

18.100A Introduction to Analysis – Practice Final 3 hours

Directions. You can use the textbook, but no other material.

To justify the arguments, **quote theorems**, by name or number, verifying their hypotheses where necessary. Work problems totalling 100. You can include just parts of some problems. Problems will be graded in their numerical order, stopping at 100 or as close above it as possible, the total scaled back to 100 if it goes over.

1. (15: 5, 10) Let $f(x) = \sum_1^{\infty} \frac{1}{x^2 + n^2}$.

- Prove that $f(x)$ is defined and continuous on $(-\infty, \infty)$. (Do not use part (b).)
- Prove $f(x)$ is differentiable on $(-\infty, \infty)$. (Hint: Use intervals of the form $[-a, a]$.)

2. (15: 10,5)

a) Let S be a compact subset of the open first quadrant $Q = \{(x, y) : x > 0, y > 0\}$. Prove there is a point \mathbf{x} in S such that the ray connecting it to the origin has maximum slope (i.e., no other point in S gives a ray with bigger slope).

b) Does (a) continue to be true if S is only assumed to be closed, not compact? If yes, show how to modify the proof in (a) to prove it; if no, give a counterexample.

3. (15) Let $f(\mathbf{x})$ be a continuous function on the plane \mathbf{R}^2 , with values in \mathbf{R} , and let I be a closed interval in \mathbf{R} . Prove that the set $f^{-1}(I)$ is a closed set, where

$$f^{-1}(I) = \{\mathbf{x} \in \mathbf{R}^2 : f(\mathbf{x}) \in I\}.$$

Indicate clearly where you are using that $f(\mathbf{x})$ is continuous.

(Suggestion: in your argument, use 25.2,(1c) as the definition of cluster point.)

4. (15: 7,3,5) The convolution $f(x) * g(x)$ of two continuous functions $f(x)$ and $g(x)$ is a function of x defined by the formula

$$f(x) * g(x) = \int_0^x f(t)g(x-t) dt$$

a) Find, with proof, a formula for its derivative $\frac{d}{dx}f * g$; the formula should be expressed only in terms of $f(x), g(x), f'(x), g'(x), f(0), g(0)$ and their convolutions.

b) Test your formula on the convolution $e^{ax} * e^{bx} = \frac{e^{ax} - e^{bx}}{a - b}$.

(Note that the convolution is linear: $(cf) * g = c(f * g)$.)

c) Using your formula, prove that $y = f(x) * \sin x$ is a solution to $y'' + y = f(x)$ on $(-\infty, \infty)$

5. (15) Let $f(x) = \int_0^{\infty} \frac{e^{-xt}}{1+t} dt$.

Prove that $f(x)$ is defined and differentiable for $x > 0$, and is a solution to the differential equation $y' - y = 1/x$, $x > 0$.

6. (15: 8,7)

- (a) Prove the set M of all square matrices with integer entries is countable.
(b) Let S be a set of real numbers having measure zero.

Prove that, for any given $\epsilon > 0$, the set $S \subseteq \bigcup_1^{\infty} I_n$, where the I_n , $n = 1, 2, 3, \dots$ are open intervals having total length $< \epsilon$.

7. (10: 5,5)

a) "If a sequence $\{x_n\}$, $n \geq 0$ is bounded above for $n \gg 1$, then it is bounded above for all $n \geq 0$." The analogous statement for a function $f(x)$ defined on $[0, \infty]$ would be:

"If $f(x)$ is bounded above for $x \gg 1$, it is bounded above for $x \geq 0$."

Show that this statement is false; strengthen the hypothesis on $f(x)$ in a reasonable way, and prove the amended statement.

b) "If an infinite series $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$." An analogous statement for functions would be:

"If $f(x)$ is continuous on $[0, \infty)$, and $\int_0^{\infty} f(x) dx$ converges, then $\lim_{n \rightarrow \infty} f(x) = 0$."

Show this statement is false; strengthen the hypothesis on $f(x)$ so the amended statement is true. (You need not prove it.)

8. (10: 6, 4) Let $f(x)$ be uniformly continuous on the interval I in \mathbf{R} ; the interval I is not assumed to be compact.

- a) Prove that if $\{x_n\}$ is a Cauchy sequence in I , then $\{f(x_n)\}$ is a Cauchy sequence.
b) Use part (a) to prove that $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.
(You can use that a sequence is a Cauchy sequence if and only if it is convergent.)

9. (10: 6, 4) Suppose $f(x)$ is a function on $[a, b]$; assume that $f(a) < f(b)$. If $f(x)$ is continuous and strictly increasing, the following statement will be true:

For each point y in the interval $[f(a), f(b)]$, there is one and only one point x in the interval $[a, b]$ such that $y = f(x)$.

- a) Prove this statement.
b) For each of the two hypotheses on $f(x)$, tell what part of the statement will not be true in general if that hypothesis is dropped, and give an example to illustrate this.

10. (10: 5, 5) a) Suppose $f(x)$ is continuous on $[a, b]$ and there is a sequence of distinct points x_n in $[a, b]$ such that $f(x_n) = 0$ for all n .

Prove there is a subsequence of x_n which converges to a point c in $[a, b]$ such that $f(c) = 0$.

b) Continuing part (a), suppose that $f'(x)$ exists on $[a, b]$ and is continuous. Prove that also $f'(c) = 0$.

11. (10) Suppose that $f(x)$ is Riemann-integrable and $f(x) = 1$ whenever x is a rational number. (Its value when x is an irrational number is not known.)

Evaluate, with proof, the integral $\int_a^b f(x) dx$.

12. (5) For what values of k does the integral $\int_{0^+}^{\infty} \frac{dx}{x^k \sqrt{x+x^2}}$ converge?