18.100A Fall 2012: Assignment 21 due Wed. Nov. 21

The rules are the same as for previous assignments.

Reading Mon.: 24.1-24.5 The Euclidean and uniform distances (norms) on $\mathbf{R}^2$.
Sequences in $\mathbf{R}^2$, limits, continuous functions on $\mathbf{R}^2$, Sequential Continuity theorem.

1. (1) In the app YouDraw, the mode Manhattan-Skyline allows only continuous paths traced out by a point moving in an x-y coordinate system, so that its motion is always in the horizontal or vertical direction – i.e., any change of direction is made by a right angle turn.

(i) Write down a suitable definition in x-y coordinates for the MS-norm $|||\ \ |||$, which gives the length of any of the shortest admissible paths connecting $\mathbf{0}$ and $\mathbf{x}$.
Draw the δ-neighborhood of $\mathbf{0}$ in this norm: $\Delta(\mathbf{0}, \delta) = \{\mathbf{x} \in \mathbf{R}^2 : |||\mathbf{x}||| < \delta\}$.

(ii) Prove that $|||\ \ |||$ satisfies the triangle inequality.

2. (2) Two norms $|\ \ |_1$ and $|\ \ |_2$ in $\mathbf{R}^2$ are called *equivalent* if there are positive constants c and d such that $|\mathbf{x}|_1 \le c|\mathbf{x}|_2$ and $|\mathbf{x}|_2 \le d|\mathbf{x}|_1$, for all $\mathbf{x} \in \mathbf{R}^2$.

a) Prove that in $\mathbf{R}^2$ the Euclidean norm $|\ \ |$ and the uniform norm $||\ \ ||$ are equivalent.

If two norms are equivalent, they give the same results when used in definitions or proofs, for example in defining the limit of a sequence, or the continuity of a function $f(\mathbf{x})$ at a point $\mathbf{x}_0$. Two exercises illustrate this: 24.2/3 ($\lim x_n$) and 24.4/2 (continuity of $f(\mathbf{x})$).

b) Work 24.2/3, to show that a sequence $\mathbf{x}_n$ is convergent in the Euclidean norm if and only if it is convergent in the uniform norm.

3. (2) Work 24.2/2 to see what convergence of a sequence in the plane looks like.
The output should be a drawing of $\mathcal{D}$, with the subregions clearly marked, and arrows on them indicating which limit the points in that subregion are tending to as $n \to \infty$. Pay particular attention to the regions on the boundary of $\mathcal{D}$.

4. (3: 1, 2)

a) Read the proof of the Bolzano-Weierstrass Theorem for $\mathbf{R}^2$ (Theorem 24.2C), using coordinate-wise convergence, and then answer Question 24.2/4, if possible without consulting the solution.

b) Prove the theorem by the bisection method: assume the given sequence lies within the box $B(\mathbf{0}, K)$, divide the box into equal quarters by a horizontal and a vertical line, and tell which quarter to select the first point $\mathbf{x}_{n_1}$ from.
Then subdivide similarly this selected quarter, and tell how to select the next point $\mathbf{x}_{n_2}$ (be careful).
Continuing, you may assume there is only one point $\mathbf{a}$ inside all of the successively chosen nested squares. Indicate why your subsequence $\mathbf{x}_{n_i}$ actually is a subsequence, and prove it converges to $\mathbf{a}$.

5. Work 24.5/5, as a good example of how the sequential continuity theorem in $\mathbf{R}^2$ is used. Give a direct argument using limits; base your argument on the ideas in 24.1-.5, without using compactness; we will save that for the next assignment.