3

The Limit of a Sequence

3.1 Definition of limit.

In Chapter 1 we discussed the limit of sequences that were monotone; this restriction allowed some short-cuts and gave a quick introduction to the concept. But many important sequences are not monotone—numerical methods, for instance, often lead to sequences which approach the desired answer alternately from above and below. For such sequences, the methods we used in Chapter 1 won't work. For instance, the sequence

$$1.1,\ .9,\ 1.01,\ .99,\ 1.001,\ .999,\ \ldots$$

has 1 as its limit, yet neither the integer part nor any of the decimal places of the numbers in the sequence eventually becomes constant. We need a more generally applicable definition of the limit.

We abandon therefore the decimal expansions, and replace them by the approximation viewpoint, in which "the limit of $\{a_n\}$ is L" means roughly

a_n is a good approximation to L , when n is large.

The following definition makes this precise. After the definition, most of the rest of the chapter will consist of examples in which the limit of a sequence is calculated directly from this definition. There are "limit theorems" which help in determining a limit; we will present some in Chapter 5. Even if you know them, don't use them yet, since the purpose here is to get familiar with the definition.

Definition 3.1 The number L is the **limit** of the sequence $\{a_n\}$ if

(1) $$\text{given } \epsilon > 0, \qquad a_n \underset{\epsilon}{\approx} L \quad \text{for } n \gg 1.$$

If such an L exists, we say $\{a_n\}$ **converges**, or *is convergent*; if not, $\{a_n\}$ **diverges**, or *is divergent*. The two notations for the limit of a sequence are:

$$\lim_{n\to\infty} \{a_n\} = L\ ; \qquad\qquad a_n \to L \ \text{ as } n \to \infty\ .$$

These are often abbreviated to: $\lim a_n = L$ or $a_n \to L$.

Statement (1) looks short, but it is actually fairly complicated, and a few remarks about it may be helpful. We repeat the definition, then build it in three stages, listed in order of increasing complexity; with each, we give its translation into English.

Definition 3.1 $\lim a_n = L$ if: given $\epsilon > 0$, $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$.

Building this up in three succesive stages:

(i) $a_n \underset{\epsilon}{\approx} L$ (a_n approximates L to within ϵ);

(ii) $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$ (the approximation holds for all a_n far enough out in the sequence;);

(iii) given $\epsilon > 0$, $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$

(the approximation can be made as close as desired, provided we go far enough out in the sequence—the smaller ϵ is, the farther out we must go, in general).

The heart of the limit definition is the approximation (i); the rest consists of the if's, and's, and but's. First we give an example.

Example 3.1A Show $\lim\limits_{n\to\infty} \dfrac{n-1}{n+1} = 1$, directly from definition 3.1.

Solution. According to definition 3.1, we must show:

$$\text{(2)} \qquad \text{given } \epsilon > 0, \qquad \frac{n-1}{n+1} \underset{\epsilon}{\approx} 1 \quad \text{for } n \gg 1 \ .$$

We begin by examining the size of the difference, and simplifying it:

$$\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} \ .$$

We want to show this difference is small if $n \gg 1$. Use the inequality laws:

$$\frac{2}{n+1} < \epsilon \quad \text{if} \quad n+1 > \frac{2}{\epsilon} \ , \quad \text{i.e., if} \quad n > N, \ \text{where } N = \frac{2}{\epsilon} - 1 \ ;$$

this proves (2), in view of the definition (2.6) of "for $n \gg 1$". □

The argument can be written on one line (it's ungrammatical, but easier to write, print, and read this way):

Solution. Given $\epsilon > 0$, $\left|\dfrac{n-1}{n+1} - 1\right| = \dfrac{2}{n+1} < \epsilon$, if $n > \dfrac{2}{\epsilon} - 1$. □

Remarks on limit proofs.

1. The heart of a limit proof is in the approximation statement, i.e., in getting a small upper estimate for $|a_n - L|$. Often most of the work will consist in showing how to rewrite this difference so that a good upper estimate can be made. (The triangle inequality may or may not be helpful here.)

Note that in doing this, you *must* use $|\ \ |$; you can drop the absolute value signs only if it is clear that the quantity you are estimating is non-negative.

2. In giving the proof, you must exhibit a value for the N which is lurking in the phrase "for $n \gg 1$". You need not give the smallest possible N; in example 3.1A, it was $2/\epsilon - 1$, but any bigger number would do, for example $N = 2/\epsilon$.

Note that N depends on ϵ: in general, the smaller ϵ is, the bigger N is, i.e., the further out you must go for the approximation to be valid within ϵ .

3. In Definition 3.1 of limit, the phrase "given $\epsilon > 0$" has at least five equivalent forms; by convention, all have the same meaning, and any of them can be used. They are:

$$\text{for all } \epsilon > 0\ , \quad \text{for every } \epsilon > 0\ , \quad \text{for any } \epsilon > 0\ ;$$
$$\text{given } \epsilon > 0\ , \quad \text{given any } \epsilon > 0\ .$$

The most standard of these phrases is "for all $\epsilon > 0$", but we feel that if you are meeting (1) for the first time, the phrases in the second line more nearly capture the psychological meaning. Think of a **limit demon** whose only purpose in life is to make it hard for you to show that limits exist; it always picks unpleasantly small values for ϵ. Your task is, given any ϵ the limit demon hands you, to find a corresponding N (depending on ϵ) such that $a_n \underset{\epsilon}{\approx} L$ for $n > N$.

Remember: the limit demon supplies the ϵ; you cannot choose it yourself.

In writing up the proof, good mathematical grammar requires that you write "given $\epsilon > 0$" (or one of its equivalents) *at the beginning*; get in the habit now of doing it. We will discuss this later in more detail; briefly, the reason is that the N depends on ϵ, which means ϵ must be named first.

4. It is not hard to show (see Problem 3-3) that if a monotone sequence $\{a_n\}$ has the limit L in the sense of Chapter 1—higher and higher decimal place agreement—then L is also its limit in the sense of Definition 3.1. (The converse is also true, but more trouble to show because of the difficulties with decimal notation.) Thus the limit results of Chapter 1, the Completeness Property in particular, are still valid when our new definition of limit is used. From now on, "limit" will always refer to Definition 3.1.

Here is another example of a limit proof, more tricky than the first one.

Example 3.1B Show $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Solution. We use the identity $A - B = \dfrac{A^2 - B^2}{A + B}$, which tells us that

$$(3) \qquad \left|(\sqrt{n+1} - \sqrt{n})\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}\ ;$$

given $\epsilon > 0$, $\qquad \dfrac{1}{2\sqrt{n}} < \epsilon \quad \text{if} \quad \dfrac{1}{4n} < \epsilon^2, \quad \text{i.e., if} \quad n > \dfrac{1}{4\epsilon^2}\ .$ $\square$

Note that here we need not use absolute values since all the quantities are positive.

It is not at all clear how to estimate the size of $\sqrt{n+1} - \sqrt{n}$; the triangle inequality is useless. Line (3) is thus the key step in the argument: the expression must first be transformed by using the identity. Even after doing this, line (3) gives a further simplifying inequality to make finding an N easier; just try getting an N without this step! The simplification means we don't get the smallest possible N; who cares?

Questions 3.1

1. Directly from the definition of limit (i.e., without using theorems about limits you learned in calculus), prove that

(a) $\dfrac{n}{n+1} \to 1$ (b) $\dfrac{\cos na}{n} \to 0$ (a is a fixed number)

(c) $\dfrac{n^2+1}{n^2-1} \to 1$ (d) $\dfrac{n^2}{n^3+1} \to 0$ (cf. Example 3.1B: make a simplifying inequality)

2. Prove that, for any sequence $\{a_n\}$, $\lim a_n = 0 \quad \Leftrightarrow \quad \lim |a_n| = 0$.
(This is a simple but important fact you can use from now on.)

3. Why does the definition of limit say $\epsilon > 0$, rather than $\epsilon \geq 0$?

3.2 The uniqueness of limits. The K-ϵ principle.

Can a sequence have more than one limit? Common sense says no: if there were two different limits L and L', the a_n could not be arbitrarily close to both, since L and L' themselves are at a fixed distance from each other. This is the idea behind the proof of our first theorem about limits. The theorem shows that if $\{a_n\}$ is convergent, the notation $\lim a_n$ makes sense; there's no ambiguity about the value of the limit. The proof is a good exercise in using the definition of limit in a theoretical argument. Try proving it yourself first.

Theorem 3.2A Uniqueness theorem for limits.

A sequence a_n has at most one limit: $a_n \to L$ *and* $a_n \to L' \;\Rightarrow\; L = L'$.

Proof. By hypothesis, given $\epsilon > 0$,

$$a_n \underset{\epsilon}{\approx} L \;\text{ for } n \gg 1, \quad \text{and} \quad a_n \underset{\epsilon}{\approx} L' \;\text{ for } n \gg 1.$$

Therefore, given $\epsilon > 0$, we can choose some large number k such that

$$L \underset{\epsilon}{\approx} a_k \underset{\epsilon}{\approx} L' .$$

By the transitive law of approximation (2.5 (8)), it follows that

$$\text{(4)} \qquad \text{given } \epsilon > 0, \quad L \underset{2\epsilon}{\approx} L' .$$

To conclude that $L = L'$, we reason indirectly (cf. Appendix A.2).

Suppose $L \neq L'$; choose $\epsilon = \frac{1}{2}|L - L'|$. We then have

$$|L - L'| < 2\epsilon, \qquad \text{by (4); i.e.,}$$
$$|L - L'| < |L - L'|, \quad \text{a contradiction.}$$

$\square$

Remarks.

1. The line (4) says that the two numbers L and L' are arbitrarily close. The rest of the argument says that this is nonsense if $L \neq L'$, since they cannot be closer than $|L - L'|$.

2. Before, we emphasized that the limit demon chooses the ϵ; you cannot choose it yourself. Yet in the proof we chose $\epsilon = \frac{1}{2}|L - L'|$. Are we blowing hot and cold?

The difference is this. Earlier, we were trying to prove a limit existed, i.e., were trying to prove a statement of the form:

$$\text{given } \epsilon > 0, \quad \text{some statement involving } \epsilon \text{ is true.}$$

To do this, you must be able to prove the truth no matter what ϵ you are given.

Here on the other hand, we don't have to prove (4)—we already deduced it from the hypothesis. It's a true statement. That means we're allowed to *use* it, and since it says something is true for every $\epsilon > 0$, we can choose a particular value of ϵ and make use of its truth for that particular value.

To reinforce these ideas and give more practice, here is a second theorem which makes use of the same principle, also in an indirect proof. The theorem is "obvious" using the definition of limit we started with in Chapter 1, but we are committed now and for the rest of the book to using the newer Definition 3.1 of limit, and therefore the theorem requires proof.

Theorem 3.2B $\{a_n\}$ *increasing*, $L = \lim a_n \Rightarrow a_n \le L$ *for all* n;
$\{a_n\}$ *decreasing*, $L = \lim a_n \Rightarrow a_n \ge L$ *for all* n.

Proof. Both cases are handled similarly; we do the first.

Reasoning indirectly, suppose there were a term a_N of the sequence such that $a_N > L$. Choose $\epsilon = \frac{1}{2}(a_N - L)$. Then since $\{a_n\}$ is increasing,

$$a_n - L \ \ge\ a_N - L \ >\ \epsilon, \quad \text{for all } n \ge N,$$

contradicting the Definition 3.1 of $L = \lim a_n$. □

The K-ϵ principle.

In the proof of Theorem 3.2A, note the appearance of 2ϵ in line (4). It often happens in analysis that arguments turn out to involve not just ϵ but a constant multiple of it. This may occur for instance when the limit involves a sum or several arithmetic processes. Here is a typical example.

Example 3.2 Let $a_n = \dfrac{1}{n} + \dfrac{\sin n}{n+1}$. Show $a_n \to 0$, from the definition.

Solution To show a_n is small in size, use the triangle inequality:

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| \ \le\ \left|\frac{1}{n}\right| \ +\ \left|\frac{\sin n}{n+1}\right| .$$

At this point, the natural thing to do is to make the separate estimations

$$\left|\frac{1}{n}\right| \ <\ \epsilon, \quad \text{for } n > \frac{1}{\epsilon}\ ; \qquad\qquad \left|\frac{\sin n}{n+1}\right| \ <\ \epsilon, \quad \text{for } n > \frac{1}{\epsilon} - 1\ ;$$

so that, given $\epsilon > 0$,

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| \ <\ 2\epsilon\ , \quad \text{for } n > \frac{1}{\epsilon}\ .$$

This is close, but we were supposed to show $|a_n| < \epsilon$. Is 2ϵ just as good?

The usual way of handling this would be to start with the given ϵ, then put $\epsilon' = \epsilon/2$, and give the same proof, but working always with ϵ' instead of ϵ. At the end, the proof shows

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| < 2\epsilon', \quad \text{for } n > \frac{1}{\epsilon'} \ ;$$

and since $2\epsilon' = \epsilon$, the limit definition is satisfied.

Instead of doing this, let's once and for all agree that if you come out in the end with 2ϵ, or 22ϵ, that's just as good as coming out with ϵ. If ϵ is an arbitrary small number, so is 22ϵ. Therefore, if you can prove something is less than 22ϵ, you have shown that it can be made as small as desired.

We formulate this as a general principle, the "K-ϵ principle". This isn't a standard term in analysis, so don't use it when you go to your next mathematics congress, but it is useful to name an idea that will recur often.

Principle 3.2 The K-ϵ principle.

Suppose that $\{a_n\}$ is a given sequence, and you can prove that

(5) $$\text{given any } \epsilon > 0, \quad a_n \underset{K\epsilon}{\approx} L \quad \text{for } n \gg 1 \ ,$$

where $K > 0$ is a fixed constant, i.e., a number not depending on n or ϵ.

Then $\lim\limits_{n\to\infty} a_n = L$.

The K-ϵ principle is here formulated for sequences, but we will use it for a variety of other limits as well. In all of these uses, the essential point is that K must truly be a constant, and not depend on any of the variables or parameters.

Questions 3.2

1. In the last (indirect) part of the proof of the Uniqueness Theorem, where did we use the hypothesis $L \neq L'$?

2. Show from the definition of limit that if $a_n \to L$, then $ca_n \to cL$, where c is a fixed non-zero constant. Do it both with and without the K-ϵ principle.

3. Show from the definition of limit that $\lim\left(\dfrac{1}{n+1} - \dfrac{2}{n-1}\right) = 0$.

3.3 Infinite limits.

Even though ∞ is not a number, it is convenient to allow it as a sort of "limit" in describing sequences which become and remain arbitrarily large as n increases. The definition is like the one for the ordinary limit.

Definition 3.3 We say the sequence $\{a_n\}$ *tends to infinity* if

(6) $$\text{given any } M \geq 0, \quad a_n > M \quad \text{for } n \gg 1 \ .$$

In symbols: $\lim\limits_{n\to\infty} \{a_n\} = \infty$, or $a_n \to \infty$ as $n \to \infty$.

As for regular limits, to establish that $\lim\{a_n\} = \infty$, what you have to do is give an explicit value for the N concealed in "for $n \gg 1$", and prove that it does the job, i.e., prove that $a_n > M$ when $n \geq N$. In general, this N will depend on M: the bigger the M, the further out in the sequence you will have to go for the inequality $a_n > M$ to hold.

As before, it is not you who chooses the M; the limit demon does that, and you have to prove the inequality in (6) for whatever positive M it gives you.

Note also that even though we are dealing with size, we do not need absolute values, since $a_n > M$ means the a_n are all positive for $n \gg 1$.

One should not think that infinite limits are associated only with increasing sequences. Consider these examples, neither of which is an increasing sequence.

Examples 3.3A Do the following sequences tend to ∞? Give reasoning.

(i) $\{a_n\} = 1,\ 10,\ 2,\ 20,\ 3,\ 30,\ 4,\ 40,\ \ldots,\ k,\ 10k,\ \ldots, \quad (k \geq 1);$

(ii) $\{a_n\} = 1,\ 2,\ 1,\ 3,\ \ldots,\ 1,\ k,\ \ldots, \quad (k \geq 1).$

Solution. (i) A formula for the n-th term is $a_n = \begin{cases} 5n, & n \text{ even;} \\ (n+1)/2, & n \text{ odd.} \end{cases}$

This shows the sequence tends to ∞ since (6) is satisfied: given $M > 0$,

$$a_n > M \quad \text{if } (n+1)/2 > M; \quad \text{i.e., if } n > 2M - 1\ .$$

(ii) The second sequence does not tend to ∞, since (6) is not satisfied for every given M: if we take $M = 10$, for example, it is not true that after some point in the sequence all $a_n > 10$, since the term 1 occurs at every odd position in the sequence.

Example 3.3B Show that $\{\ln n\} \to \infty$.

Solution. We use the fact that $\ln x$ is an increasing function, that is,

$$\ln a > \ln b \quad \text{if } a > b;$$

therefore, given $M > 0$, $\quad \ln n > \ln(e^M) = M \quad \text{if } n > e^M.$

Questions 3.3

1. (a) Formulate a definition for $\lim\limits_{n\to\infty} a_n = -\infty$: "a_n tends to $-\infty$".

   (b) Prove $\ln(1/n) \to -\infty$.

2. Which of these sequences tend to ∞? For those that do, prove it.

   (a) $(-1)^n n$ (b) $n|\sin n\pi/2|$ (c) $\sqrt{n}$ (d) $n + 10\cos n$

3. Prove: if $a_n \to \infty$, then a_n is positive for large n.

3.4 An important limit.

As a good opportunity to practice with inequalities and the limit definition, we prove an important limit that will be used constantly later on.

Theorem 3.4 The limit of a^n .

(7) $$\lim_{n\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1; \\ 1, & \text{if } a = 1; \\ 0, & \text{if } |a| < 1. \end{cases}$$

Proof. We consider the case $a > 1$ first. Since $a > 1$, we can write

$$a = 1 + k, \quad k > 0.$$

Thus $$a^n = (1+k)^n, \quad \text{which by the binomial theorem}$$
$$= 1 + nk + \frac{n(n-1)}{2!}k^2 + \frac{n(n-1)(n-2)}{3!}k^3 + \ldots + k^n.$$

Since all the terms on the right are positive,

(5) $$a^n > 1 + nk\ ;$$
$$> M, \quad \text{for any given } M > 0, \ \text{if } n > M/k, \ \text{say.}$$

This proves that $\lim a^n = \infty$ if $a > 1$, according to Definition 3.3. □

The second case $a = 1$ is obvious. For the third, in outline the proof is:

$$|a| < 1 \quad \Rightarrow \quad \frac{1}{|a|} > 1 \quad \Rightarrow \quad \left(\frac{1}{|a|}\right)^n \to \infty \quad \Rightarrow \quad a^n \to 0\ .$$

Here the middle implication follows from the first case of the theorem. The last implication uses the definition of limit; namely, by hypothesis,

$$\text{given } \epsilon > 0, \qquad \left(\frac{1}{|a|}\right)^n > \frac{1}{\epsilon} \quad \text{for } n \text{ large;}$$

by the reciprocal law of inequalities (2.1) and the multiplication law for $|\ \ |$,

$$|a^n| < \epsilon \quad \text{for } n \text{ large.}$$ □□

Why did we begin by writing $a = 1 + k$? Experimentally, you can see that when $a > 1$, but very close to 1 (like $a = 1.001$), a increases very slowly at first when raised to powers. This is the worst case, therefore, and it suggests writing a in a form which shows how far it deviates from 1.

The case $a \leq -1$ is not included in the theorem; here the a^n alternate in sign without getting smaller, and the sequence has no limit. A formal proof of this directly from the definition of limit is awkward; instead we will prove it at the end of Chapter 5, when we have more technique.

Questions 3.4

1. Find (a) $\lim\limits_{n\to\infty} \cos^n a$; (b) $\lim\limits_{n\to\infty} \ln^n a$, for $a \geq 1$.

2. Suppose one tries to prove the theorem for the case $0 < a < 1$ directly, by writing $a = 1 - k$, where $0 < k < 1$ and imitating the argument given for the first

case $a > 1$. Where does the argument break down? Can one prove a^n is small by dropping terms and estimating? Could one use the triangle inequality?

3.5 Writing limit proofs.

Get in the habit of writing your limit proofs using correct mathematical grammar. The proofs in the body of the text and some of the answers to questions (those answers which aren't just brief indications) are meant to serve as models. But it may also help to point out some common errors.

One frequently sees the following usages involving "for n large" on student papers. Your teacher may know what you mean, but the mathematical grammar is wrong, and technically, they make no sense; avoid them.

Wrong	*Right*
$a_n \to 0$ for $n \gg 1$;	$a_n \to 0$ as $n \to \infty$;
$\lim 2^n = \infty$ for $n \gg 1$;	$\lim 2^n = \infty$;
$\lim(1/n) = 0$ if $n > 1/\epsilon$;	$\begin{cases} \lim(1/n) = 0; \\ (1/n) \underset{\epsilon}{\approx} 0 \quad \text{if } n > 1/\epsilon. \end{cases}$

In the first two, the limit statement applies to the sequence as a whole, whereas "for $n >$ some N" can only apply to individual terms of the sequence. The third is just a general mess; two alternatives are offered, depending on what was originally meant.

As we said earlier, "given $\epsilon > 0$" or "given $M > 0$" must come first:

Poor: $1/n < \epsilon$, for $n \gg 1$ (what is ϵ, and who picked it?)

Wrong: For $n \gg 1$, given $\epsilon > 0$, $1/n < \epsilon$.

This latter statement is wrong, because according to mathematical conventions, it would mean that the N concealed in "for $n \gg 1$" should not depend on ϵ. This point is more fully explained in Appendix B; rather than try to study it there at this point, you will be better off for now just remembering to first present ϵ or M, and then write the rest of the statement.

Another point: write up your arguments using plenty of space on your paper (sorry, clarity is worth a tree). Often in the book's examples and proofs, the inequality and equality signs are lined up below each other, rather than strung out on one line; it is like properly-written computer code. See how it makes the argument clearer, and imitate it in your own work. If equalities and inequalities both occur, the convention we will follow is:

$$\begin{aligned} A &< n(n+1) \\ &= n^2 + n\ , \end{aligned} \qquad \text{rather than} \qquad \begin{aligned} A &< n(n+1) \\ &< n^2 + n\ . \end{aligned}$$

The form on the right doesn't tell you explicitly where the second line came from; in the form on the left, the desired conclusion $A < n^2 + n$ isn't explicitly stated, but it is easily inferred.

3.6 Some limits involving integrals.

To broaden the range of applications and get you thinking in some new directions, we look at a different type of limit which involves definite integrals.

Example 3.6A Let $a_n = \int_0^1 (x^2+2)^n dx$. Show that $\lim_{n\to\infty} a_n = \infty$.

The way *not* to do this is to try to evaluate the integral, which would just produce an unwieldy expression in n that would be hard to interpret and estimate. To show that the integral tends to infinity, all we have to do is get a lower estimate for it that tends to infinity.

Solution. We estimate the integral by estimating the integrand.

$$x^2+2 \ \geq\ 2 \qquad \text{for all } x;$$

therefore,

$$(x^2+2)^n \ \geq\ 2^n \qquad \text{for all } x \text{ and all } n \geq 0.$$

Thus

$$\int_0^1 (x^2+2)^n dx \ \geq\ \int_0^1 2^n dx \ =\ 2^n \ .$$

Since $\lim 2^n = \infty$ by Theorem 3.4, the definite integral must tend to ∞ also:

$$\text{given } M > 0, \quad \int_0^1 (x^2+2)^n dx \ \geq\ 2^n \ \geq\ M, \ \text{for } n > \log_2 M. \qquad \square$$

Example 3.6B Show $\lim_{n\to\infty} \int_0^1 (x^2+1)^n dx = \infty$.

Solution. Once again, we need a lower estimate for the integral that is large. The previous argument gives the estimate $(x^2+1)^n \geq 1^n = 1$, which is useless. However, it may be modified as follows.

Since x^2+1 is an increasing function which has the value $A = 5/4$ at the point $x = .5$ (any other point on $(0,1)$ would do just as well), we can say

$$x^2+1 \ \geq\ A \ >\ 1 \qquad \text{for } .5 \leq x \leq 1;$$

therefore,

$$(x^2+1)^n \ \geq\ A^n \qquad \text{for } .5 \leq x \leq 1;$$

since $\lim A^n = \infty$ by Theorem 3.4, the definite integral must tend to ∞ also:

$$\text{given } M > 0, \quad \int_0^1 (x^2+1)^n dx \ \geq\ \int_{.5}^1 A^n dx \ =\ \frac{A^n}{2} \ \geq\ M, \ \text{for } n \text{ large.} \qquad \square$$

Questions 3.6

1. By estimating the integrand, show that: $\dfrac{1}{3} \leq \displaystyle\int_0^1 \frac{x^2+1}{x^4+2}\, dx \leq 1$.

2. Show without integrating that $\lim_{n\to\infty} \displaystyle\int_0^1 x^n(1-x)^n dx = 0$.

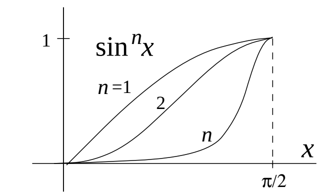
3.7 Another limit involving an integral.

We give a third example, this one a little more sophisticated than the other two. You can skip it, but it is worth studying and understanding.

Example 3.7 Let $a_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$. Determine $\lim a_n$.

Remarks. As before, attempting to evaluate the integral will lead to an expression in n whose limit is not so easy to determine.

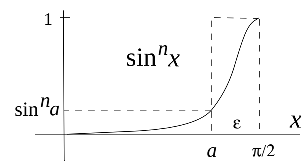
The integral represents an area, so it helps to have some idea of how the curves $\sin^n x$ look. Since $\sin x < 1$ on the interval $0 \le x < \frac{\pi}{2}$, by Theorem 3.4 the powers $\sin^n x \to 0$. Thus as n increases, the successive curves get closer and closer to the x-axis, except that the right-hand end always passes through the point $(\frac{\pi}{2}, 1)$.



The area under the curve seems to get small, as n increases, so the limit should be 0. But how do we prove this?

For a big value of n, the area really is composed of two parts, both of which are small, but for different reasons. The vertical strip on the right is small in area because it is thin. The horizontal strip below is small in area because the curve is near the x-axis.

This suggests the following procedure.



Solution. Given $\epsilon > 0$, we show:

$$\text{area under } \sin^n x \text{ and over } [0, \tfrac{\pi}{2}] \ < \ 2\epsilon, \ \text{ for } n \gg 1.$$

(This suffices, by the K-ϵ principle of section 3.2.)

Mark off the the point a shown, where $a = \frac{\pi}{2} - \epsilon$. Divide up the area under the curve into two pieces as shown, and draw in the two rectangles.

$$\begin{aligned} \text{right-hand area} \ &< \ \text{area of right-hand rectangle} \ = \ \epsilon \ ; \\ \text{left-hand area} \ &< \ \text{area of horizontal rectangle} \ = a \sin^n a \\ &< \ \epsilon, \quad \text{for } n \gg 1, \end{aligned}$$

since $|\sin a| < 1$ (cf. Theorem 3.4). Therefore,

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx \ = \ \begin{matrix}\text{total area} \\ \text{under curve}\end{matrix} \ < \ \begin{matrix}\text{area of the} \\ \text{two rectangles}\end{matrix} \ < \ 2\epsilon, \quad \text{for } n \gg 1. \qquad \square$$

It would be easy to get rid of the pictures and just use integral signs everywhere, but it wouldn't make the argument more rigorous, just more obscure.

Exercises

3.1

1. Show that the following sequences have the indicated limits, directly from the definition of limit.

(a) $\lim_{n\to\infty} \dfrac{\sin n - \cos n}{n} = 0$ (b) $\lim_{n\to\infty} \dfrac{2n-1}{n+2} = 2$

(c) $\lim_{n\to\infty} \dfrac{n}{n^2+3n+1} = 0$ (d) $\lim_{n\to\infty} \dfrac{n}{n^3-1} = 0$

(e) $\lim_{n\to\infty} \sqrt{n^2+2} - n = 0$

2. Prove that if a_n is a non-negative sequence, $\lim a_n = 0 \Rightarrow \lim \sqrt{a_n} = 0$.

3.2

1. Prove that if $a_n \to L$ and $b_n \to M$, then $a_n + b_n \to L + M$.

Do this directly from the definition 3.1 of limit.

2. Suppose $\{a_n\}$ is a convergent increasing sequence, and $\lim a_n = L$.

Let $\{b_n\}$ be another sequence "interwoven" with the first, i.e., such that

$$a_n < b_n < a_{n+1} \quad \text{for all } n.$$

Prove from the definition of limit that $\lim\ b_n = L$ also.

3. (a) Prove the sequence $a_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{2n}$ has a limit.

(b) Criticize the following "proof" that its limit is 0:

Given $\epsilon > 0$, then for $i = 1, 2, 3, \ldots$, we have

$$\frac{1}{n+i} < \epsilon, \quad \text{if } \frac{1}{n} < \epsilon, \quad \text{i.e., if } n > 1/\epsilon .$$

Adding up these inequalities for $i = 1, \ldots, n$ gives

$$0 < a_n < n\epsilon, \quad \text{for } n > 1/\epsilon ;$$

therefore, $$a_n \underset{n\epsilon}{\approx} 0, \quad \text{for } n \gg 1 .$$

By the definition of limit and the K-ϵ principle, $\lim\ a_n = 0$. □

4. Prove that $\lim_{n\to\infty} \left(\dfrac{1}{n^2+1} + \dfrac{1}{n^2+2} + \ldots + \dfrac{1}{n^2+n} \right) = 0$.

(Modify the incorrect argument in the preceding exercise.)

5. Let $\{a_n\}$ be a convergent sequence of integers, having the limit L. Prove that it is "eventually constant", that is, $a_n = L$ for large n.

(Apply the limit definition, taking $\epsilon = 1/4$, say. Why is it legal for you to choose the ϵ in this case?)

3.3

1. For each sequence $\{a_n\}$, tell whether or not $a_n \to \infty$; if so, prove it directly from Definition 3.3 of infinite limit.

(a) $a_n = \dfrac{n^2}{n-1}$ (b) $a_n = n^2|\cos n\pi|$ (c) $a_n = \dfrac{n^3}{n^2+2}$

(d) $a_n = 1,\ 2,\ 3,\ \ 2,\ 3,\ 4,\ \ 3,\ 4,\ 5,\ \dots$ (e) $\sqrt{n}$ (f) $\ln\ln(n)$

2. Prove: if $a_n < b_n$ for $n \gg 1$, and $a_n \to \infty$, then $b_n \to \infty$. Base the proof on Definition 3.3.

3. Prove: if $\{a_n\} \to \infty$, then $\{a_n\}$ is not bounded above. (This is "obvious"; the point is to get practice in using the definitions to construct arguments. Give an indirect proof; see Appendix A.2.)

3.4

1. Define a_n recursively by $a_{n+1} = ca_n$, where $|c| < 1$. Prove $\lim a_n = 0$, using Theorem 3.4 and Definition 3.1.

2. Let $a_n = r^n/n!$. Prove that for all $r \in \mathbb{R}$, we have $a_n \to 0$.

(If $|r| \le 1$, this is easy. If $|r| > 1$, it is more subtle. Compare two successive terms of the sequence, and show that if $n \gg 1$, then $|a_{n+1}|$ is less than half of $|a_n|$. Then complete the argument.)

3. Prove that if $a > 1$, then $a^n/n \to \infty$.

(Hint: imitate the proof in theorem 3.4, but use a different term in the binomial expansion.)

4. Prove that $na^n \to 0$ if $0 < a < 1$, using the result in the preceding exercise.

5. Prove that $\lim a^{1/n} = 1$, if $a > 0$.

(Here $a^{1/n}$ means the real positive n-th root of a. Nothing is said about $a < 0$, since then $a^{1/n}$ is not a real number if n is even.

Note how $\{a^{1/n}\}$ and $\{a^n\}$ have opposite behavior as sequences. As we take successive n-th roots, all positive numbers *approach* 1; in contrast, as we take successive n-th powers, all positive numbers $\neq 1$ *recede from* 1.)

(a) Consider first the case $a > 1$. For each n, put $a^{1/n} = 1 + h_n$, and show that $h_n \to 0$, by reasoning like that in Theorem 3.4.

(b) Now consider the case $a < 1$. If $a < 1$, then $1/a > 1$; use this and the definition of limit to deduce this case from the previous one. (Use only the definition of limit in this chapter, not other "obvious" facts about limits.)

3.6

1. Modeling your arguments on the two examples given in this section, prove the following without attempting to evaluate the integrals explicitly.

(a) $\displaystyle\lim_{n\to\infty} \int_1^2 \ln^n x\, dx = 0$ (b) $\displaystyle\lim_{n\to\infty} \int_2^3 \ln^n x\, dx = \infty$

3.7

1. Show that $\lim_{n\to\infty} \int_0^1 (1-x^2)^n dx = 0$, without attempting to evaluate the integral explicitly.

Problems

3-1 Let $\{a_n\}$ be a sequence and $\{b_n\}$ be its sequence of averages:

$$b_n = (a_1 + \ldots + a_n)/n \qquad \text{(cf. Problem 2-1).}$$

(a) Prove that if $a_n \to 0$, then $b_n \to 0$.

(Hint: this uses the same ideas as example 3.7. Given $\epsilon > 0$, show how to break up the expression for b_n into two pieces, both of which are small, but for different reasons.)

(b) Deduce from part (a) in a few lines without repeating the reasoning that if $a_n \to L$, then also $b_n \to L$.

3-2 To prove a^n was large if $a > 1$, we used "Bernoulli's inequality":

$$(1+h)^n \geq 1 + nh, \quad \text{if } h \geq 0 .$$

We deduced it from the binomial theorem. This inequality is actually valid for other values of h however. A sketch of the proof starts:

$$(1+h)^2 = 1 + 2h + h^2 \geq 1 + 2h, \qquad \text{since } h^2 \geq 0 \text{ for all } h;$$

$$\begin{aligned}(1+h)^3 &= (1+h)^2(1+h) \geq (1+2h)(1+h), \quad \text{by the previous case,}\\ &= 1 + 3h + 2h^2,\\ &\geq 1 + 3h .\end{aligned}$$

(a) Show in the same way that the truth of the inequality for the case n implies its truth for the case $n+1$. (This proves the inequality for all n by mathematical induction, since it is trivially true for $n = 1$.)

(b) For what h is the inequality valid? (Try it when $h = -3$, $n = 5$.) Reconcile this with part (a).

3-3 Prove that if a_n is a bounded increasing sequence and $\lim a_n = L$ in the sense of Definition 1.3A, then $\lim a_n = L$ in the sense of Definition 3.1.

3-4 Prove that a convergent sequence $\{a_n\}$ is bounded.

3-5 Given any $c \in \mathbb{R}$, prove there is a strictly increasing sequence $\{a_n\}$ and a strictly decreasing sequence $\{b_n\}$, both of which converge to c, and such that all the a_n and b_n are

(i) rational numbers; (ii) irrational numbers.

(Theorem 2.5 is helpful.)

Answers

3.1

1. We write these up in four slightly different styles; take your pick.

(a) Given $\epsilon > 0$, $\left|\dfrac{n}{n+1} - 1\right| = \dfrac{1}{n+1}$,

which is $< \epsilon$ if $n > 1/\epsilon - 1$, or $n > 1/\epsilon$.

(b) Given $\epsilon > 0$, $\left|\dfrac{\cos na}{n}\right| \le \dfrac{1}{n}$, since $|\cos x| \le 1$ for all x;

and $1/n < \epsilon$ if $n > 1/\epsilon$.

(c) Given $\epsilon > 0$, $\left|\dfrac{n^2+1}{n^2-1} - 1\right| = \left|\dfrac{2}{n^2-1}\right|$,

$< \epsilon$ if $n^2 - 1 > 2/\epsilon$, or $n > \sqrt{2/\epsilon + 1}$.

(d) Given $\epsilon > 0$, $\dfrac{n^2}{n^3+1} < \dfrac{1}{n} < \epsilon$ if $n > 1/\epsilon$.

2. $\lim a_n = 0$ means: given $\epsilon > 0$, $|a_n - 0| < \epsilon$ for $n \gg 1$.

$\lim |a_n| = 0$ means: given $\epsilon > 0$, $\big||a_n| - 0\big| < \epsilon$ for $n \gg 1$.

But these two statements are the same, since

$$\big||a_n| - 0\big| = \big||a_n|\big| = |a_n| = |a_n - 0| \;.$$

3. If the limit demon were allowed to give you $\epsilon = 0$, then since $\underset{0}{\approx}$ is the same as equality $=$, it would have to be true that $a_n = L$ for $n \gg 1$; in other words, any sequence which had the limit L would from some point on have to be constant and equal to L. This would be too restricted a notion of limit. (The sequences which do behave this way are said to be "eventually constant"; cf. Exercise 3.2/5.)

3.2

1. $L \ne L' \;\Rightarrow\; \epsilon = |L - L'|/2 > 0$, which is essential (cf. Question 3.1/3).

2. (a) Given $\epsilon > 0$, $|a_n - L| < \epsilon$ for $n > N$, say. Therefore,

$$|ca_n - cL| = |c||a_n - L| < |c|\epsilon \quad \text{for } n > N,$$

so we're done by the K-ϵ principle.

(b) Given $\epsilon > 0$, $|a_n - L| < \epsilon/|c|$ for $n > N_1$, say. Therefore,

$$|ca_n - cL| = |c||a_n - L| < |c|\epsilon/|c| = \epsilon \quad \text{for } n > N_1.$$

3. Given $\epsilon > 0$,

$$\left|\frac{1}{n+1} - \frac{2}{n-1}\right| \le \left|\frac{1}{n+1}\right| + \left|\frac{2}{n-1}\right| \qquad \text{(triangle inequality)}$$

$$< \epsilon + \epsilon, \quad \text{if } n+1 > 1/\epsilon, \text{ and } n-1 > 2/\epsilon;$$

$$< 2\epsilon, \quad \text{if } n > 1 + 2/\epsilon;$$

so we're done by the K-ϵ principle.

3.3

1. (a) Given $-M < 0$, $a_n < -M$ for $n \gg 1$. (The $-$ signs can be omitted.)

(b) $\ln(1/n) = -\ln n < -M$ if $\ln n > M$, i.e., if $n > e^M$.

2. (a) no; alternate terms are negative;

(b) no; alternate terms are 0;

(c) yes; given $M > 0$, $\sqrt{n} > M$ if $n > M^2$;

(d) yes; given $M > 0$, $n + 10\cos n > n - 10 > M$, if $n > M + 10$.

3. Given M, $a_n > M$ for $n \gg 1$. Take $M = 0$: $a_n > 0$ for $n \gg 1$.

3.4

1. (a) limit is 0, except: limit is 1 if $a = 2n\pi$, no limit if $a = (2n+1)\pi$.

(b) limit is: $\begin{cases} 0, & \text{if } 1 \le a < e; \\ 1, & \text{if } a = e; \\ \infty, & \text{if } a > e. \end{cases}$

2. Since the terms alternate in sign, one cannot get an inequality after dropping most of the terms to simplify the expression. Basically, a^n is small not because the individual terms are small, but because they cancel each other out. Thus the triangle inequality cannot help either. So this approach doesn't lead to a usable estimation that would show a_n is small.

3.6

1. Over the interval $0 \le x \le 1$,

$$1 \le x^2 + 1 \le 2; \qquad 2 \le x^4 + 2 \le 3.$$

By standard reasoning (see Example 2.2B for instance),

$$\frac{1}{3} \le \frac{x^2+1}{x^4+2} \le \frac{2}{2}\ ;$$

therefore its integral over the interval $[0,1]$ of length 1 also lies between these bounds.

2. On $[0,1]$, the maximum of $x(1-x)$ occurs at $x = 1/2$, therefore

$$0 \le x(1-x) \le 1/4 \quad \text{on } [0,1];$$

$$\text{given } \epsilon > 0, \quad 0 \le \int_0^1 x^n(1-x)^n dx \le (1/4)^n < \epsilon, \quad \text{if } 4^n > 1/\epsilon\ ;$$

therefore, denoting the integral by I_n, we see that

$$\text{given } \epsilon > 0, \quad |I_n| < \epsilon \quad \text{if } n > \ln(1/\epsilon)/\ln 4\ ,$$

which proves that $\lim I_n = 0$.