25.4 Compactness via Open Sets

In this book we have defined compact sets as those which satisfy the sequential compactness property, and we have proved in **25.2** that in $\mathbf{R}^2$ (and more generally in $\mathbf{R}^n$), these are exactly the sets which are closed and bounded.

In topology and more advanced analysis, compactness is usually given an equivalent definition, phrased entirely in terms of open sets, rather than convergent sequences. There are two reasons: this alternate definition applies to more general spaces than $\mathbf{R}^n$ — for example, infinite-dimensional spaces whose points represent functions. Also, having this definition available gives you more flexibility in constructing proofs for statements that involve compact sets.

In this section, we will give this alternate definition of compactness and an example of a proof using it. We will mostly work in $\mathbf{R}^2$ for concreteness, but all the definitions, arguments, and ideas are essentially the same in $\mathbf{R}^n$, and some of the examples will use $\mathbf{R}$, i.e, the real line $\mathbf{R}^1$.

Definition 25.4A Let S be a subset of $\mathbf{R}^2$. An **open cover** $\mathcal{C}$ of S is a collection $\{U_i\}$ of open subsets U_i of $\mathbf{R}^2$ which together cover S, i.e., every point $x \,\epsilon\, S$ lies in at least one of the sets U_i.

The cover $\mathcal{C}$ is called *finite* if there are only a finite number of the U_i, otherwise *infinite*.

A **subcover of** $\mathcal{C}$ is a selection $\{U_{i_k}\}$ from the subsets $\{U_i\}$ which by themselves also cover S.

Example 25.4A Open Covers

(a) In $\mathbf{R}^1$, the interval $[0,4]$ has finite open covers $\{(-1,5)\}$ and $\{(-1,1),(0,4),(3,5)\}$;

(b) The infinite collection $\mathcal{C} = \{(1/n, 4) : n = 1,2,3,\dots\}$ is not an open cover of $[0,4]$ since the points 0 and 4 are not covered.

(c) In the previous example, if some $\epsilon > 0$ is chosen and $(-\epsilon,\epsilon)$ and $(4-\epsilon, 4+\epsilon)$ are added to $\mathcal{C}$, then it becomes an infinite open cover of $(0,4)$. It has the finite subcover $(-\epsilon,\epsilon)$, $(1/n,4)$, $(4-\epsilon,4+\epsilon)$, where n is any integer such that $1/n < \epsilon$.

(d) The open boxes $\{B(\mathbf{a},1) : \mathbf{a} = (a,b),\ a,b \,\epsilon\, \mathbf{Z}\}$ give an infinite covering of $\mathbf{R}^2$.

Definition 25.4B A set S in $\mathbf{R}^2$ is called **t-compact** if every open cover $\mathcal{C}$ of S has a finite subcover.

(We use t-compact (topologically compact) to temporarily distinguish it from our earlier use of "compact" to mean "sequentially compact". In ordinary parlance only "compact" is used.)

Example 25.4A(c) above illustrates: the interval $[0,4]$ is t-compact; and the collection $\mathcal{C}$ with the two added open intervals around 0 and 4 give an infinite open covering which has a finite subcovering.

Theorem 25.4 The Heine-Borel Theorem

The closed box $B = [-k,k] \times [-k,k]$ *in* $\mathbf{R}^2$ *is t-compact.*

Proof. We give an indirect proof using bisection. Suppose that B is not t-compact. Then it has a open covering $\mathcal{C}$ which has no finite subcovering.

Divide B into four identical closed boxes by the two lines which bisect it horizontally and vertically. Then at least one of these boxes also has no finite subcovering by open sets of $\mathcal{C}$ — choose one and call it B_0. Its dimensions are $k \times k$.

Repeat the process with B_0, getting a closed box B_1 with dimensions $\frac{k}{2} \times \frac{k}{2}$. Continuing, we get an infinite sequence of nested closed boxes B_i of dimensions $\frac{k}{2^i} \times \frac{k}{2^i}$.

By the theorem on nested intervals, there is a unique point $\mathbf{a} = (a, b)$ inside all the B_i. Since $\mathcal{C}$ is a covering of the starting box B, it has an open set U which contains $\mathbf{a}$. This open set must also contain for some $\delta > 0$ the box $B(\mathbf{a}, \delta)$, by the definition of "open set". Therefore U contains the box B_i, if $\frac{k}{2^i} < \frac{\delta}{2}$.

But this is a contradiction: such a B_i is covered by the single open set U from the covering $\mathcal{C}$, yet it was selected as having no finite subcovering from $\mathcal{C}$. This completes the indirect proof, showing that the original square B is t-compact.

A similar bisection argument works for the corresponding box in $\mathbf{R}^n$.

Corollary 25.4B. *A closed and bounded subset S of $\mathbf{R}^2$ is t-compact.*

Proof. Since S is bounded, it is contained in some closed box $\bar{B}(\mathbf{0}, k)$.
Let $\mathcal{C}$ be an open covering of S; add the complementary open set S' to the collection $\mathcal{C}$, you get an open covering of $\bar{B}(\mathbf{0}, k)$. By the Heine-Borel Theorem, this has a finite subcovering, which after removing the added open set S' becomes a finite subcovering of S selected from the original covering $\mathcal{C}$.

The corollary holds in $\mathbf{R}^n$, with the same proof. Its converse is also true in $\mathbf{R}^n$, though we will not prove it here. The end result from combining this with the Compactness Theorem 25.2 is that the following are all equivalent in $\mathbf{R}^2$ (and $\mathbf{R}^n$):

(9) $\quad S$ is sequentially compact $\iff$ S is closed and bounded $\iff$ S is t-compact

Here is an example of a proof using t-compactness, rather than sequential compactness. We prove a weaker version of the boundedness theorem of Chapter 24.

Boundedness Theorem 24.7A.

If $f(\mathbf{x})$ is a continuous function on $\mathbf{R}^2$, then it is bounded on any closed and bounded subset S of $\mathbf{R}^2$.

Proof. Since $f(\mathbf{x})$ is continuous, it is locally bounded at every point $\mathbf{a} \,\epsilon\, S$, by the function location theorem: there is an open disc $D(\mathbf{a}, \delta_a)$ surrounding each $\mathbf{a}\,\epsilon S$ such that

(10) $$f(\mathbf{a}) - 1 < f(\mathbf{x}) < f(\mathbf{a}) + 1 \quad \text{for all } \mathbf{x} \,\epsilon\, D(\mathbf{a}, \delta_a)\ .$$

These discs give an open covering of S, therefore by Corollary 25.4, S has a subcovering using just the discs centered at a finite number of points $a_1, \ldots, \mathbf{a}_n$. It follows from (10) that

$$\min(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_n)) - 1 < f(\mathbf{x}) < \max(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_n)) + 1 \quad \text{for all } \mathbf{x} \,\epsilon\, S\ ,$$

which shows that $f(\mathbf{x})$ is bounded on S.