18.100A Fall 2012: Assignment 19 due Fri. Nov. 16

Directions: Same as for previous assignments.

Reading: 22.3,.4 Continuity of uniform limits. Integration of series term-by-term.

Problem 1. (2) Prove the series $\sum_{n=1}^{\infty} \frac{e^{-nx}}{(x+n)^2}$ converges for $x \epsilon [0, \infty)$, and its sum $f(x)$ is continuous on this interval.

(Use simple comparison-type arguments, not tests meant primarily for power series. Verify hypotheses of any theorems being used.)

Problem 2. (2) Prove the series $\sum_{n=0}^{\infty} e^{-nx}$ converges on $(0, \infty)$ to a continuous function.

(The convergence is not uniform on this interval. Give carefully the argument which gets around this difficulty. A similar argument is in given in section 22.3.)

Problem 3. (3:2,1) Let $f(x) = \sum_{1}^{\infty} \frac{\sin nx}{(n-1)!}$.

a) Prove carefully that the integral $\int_0^{\pi} f(x)\, dx$ exists, obtaining its numerical value as the sum of an infinite series. (Cite theorems which justify each step; verify hypotheses.)

b) Express the value of the integral in terms of the function: $\sinh x = \frac{e^x - e^{-x}}{2}$.

(Get the series for $\sinh x$ ("cinch x", the hyperbolic sine) from the known series for e^x.)

Problem 4. (3) Work Problem 22-3b.

Earlier in the semester, in class, $J_0(x)$ was introduced as the function whose graph gives at a single moment in time the diametral cross-sectional shape of a vibrating bass drumhead in a marching band, shortly after the drummer hits it exactly in the middle.

From this, one can derive a differential equation that this shape must satisfy – Bessel's ODE, given in Problem 22-2 – and by finding a series solution to this ODE, get the power series representing $J_0(x)$.

Actually, Bessel was an astronomer, and he discovered the function in a different context, and in the strange-looking form given in P22-3b. So the purpose of this problem is to show that the two representations of $J_0(x)$ are equivalent.

To do this, you will need the power series form of $J_0(x)$, as given in P22-2, and also the standard definite integrals in 22-3a. (These are used in 18.01 and 18.02 for evaluating various multiple integrals – moments of inertia of solids having an axis of symmetry, for instance.)

The series to start with is the power series for $\cos u$.

After $x \sin\theta$ is substituted for u, it is no longer a power series, but the theorems in 22.3,.4 are still usable, since they are phrased in terms of general functions $u_k(x)$.

If you mind your θ's, x's, and u's – the usual variable x used in the problem does not play the same role as the x in the book, for instance – and don't make any errors in calculation, it should all work out in the end, despite some possible dark moments in the middle.

Cite the key theorems you are using; verify their hypotheses hold here.