18.100A Practice Questions for Exam 1

This could take 2 hours or more to complete and check over; the actual exam 1 will be shorter and easier, about 2/3 the length. You can use the material in the book, Chs. 1-8, but no other material (notes, problem sets, calculator, etc.)

To justify steps, cite (by number or by name) Theorems or Examples in the book. You should not cite anything else: Questions, Exercises, or Problems.

1. Prove directly from the definition of limit (i.e., without using limit theorems), that

$$\lim_{n\to\infty}\left(\frac{3n^2-1}{n^2+n}\right) \quad = \quad 3 .$$

(Use $|\ \ |$, replacing complicated expressions by simpler ones, paying due attention to which way the inequalities have to go.)

2. Find the radius of convergence R of $\displaystyle\sum_0^\infty \frac{(2n)!\,x^n}{n!(n+1)!}$.

3. a) Prove that if $c > 1$, the sequence $a_n = \dfrac{c^n}{n!}$ is decreasing for $n \gg 1$.

b) Prove $\lim\limits_{n\to\infty} a_n = 0$, by starting at some suitable point N in the sequence and give an estimate of the size of the factors which allows you to use Theorem 3.4 .)

c) Prove part (b) differently by considering the series $\displaystyle\sum_0^\infty a_n$.

4. Prove by elementary reasoning (i.e., without using any theorems from calculus), that if $a > 1$, then $\lim\limits_{n\to\infty} \dfrac{a^n}{n} = \infty$. (Adapt suitably the proof of Theorem 3.4 .)

5. Prove that if $a_n \geq 0$ for all n, and $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L > 1$, then $\sum a_n$ diverges.
(You can use theorems in Chapter 7, but not the n-th root test (Theorem 7.4B).)

6. Let $h(n)$ be the largest prime factor of the integer $n > 1$, and $s(n)$ be the sum of its prime factors, so $h(12) = 3$, $s(12) = 7$.

Prove the sequence $\{h(n)/s(n)\}$, $n = 2, 3, 4, \ldots$ has $1/k$ as a cluster point for every positive integer k, but no limit.

7. Let S be a non-empty set of real numbers which has no upper bound. Prove there is a sequence $\{x_n\}$ of elements in S such that $x_n \to \infty$ as $n \to \infty$.

(Construct the sequence step-by-step, proving each step is possible, and then show its limit is ∞ by using Defn 3.3 .)

8. Prove that the sequence $\{\tan n\}, n = 0, 1, 2, 3, \ldots$ has a convergent subsequence.

(Section 5.4 is useful. The function $\tan x$ is unbounded; it has the vertical asymptotes $x = n\pi/2$, where n is an odd integer.)