18.100A Practice Problems for Exam 2 F-12 120 minutes (exam is 80 min.)

Directions: You can only use the book; cite relevant theorems when asked to.

1. A function $f(x)$ has three distinct zeros $a_0 < a_1 < a_2$ on an interval I, and in addition $f'(a_2) = 0$. Assume $f(x)$ has a third derivative $f'''(x)$ at all points of I.

Prove there is a point $c \in I$ such that $f'''(c) = 0$.

2. Let $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$. Find the best lower and upper quadratic estimates of the form $1 - x/2 + Kx^2$, valid on $[0, 3]$, given by the Linearization Error Theorem.

3. For what values of the constant $k > 0$ will the equation $\dfrac{\ln x}{x} = k$ have no solutions?

(You can use instead the alternative form $\ln x = kx$; in either case, make a good drawing, show calculations and brief reasoning; cite relevant theorems.

4. Prove from a definition of continuity: $f(x) = \displaystyle\int_0^1 \frac{dt}{1 + xt^4}$ is continuous at 0.

5. a) A function $f(x)$ is defined on all of $\mathbf{R}$, and its secants (i.e., line segments joining two arbitrary points $(x', f(x'))$ and $(x'', f(x''))$ on its graph) have bounded slope.

Prove $f(x)$ is uniformly continuous on I.

(Note that I is not compact, and $f(x)$ is not assumed to be differentiable.)

b) Give an example of an $f(x)$ which satisfies all the hypotheses of part (a) but is not differentiable on I; give with proof explicit bounds on the secant slopes of $f(x)$.

6. Using the two different methods given below, prove that if $f(t)$ is continuous on $[a, b]$,

$$\int_a^b f(t)\, dt \;=\; f(c)(b-a) \quad \text{for some } c \text{ between } a \text{ and } b.$$

Cite theorems used for each step, and indicate where the continuity of $f(t)$ is being used.

a) Prove it by considering the function $F(x) = \displaystyle\int_a^x f(t)\, dt$.

b) Prove it by using the intermediate value theorem, making use of points t' and t'' where $f(t)$ attains respectively its minimum and maximum values on $[a, b]$.

(Assume say $t' < t''$; begin by writing the resulting inequalities $f(t)$ satisfies on $[a, b]$.)

7. Let $f(x)$ be continuous on $[0, \infty)$, and assume $\lim\limits_{x\to\infty} f(x) = L \quad (L \neq \pm\infty)$.

Prove $f(x)$ is bounded on $[0, \infty)$. (Cite theorems; note that the interval is not compact.)

8. Let $f(x)$ be an integrable function on the interval $[a, b]$ of positive length, and assume that $f(x) = 0$ whenever x is a rational number. Prove that $\displaystyle\int_a^b f(x)\, dx = 0$.

(Nothing is known about the value of $f(x)$ when x is irrational. For a weaker result, you can add the assumption: $f(x) \geq 0$ for all $x \in [a, b]$.)

9. For what values of k does $\displaystyle\int_{0^+}^{1^-} \frac{x^k}{\sqrt{x - x^3}} dx$ converge? Give reasoning, cite theorems.

10. Suppose $f'(x)$ exists on $[0, \infty)$ and is continuous, $f(0) = 0$, and $0 \leq f(x) \leq e^{kx}$ for some constant $k < 1$. Citing theorems, prove $\displaystyle\int_0^\infty f'(x)\, e^{-x} dx = \int_0^\infty f(x)\, e^{-x} dx$.