge to c. Give a reason why each step is actually possible, and verify the subsequence has the right properties. (If stuck, you can look at the book's proof to get an idea, but try to write it up in your own words.)

Problem 2. (3) Use the Bolzano-Weierstrass theorem (one way or another) in all of the following. In them, the sequence $\{x_n\}$ denotes an arbitrary sequence, not assumed to converge (i.e., have a limit L) or have any other special properties, other than that the expression in the problem be defined for all $n \geq 0$.

a) Let $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ be a polynomial with real coefficients. Prove the sequence $p(\sin x_n)$ has a convergent subsequence. (Use the triangle inequality.)

b) Prove that $\dfrac{x_n^2 + 5}{1 + 2x_n + x_n^2}$ does not always have a convergent subsequence.
(Find a counterexample, and prove it is one.)

c) Prove that the sequence $\dfrac{1}{\sin n}$, $n \geq 1$ has a convergent subsequence. (It is not bounded. Reread Example 5.4C).

Problem 3. (2)

a) Prove the converse of Theorem 6.4: a convergent sequence $\{a_n\}$ is a Cauchy sequence. (Thus the set of convergent sequences is the same as the set of Cauchy sequences.)

b) The sequence $a_n = \sqrt{n}$ satisfies the condition: given $\epsilon > 0$, $|a_n - a_{n+1}| < \epsilon$ for $n \gg 1$; yet a_n is not a Cauchy sequence. Prove these two statements; use theorems about Cauchy sequences, don't go back to the definition.
(The identity $(A - B)(A + B) = A^2 - B^2$) is useful.)

Problem 4. (2) Work Exercise 6.4/2.

The last three problems depend on Wednesday's lecture.

Problem 5. (1) Work 6.5/1ac

Problem 6. (2) Work 6.5/3ag
(For example, to do (a), show $\sup B$ satisfies sup-1 for A.)

Problem 7. (4) Work P6-2ab. This is a significant problem; follow the hints for both parts.
For part (a), your work in Problem 1 above should give some ideas.