

18.100A Fall 2012: Assignment 13 due Mon., Oct. 22

Directions: Same as for previous assignments. If you collaborate, list collaborators and write up independently. No consulting previous semesters' problem sets. Cite significant theorems.

Reading: 14.1-3 Differentiation: Local Properties. Definitions, formulas, local theorems, finding local extrema.

1. (1.5) Let $f(x) = \ln x$, $x > 0$. Assuming that $f'(1) = 1$, prove that $f'(x) = 1/x$.
You'll need the standard algebraic formulas: $\ln ab = \ln a + \ln b$, $\ln 1/a = -\ln a$.

2. (1) Work: 14.3/1.

3. (1.5) Work: 14.3/2, for the function $f(x) = \frac{\ln x}{x^n}$, $n = 1, 2, 3, \dots$

4. (3: .5,.5,1,1) The functions $x^k \sin(1/x)$ for $k = 0, 1, 2, \dots$ are a useful source of examples and counterexamples in elementary analysis.

They are defined, continuous, and differentiable when $x \neq 0$, according to Theorems 11.4A and D and the differentiation formulas 14.2A and B. So the interest centers on their properties at the point $x = 0$.

In what follows, complete the domain of $x^k \sin(1/x)$ and $x^k \cos(1/x)$ by defining their value at $x = 0$ to be 0.

Part of what follows is done in Examples in the book, but using the above definition for $x = 0$ simplifies somewhat the arguments in the book. Try to do them without referring to the book; if you do look, don't mindlessly copy what's there.

Prove in turn the following, by using limit theorems, the definition of derivative, and the differentiation rules. For each part you can use if needed the results in previous parts. In each case, draw a sketch of the function illustrating its behavior near 0 (cf. p. 154).

- $\sin(1/x)$ is not continuous at 0; (give an indirect proof using sequential continuity).
- $x \sin(1/x)$ is continuous at 0, but not differentiable at 0;
- $x^2 \sin(1/x)$ is differentiable at 0, but its derivative is not continuous at 0.

A function on an interval I is called **smooth** on I if it is differentiable on I and its derivative is continuous on I . Prove

- $x^3 \sin(1/x)$ is smooth on \mathbf{R} .

5. (1.5) Work Problem 14-1.

6. (1.5: 1,.5) Work Problem 14-6, both parts. No proofs required. Sketch both functions. For part (a), the hint refers to Problem 4c in this assignment.

For part (b), the Dirichlet function $\mathcal{D}(x)$, which has the value 1 if x is rational, 0 otherwise, is helpful.