NON-DECOMPOSABILITY OF THE DE RHAM COMPLEX AND NON-SEMISIMPLICITY OF THE SEN OPERATOR

ALEXANDER PETROV

Abstract. We describe the obstruction to decomposing in degrees $\leq p$ the de Rham complex of a smooth variety over a perfect field k of characteristic p that lifts over $W_2(k)$, and show that there exist liftable smooth projective varieties of dimension $p+1$ whose Hodge-to-de Rham spectral sequence does not degenerate at the first page. We also describe the action of the Sen operator on the de Rham complex in degrees $\leq p$ and give examples of varieties with a non-semisimple Sen operator.

Our methods rely on the commutative algebra structure on de Rham and Hodge-Tate cohomology, and are inspired by the properties of Steenrod operations on cohomology of cosimplicial commutative algebras. The example of a non-degenerate Hodge-to-de Rham spectral sequence relies on a non-vanishing result on cohomology of groups of Lie type. We give applications to other situations such as describing extensions in the canonical filtration on de Rham, Hodge, and étale cohomology of an abelian variety equipped with a group action. We also show that the de Rham complex of a smooth variety over k is formal as an E_∞-algebra if and only if the variety lifts to $W_2(k)$ together with its Frobenius endomorphism.

Contents

1. Introduction

1.1. **The main results.** A celebrated theorem of Deligne and Illusie provides an analog of Hodge decomposition on de Rham cohomology in degrees $< p$ of varieties over a perfect field k of positive characteristic p that lift to $W_2(k)$. In [DI87] they proved that for a smooth variety over X_0 over k that admits a lift to a smooth scheme X_1 over the ring $W_2(k)$ of length 2 Witt vectors, the canonical truncation of the de Rham complex $\mathrm{dR}_{X_0/k} := (F_{X_0/k*}\Omega^\bullet_{X_0/k}, d)$ in degrees $\leq p-1$ decomposes in the derived category $D(X_0^{(1)})$ of $\mathcal{O}_{X_0^{(1)}}$-modules:

$$\tau^{\leq p-1}\mathrm{dR}_{X_0/k} \simeq \bigoplus_{i=0}^{p-1} \Omega^i_{X_0^{(1)}/k}[-i] \tag{1.1}$$

Passing to cohomology, this gives decompositions for $n < p$:

$$H^n_{\mathrm{dR}}(X_0/k) \simeq \bigoplus_{i+j=n} H^j(X_0, \Omega^i_{X_0/k})^{(1)} \tag{1.2}$$

In this paper we investigate whether the de Rham complex of a variety liftable to $W_2(k)$ decomposes in further degrees. We prove that in general it does not decompose, and decomposition (1.2) might not exist, answering a question of Deligne and Illusie [DI87, Remarque 2.6(iii)]:

Theorem 1.1 (Corollary 10.7)**.** *There exists a smooth projective variety X_0 over k of dimension $p+1$ that lifts to $W(k)$ such that* $\dim_k H^p_{\mathrm{dR}}(X_0/k) < \sum_{i+j=p} \dim_k H^j(X_0, \Omega^i_{X_0/k})$*. In particular, the Hodge-to-de Rham spectral sequence of X_0 does not degenerate at the first page, and $\mathrm{dR}_{X_0/k}$ is not quasi-isomorphic to $\bigoplus_{i\geq 0} \Omega^i_{X_0^{(1)}/k}[-i]$.*

Moreover, we compute the obstruction to decomposing the truncation $\tau^{\leq p}\mathrm{dR}_{X_0/k}$ of the de Rham complex in degrees $\leq p$ in terms of other invariants of the variety X_0 and its lift X_1. Given that $\tau^{\leq p-1}\mathrm{dR}_{X_0/k}$ decomposes, the next truncation fits into an exact triangle in the derived category $D(X_0^{(1)})$:

$$\bigoplus_{i=0}^{p-1} \Omega^i_{X_0^{(1)}/k}[-i] \to \tau^{\leq p}\mathrm{dR}_{X_0/k} \to \Omega^p_{X_0^{(1)}/k}[-p] \tag{1.3}$$

and hence gives an extension class

$$e_{X_1,p} \in H^{p+1}(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k}) \oplus H^p(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k} \otimes \Omega^1_{X_0^{(1)}/k}) \oplus \dots$$

that vanishes if and only if (1.3) splits in $D(X_0^{(1)})$. We compute this class:

Theorem 1.2 (Theorem 5.12 if X_1 lifts to $W(k)$, Corollary 7.5 in general)**.** *All components of $e_{X_1,p}$ except for the one in $H^{p+1}(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k})$ vanish, and that component is equal to*

$$\mathrm{Bock}_{X_1^{(1)}}(\mathrm{ob}_{F,X_1} \cdot \alpha(\Omega^1_{X_0^{(1)}})) \in H^{p+1}(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k}) \tag{1.4}$$

where

- $\mathrm{Bock}_{X_1^{(1)}} : H^p(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k}) \to H^{p+1}(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}/k})$ *is the Bockstein homomorphism, that is the connecting morphism arising from the short exact sequence of sheaves on X_1: $0 \to \Lambda^p T_{X_0^{(1)}/k} \to \Lambda^p T_{X_1^{(1)}/W_2(k)} \to \Lambda^p T_{X_0^{(1)}/k} \to 0$.*

- $\mathrm{ob}_{F,X_1} \in H^1(X_0^{(1)}, F^*_{X_0^{(1)}} T_{X_0^{(1)}})$ *is the obstruction to lifting the Frobenius morphism* $F_{X_0} : X_0 \to X_0$ *to an endomorphism of* X_1*,*
- $\alpha(\Omega^1_{X_0^{(1)}}) \in H^{p-1}(X_0^{(1)}, F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}} \otimes (\Lambda^p \Omega^1_{X_0^{(1)}})^\vee)$ *is a certain 'characteristic class' of the cotangent bundle, defined in Definition 3.4.*

The class $\alpha(E) \in H^{p-1}(X_0, F^*_{X_0}E \otimes (\Lambda^p E)^\vee)$ is defined for any vector bundle E on X_0, and it can be made especially explicit when $p = 2$, in which case it is the class of the extension

$$0 \to F^*_{X_0}E \to S^2E \xrightarrow{e_1 \cdot e_2 \mapsto e_1 \wedge e_2} \Lambda^2 E \to 0. \tag{1.5}$$

The fact that $e_{X_1,p}$ has only one potentially non-zero component is a shadow of an additional $\mathbb{Z}/p$-grading on the de Rham complex $\mathrm{dR}_{X_0/k}$ in the presence of a lift X_1, discovered by Drinfeld through his work on the prismatization [Dri21]. More generally, Bhatt-Lurie [BL22a] and Li-Mondal [LM21] showed that a lift of X_0 to a smooth $W_2(k)$-scheme X_1 induces an endomorphism Θ_{X_1} of $\mathrm{dR}_{X_0/k}$ in the derived category $D(X_0^{(1)})$, called the *Sen operator*.

The action of Θ_{X_1} on the ith cohomology sheaf $H^i(\mathrm{dR}_{X_0/k}) \simeq \Omega^i_{X_0^{(1)}/k}$ is given by multiplication by $(-i)$. Therefore generalized eigenspaces for this operator form a decomposition

$$\mathrm{dR}_{X_0/k} \simeq \bigoplus_{i=0}^{p-1} \mathrm{dR}_{X_0/k,i} \tag{1.6}$$

such that the object $\mathrm{dR}_{X_0/k,i}$ has non-zero cohomology only in degrees congruent to i modulo p. In particular, $\tau^{\leq p}\mathrm{dR}_{X_0/k,0}$ is an extension of $\Omega^p_{X_0^{(1)}/k}[-p]$ by $\mathcal{O}_{X_0^{(1)}}$ that splits off as a direct summand from the extension (1.3), corroborating the fact that the components of $e_{X_1,p}$ in $H^{p+1-i}(X_0, \Lambda^p T_{X_0} \otimes \Omega^i_{X_0})$ vanish for $i > 0$. It was observed earlier by Achinger and Suh [AS20] that these components vanish for $i > 1$, for a purely homological-algebraic reason which is related to our method.

The Sen operator Θ_{X_1} not only induces the decomposition (1.6) but also equips each direct summand $\mathrm{dR}_{X_0/k,i}$ with an additional structure of the nilpotent endomorphism $\Theta_{X_1} + i$. Our methods allow to describe the action of Θ_{X_1} on $\tau^{\leq p}\mathrm{dR}_{X_0,0}$. Since $\tau^{\leq p}\mathrm{dR}_{X_0/k,0}$ has only two non-zero cohomology sheaves and Θ_{X_1} acts on them by zero, it naturally defines a map $\Omega^p_{X_0^{(1)}}[-p] \to \mathcal{O}_{X_0^{(1)}}$ in the derived category $D(X_0^{(1)})$ whose cohomology class we denote by $c_{X_1^{(1)},p} \in H^p(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}})$. This class can be described as:

Theorem 1.3 (Theorem 7.1)**.** *For a smooth $W_2(k)$-scheme X_1 with the special fiber $X_0 = X_1 \times_{W_2(k)} k$ we have*

$$c_{X_1,p} = \mathrm{ob}_{F,X_1} \cdot \alpha(\Omega^1_{X_0}) \tag{1.7}$$

where the right hand side is the product of classes $\mathrm{ob}_{F,X_1} \in H^1(X_0, F^*_{X_0}T_{X_0/k})$ *and* $\alpha(\Omega^1_{X_0/k}) \in H^{p-1}(X_0, F^*_{X_0}\Omega^1_{X_0/k} \otimes \Lambda^p T_{X_0/k})$

It is not a coincidence that $e_{X_1,p}$ is the result of applying the Bockstein homomorphism to $c_{X_1,p}$ – this follows from the basic properties of the action of the Sen operator on the diffracted Hodge cohomology of X_1, as we prove in Lemma 6.12. In particular, if $\tau^{\leq p}\mathrm{dR}_{X_0/k}$ is not decomposable then the Sen operator on $\tau^{\leq p}\mathrm{dR}_{X_0/k}$ must be non-semi-simple. Thus Theorem 1.1 also provides an example of a liftable variety with a non-semisimple Sen operator on its de Rham cohomology, answering a question of Bhatt.

In fact, the non-vanishing of $c_{X_1,p}$ is a more frequent phenomenon than that of $e_{X_1,p}$ as we demonstrate by the following:

Corollary 1.4 (Proposition 8.4)**.** *There exists a smooth projective variety X_0 of dimension p over k equipped with a lift X_1 over $W_2(k)$ such that the Sen operator Θ_{X_1} on $\mathrm{dR}_{X_0/k}$ is not semisimple.*

The examples we construct admit a smooth proper morphism to a curve, and for such varieties the class $\alpha(\Omega^1_{X_0})$ can be expressed in terms of the Kodaira-Spencer class of this fibration, giving a more tangible reformulation (Theorem 8.1) of Theorem 1.3.

We give two different, but similar in spirit, proofs of Theorem 1.2, both of which use crucially the multiplicative structure on de Rham and Hodge-Tate complexes.

1.2. **Proof of Theorem 1.2.** The first proof, for which we additionally need to assume that X_0 lifts to a smooth (formal) scheme X over $W(k)$, takes place in homotopical algebra rather than algebraic geometry. To prove Theorem 1.2 in full generality, in a situation when only a lift over $W_2(k)$ exists, we appeal to Theorem 1.3, which implies it by Corollary 7.5.

To describe the idea, recall the structure of the proof of [DI87]. Given the lift X_1 over $W_2(k)$, they produce a map $s:\Omega^1_{X_0^{(1)}}[-1]\to\mathrm{dR}_{X_0/k}$ in the derived category $D(X_0^{(1)})$ inducing an isomorphism on cohomology sheaves in degree 1. Using that $\mathrm{dR}_{X_0/k}$ can be represented by a cosimplicial commutative algebra in quasi-coherent sheaves on X_0 via the Čech resolution, s gives rise to maps

$$s_i : S^i(\Omega^1_{X_0^{(1)}/k}[-1]) \to \mathrm{dR}_{X_0/k} \tag{1.8}$$

for all $i\geq 0$ where S^i denotes the derived symmetric power functor in the sense of [Ill71]. When $i<p$, the derived symmetric power $S^i(\Omega^1_{X_0^{(1)}/k}[-1])$ is identified with $\Omega^i_{X_0^{(1)}/k}[-i]$, and maps s_i produce a quasi-isomorphism $\bigoplus s_i : \bigoplus\limits_{i=0}^{p-1}\Omega^i_{X_0^{(1)}/k}[-i]\simeq\tau^{\leq p-1}\mathrm{dR}_{X_0/k}$.

This method of decomposing $\mathrm{dR}_{X_0/k}$ stops working for $i=p$, because $S^p(\Omega^1_{X_0^{(1)}/k}[-1])$ is now different from $\Omega^p_{X_0^{(1)}/k}[-p]$. It has non-zero cohomology sheaves in several degrees, and we analyze this object in detail in Section 3. Specifically, there is a natural map $N_p : S^p(\Omega^1_{X_0^{(1)}/k}[-1])\to\Omega^p_{X_0^{(1)}/k}[-p]$ but the map s_p need not factor through it. In Lemma 3.10 we compute the map s_p on the fiber of N_p, and thus relate the obstruction to splitting $\tau^{\leq p}\mathrm{dR}_{X_0/k}$ to the obstruction to the existence of a section of the map N_p, the latter being precisely the class $\alpha(\Omega^1_{X_0})$.

This argument is not specific to the de Rham complex and applies more generally to derived commutative algebras whose cohomology algebra is freely generated by H^1. In the body of the paper we work with derived commutative algebras in the sense of Mathew, but here we state the main algebraic result for the more classical notion of cosimplicial commutative algebras:

Theorem 1.5 (Theorem 4.1)**.** *Let A be a cosimplicial commutative algebra in quasi-coherent sheaves on a flat $\mathbb{Z}_{(p)}$-scheme X. Suppose that $H^0(A)=\mathcal{O}_X$, the sheaf $H^1(A)$ is* *a locally free sheaf of $\mathcal{O}_X$-modules, and the multiplication on the cohomology of A induces an isomorphism $\Lambda^\bullet H^1(A) \simeq H^\bullet(A)$. If there exists a map $s : H^1(A)[-1] \to A$ in $D(X)$ inducing an isomorphism on cohomology in degree 1, then*

(1) $\tau^{\leq p-1}A$ *is quasi-isomorphic to* $\bigoplus_{i=0}^{p-1} H^i(A)[-i]$.

(2) The connecting map $H^p(A) \to (\tau^{\leq p-1}A)[p+1]$ *corresponding to the fiber sequence* $\tau^{\leq p-1}A \to \tau^{\leq p}A \to H^p(A)[-p]$ *can be described as the composition*

(1.9)

$$H^p(A) = \Lambda^p H^1(A) \to \Lambda^p H^1(A)/p \xrightarrow{\alpha(H^1(A)/p)} F^*_{X_0}(H^1(A)/p)[p-1] \xrightarrow{F^*_{X_0}s[p]} (\tau^{\leq 1}F^*_{X_0}(A/p))[p]$$
$$\xrightarrow{\varphi_{A/p}} (\tau^{\leq 1}A/p)[p] \xrightarrow{\tau^{\leq 1}\mathrm{Bock}_A} (\tau^{\leq 1}A)[p+1] \xrightarrow{\oplus} (\tau^{\leq p-1}A)[p+1]$$

Here $\varphi_{A/p}$ is the Frobenius morphism of the cosimplicial commutative algebra A/p, and $\mathrm{Bock}_A : A/p \to A[1]$ is the morphism corresponding to the fiber sequence $A \xrightarrow{p} A \to A/p$ on X.

The fact that the map $s_p : S^p(\Omega^1_{X_0^{(1)}}[-1]) \to \mathrm{dR}_{X_0/k}$ need not factor through $\Omega^p_{X_0^{(1)}}[-p]$ is an incarnation of the classical phenomenon responsible for the existence of non-trivial Steenrod operations on mod p cohomology of topological spaces. If the map $s : \Omega^1_{X_0^{(1)}/k}[-1] \to \mathrm{dR}_{X_0/k}$ in the derived category $D(X_0^{(1)})$ was represented by a genuine map between these complexes, all the maps s_i would have to factor through $\Omega^i_{X_0^{(1)}/k}[-i]$ because of the commutative DG algebra structure on the de Rham complex. However, s can generally only be represented by a chain-level map into a model of $\mathrm{dR}_{X_0/k}$ where multiplication is commutative up to a set of coherent homotopies, rather than commutative on the nose.

Theorem 1.5 draws inspiration from Steenrod's construction [Ste53], [May70] of cohomology operations from homology classes of symmetric groups, and our method of proof implies the following statement about operations on cohomology of cosimplicial commutative $\mathbb{F}_p$-algebras, defined by Priddy [Pri73]. The appearance of Frobenius and Bockstein homomorphisms in the description of the degree 1 Steenrod operation is directly related (Lemma 3.10 being the common reason for both) to their appearance in (1.9).

Proposition 1.6 (Proposition 3.14)**.** *For a cosimplicial commutative $\mathbb{F}_p$-algebra B,*

(1) the degree zero Steenrod operation $P^0 : H^i(B) \to H^i(B)$ is equal to the Frobenius endomorphism φ_B of B,

(2) the degree 1 Steenrod operation $P^1 : H^i(B) \to H^{i+1}(B)$ is equal to Serre's Witt vector Bockstein homomorphism induced by the exact sequence $B \xrightarrow{V} W_2(B) \to B$. In particular, for any cosimplicial commutative flat $\mathbb{Z}/p^2$-algebra $\widetilde{B}$ lifting B, P^1 can be described as the composition $H^i(B) \xrightarrow{\varphi_B} H^i(B) \xrightarrow{\mathrm{Bock}_{\widetilde{B}}} H^{i+1}(B)$.

We apply Theorem 1.5 to the diffracted Hodge complex $A := \Omega^{\not D}_X$ of [BL22a] (it coincides with the Hodge-Tate cohomology of [BS22] over an appropriately chosen prism), and deduce Theorem 1.2 by reducing modulo p, using that $\mathrm{dR}_{X_0/k}$ is the reduction of $\Omega^{\not D}_{X^{(1)}}$. We thus along the way obtain in Theorem 5.9 a similar description of the extension class corresponding to $\tau^{\leq p}\Omega^{\not D}_X$ as well. Theorem 1.5 also applies to a number of cohomology theories evaluated on abelian varieties: in Section 9 we use it to study extensions in the canonical filtration on coherent (Proposition 9.1), de Rham (Proposition 9.3), and étale cohomology (Proposition 9.4) of an abelian scheme being acted on by a group. The results on coherent and de Rham equivariant cohomology of abelian schemes play the key role in our example of a liftable variety with a non-degenerate conjugate spectral sequence.

A related approach to decomposability of the de Rham complex has been previously used by Beuchot [Beu19], who used a description of cohomology sheaves of $(\Omega^1_{X_0^{(1)}}[-1]^{\otimes p})_{hS_p}$ to prove decomposability of the truncation $\tau^{\leq p}\mathrm{dR}_{X_0/k}$ for a liftable variety X_0 under the assumption that certain cohomology groups of X_0 vanish. The consequence of Theorem 1.5 for the de Rham complex was also obtained independently by Robert Burklund, by a similar method.

At the moment it is unclear how to generalize Theorem 1.5, and therefore Theorem 1.2, to describe extensions in degrees $> p$. We make some preliminary observations on this in Section 11.

Computations of the map $S^p(\Omega^1_{X_0^{(1)}}[-1]) \to \mathrm{dR}_{X_0/k}$ that we perform for the proof of Theorem 1.2 also allow us to determine when the de Rham complex can be decomposed compatibly with the algebra structure. The following result is to be contrasted with the formality result of [DGMS75] for smooth projective varieties in characteristic zero:

Proposition 1.7 (Proposition 5.14)**.** *For a smooth variety X_0 over k the de Rham complex $\mathrm{dR}_{X_0/k}$ is quasi-isomorphic to $\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]$ as an E_∞-algebra in $D(X_0^{(1)})$ if and only if X_0 admits a lift over $W_2(k)$ together with its Frobenius endomorphism.*

The 'if' direction was proven in [DI87] and is the starting point of their proof of the decomposition (1.1).

1.3. **Liftable variety with a non-degenerate conjugate spectral sequence.** It is rather non-obvious that the formula (1.4) ever takes non-zero values. Our example for Theorem 1.1 is an approximation (in the sense of [ABM21]) of the classifying stack of a finite flat non-commutative group scheme G over $W(k)$ defined as follows. Let E be an elliptic curve over $W(k)$ whose reduction is supersingular. The group scheme G is

$$G := SL_p(\mathbb{F}_{p^2}) \ltimes (E[p]\otimes_{\mathbb{F}_p}\mathbb{F}_{p^2}^{\oplus p}). \tag{1.10}$$

Here $E[p]\otimes_{\mathbb{F}_p}\mathbb{F}_{p^2}^{\oplus p}$ is the product of $2p$ copies of the p-torsion group scheme $E[p]$, and the discrete group $SL_p(\mathbb{F}_{p^2})$ acts on it via the tautological representation on $\mathbb{F}_{p^2}^{\oplus p}$.

We do not apply formula (1.4) to the stack BG directly, but rather study the conjugate spectral sequence of an auxiliary quotient stack by applying Theorem 1.5 to equivariant Hodge and de Rham cohomology of the abelian scheme $E^{\times d}$ for a certain d, with respect to a certain discrete group.

First, we show that the class $\alpha(-)$ does not vanish in the universal example:

Proposition 1.8 (Proposition 12.1)**.** *Denote by V the tautological p-dimensional representation of the algebraic group $GL_{p,k}$. The class $\alpha(V)\in \mathrm{Ext}^{p-1}_{GL_{p,k}}(\Lambda^p V, V^{(1)})$ is non-zero.*

For the proof, we find an explicit representation for $\alpha(V)$ as a Yoneda extension and apply Kempf's vanishing theorem to compare spectral sequences associated to bête and canonical filtrations on this explicit complex. We also prove an enhancement of this non-vanishing: the class $\alpha(V)$ remains non-zero after being restricted to the discrete group

$GL_p(\mathbb{F}_{p^r})$ whenever $r > 1$ (Proposition 13.1), and that the Bockstein homomorphism applied to $\alpha(V)$ gives a non-zero class in the cohomology of the special linear group $SL_p(\mathcal{O}_F)$ of the ring of integers in an appropriately chosen number field F (Proposition 14.1). These enhancements use the technique of Cline-Parshall-Scott-van der Kallen [CPSvdK77] for comparing the cohomology a reductive group with that of its group of $\mathbb{F}_q$-points.

With these non-vanishing results in hand, we apply Theorem 1.5 to de Rham and coherent cohomology of an abelian scheme being acted on by a group. Our example in Theorem 1.1 stems from the following potential discrepancy between de Rham and coherent cohomology:

Proposition 1.9 (Proposition 10.3)**.** *There exists an abelian scheme A over $W(k)$ with an action of a discrete group Γ such that the truncation $\tau^{\leq p}\mathrm{R}\Gamma(A_0,\mathcal{O})$ of the cohomology of the structure sheaf of the special fiber $A_0 = A\times_{W(k)} k$ is Γ-equivariantly decomposable, while its de Rham cohomology $\tau^{\leq p}\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)$ is not.*

The ultimate reason for different behavior of de Rham and coherent cohomology in Proposition 1.9 is that the Frobenius morphism appearing in (1.9) is zero on $H^1(A_0,\mathcal{O})$, but is non-zero on $H^1_{\mathrm{dR}}(A_0/k)$. We moreover arrange that the stack $[A_0/\Gamma]$ has a non-degenerate conjugate spectral sequence. We then construct a map $[A_0/\Gamma]\to BG$ and deduce that the conjugate spectral sequence of BG does not degenerate either, as desired.

The dependence of the extension class (1.9) on the Frobenius action yields the following criterion for supersingularity of an elliptic curve:

Proposition 1.10 (Corollary 9.2)**.** *Suppose that $n \geq 2p$. For any elliptic curve E_0 over k, the discrete group $GL_n(\mathbb{Z})$ naturally acts on $E_0^{\times n}$ and therefore on $\mathrm{R}\Gamma(E_0^{\times n},\mathcal{O})$. The complex $\tau^{\leq p}\mathrm{R}\Gamma(E_0^{\times n},\mathcal{O})$ is $GL_n(\mathbb{Z})$-equivariantly quasi-isomorphic to the direct sum $\bigoplus\limits_{i=0}^{p} H^i(E_0^{\times n},\mathcal{O})[-i]$ of its cohomology if and only if E_0 is supersingular.*

1.4. **Computation of the Sen operator.** Finally, let us describe the idea of our proof of Theorem 1.3. It might be possible to upgrade Theorem 1.5 to work in the category of sheaves on X equipped with an endomorphism, but we instead evaluate the Sen operator on $\tau^{\leq p}\mathrm{dR}_{X_0/k}$ using the technique of descent to quasiregular semiperfectoid rings developed by [BMS18]. We set up the foundation for our computation in Section 6, defining in particular a version of diffracted Hodge cohomology equipped with a Sen operator relative to the base $\mathbb{Z}/p^n$ for $n\geq 2$ (Theorem 6.10). We then prove formula (1.7) in Theorem 7.1.

For a quasiregular semiperfectoid $W(k)$-algebra S with mod p reduction $S_0 = S/p$ the derived de Rham complex $\mathrm{dR}_{S_0/k}$ is concentrated in degree zero so the Sen operator in question is an endomorphism of a classical module, rather than an object of the derived world. To compute it, we consider the natural map

$$\bigoplus_{i\geq 0} S^i(L\Omega^1_S[-1]) \to \Omega^{\not{D}}_S \tag{1.11}$$

from the symmetric algebra on the shifted cotangent complex of S, comprised of the analogs of the maps (1.8).

Since S is quasiregular semiperfectoid, the map (1.11) is an injection of flat $W(k)$-modules with torsion cokernel. As Θ_S acts on the source of this map via multiplication by $-i$ on the summand $S^i(L\Omega^1_S[-1])$, this allows us to pin down exactly how Θ_S acts on $\Omega^{\not{D}}_S$. A slight variation of this argument works for a semiperfectoid $W_2(k)$-algebra S_1

lifting S_0, even if there is no lift over $W(k)$, as we prove in Lemma 7.2. Though as a result we only compute the Sen operator on dR_{S_0} rather than $\Omega^{\not D}_{S_1/\mathbb{Z}/p^2}$.

Note that, as in the proof of Theorem 1.2, we do not directly use here the construction of the Sen operator via the Cartier-Witt stack, but rather rely crucially on its basic properties: that it is a derivation with respect to the multiplication on $\Omega^{\not D}_X$ and that it acts by scalar multiplication on graded quotients of the conjugate filtration.

Notation. We use the machinery of ∞-categories in the sense of [Lur09]. When referring to an ordinary category we always view it as an ∞-category via the simplicial nerve construction [Lur09, 1.1.2].

For an object $\mathcal{F} \in D(X)$ of the derived category of quasi-coherent sheaves on a prestack X we denote by $H^i(\mathcal{F}) \in \mathrm{QCoh}(X)$ the cohomology sheaf of $\mathcal{F}$ in degree i, and by $H^i(X, \mathcal{F})$ the abelian group of the degree i cohomology of the derived global sections complex $\mathrm{R}\Gamma(X, \mathcal{F})$. By $\tau^{\leq i}\mathcal{F}$ we denote the canonical truncation: it is an object with a map $\tau^{\leq i}\mathcal{F} \to \mathcal{F}$ such that $H^j(\tau^{\leq i}\mathcal{F}) \simeq H^j(\mathcal{F})$ for $j \leq i$, and $H^j(\tau^{\leq i}\mathcal{F}) = 0$ for $j > i$.

When working in a stable ∞-category $\mathcal{C}$ linear over a commutative ring R, for objects $M, N \in \mathcal{C}$ we denote by $\mathrm{RHom}_{\mathcal{C}}(M, N) \in D(R)$ the corresponding object of morphisms. The mapping space $\mathrm{Map}_{\mathcal{C}}(M, N)$ is equivalent to the image of the truncation $\tau^{\leq 0}\mathrm{RHom}_{\mathcal{C}}(M, N)$ under the forgetful functor $D(R) \to \mathrm{Sp}$ to the ∞-category of spaces.

For an algebraic stack X over a ring R we denote by $L\Omega^i_{X/R}$ the ith exterior power of its cotangent complex, viewed as an object of the derived category $D(X)$ of quasicoherent sheaves on X. If $pR = 0$ we denote by $\mathrm{dR}_{X/R}$ the (derived) de Rham complex of X relative to R, viewed as an object of $D(X \times_{R, F_R} R)$, cf. [Bha12, §3]. We sometimes drop the base R from the notation if it is clear from the context, especially if R is perfect.

Acknowledgements. I am very grateful to Vadim Vologodsky for many conversations on the surrounding topics throughout several years, I learned most of the techniques used in this paper from him. I am thankful to Luc Illusie for numerous stimulating conversations, encouragement, and many very helpful comments and corrections on the previous versions of this text.

I am also grateful to Bhargav Bhatt, Lukas Brantner, Robert Burklund, Sanath Devalapurkar, Dmitry Kubrak, Shizhang Li, Shubhodip Mondal, Arthur Ogus, Alexei Skorobogatov, and Bogdan Zavyalov for helpful conversations, suggestions, and explanations. Special thanks to Dmitry and Shizhang for pointing out errors in the previous versions of some of the arguments.

This work was completed during my stay at the Max Planck Institute for Mathematics in Bonn, I am grateful to MPIM for hospitality. This research was partially conducted during the period the author served as a Clay Research Fellow.

2. Preliminaries on homotopical algebra

The key to our approach to the study of the de Rham and Hodge-Tate cohomology is the fact that the de Rham complex and the diffracted Hodge complex admit a structure of a commutative algebra, in the appropriate derived sense. In this expository section we summarize the necessary facts about non-abelian derived functors, derived commutative algebras in the sense of Mathew, and cosimplicial commutative algebras. The specific piece of structure that will be used in Section 4 is a map $S^pA \to A$ where S^p is the derived symmetric power functor in the sense of [Ill71]. Such a map is naturally induced both by the structure of a derived commutative algebra and a cosimplicial commutative algebra structure on A. We choose to work with the former, but comment in Subsection 2.3 on how to translate the results in the language of cosimplicial commutative algebras.

Our exposition here definitely does not introduce any original mathematics but, on the contrary, is aiming to convince the reader that the part of this machinery relevant for us is powered by a fairly classical piece of homolog(top)ical algebra.

2.1. **Non-abelian derived functors.** Let X be an arbitrary prestack, that is a functor $X : \mathrm{Ring}^{\mathrm{op}} \to \mathrm{Sp}$ from the category of ordinary commutative rings to the ∞-category of spaces. We denote by PreStk the ∞-category of prestacks. For a prestack X, we denote by $D(X)$ the derived ∞-category of quasi-coherent sheaves on X defined as the limit $\lim\limits_{\mathrm{Spec}\, R \to X} D(R)$ of derived categories of R modules, cf. [GR17, Chapter I.3]. We work in this large generality simply because all actual proofs in this section take place over affine schemes and the language of prestacks provides a convenient framework for generalizing the result to more general geometric objects via descent.

By an algebraic stacks we will always mean a 1-Artin stack. In all of our applications, X will either be a scheme, a p-adic formal scheme, or a global quotient of a scheme by a group scheme. If X is a quasi-compact scheme with affine diagonal then $D(X)$ is equivalent to the derived category (in the sense of [Lur17, Definition 1.3.5.8]) of the usual abelian category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. If such an X is being acted on by an affine flat group scheme G then $D([X/G])$ is equivalent to the left completion of the derived category of the abelian category $\mathrm{QCoh}_G(X)$ of G-equivariant quasicoherent sheaves on X.

For each $n \geq 0$ we have the derived functors $S^n, \Gamma^n, \Lambda^n : D(X) \to D(X)$ of symmetric, divided, and exterior power functors, respectively. These are the ∞-categorical enhancements for the non-abelian derived functors defined by Illusie [Ill71]. We refer to [BM19, Section 3] and [BGMN21, Section 2.3] for a treatment in the case $X = \mathrm{Spec}\, R$ is an affine scheme, and [KP21, Section A.2] for the general case. We will now briefly recall the construction of these functors and their basic properties. Denote by Mod_R the ordinary abelian category of R-modules. These functors are uniquely characterized by the following properties:

(1) If $X = \mathrm{Spec}\, R$ is affine then for a flat module $M \in \mathrm{Mod}_R \subset D(R)$, the values

$$S^n(M) = (M^{\otimes n})_{S_n}, \Gamma^n(M) = (M^{\otimes n})^{S_n},$$
$$\Lambda^n(M) = (M^{\otimes n})/\langle m_1 \otimes \ldots \otimes m_n | m_i = m_j \text{ for some } i \neq j\rangle$$

are the usual symmetric, divided, and exterior powers.

(2) If X is an affine scheme, the functors S^n, Λ^n, Γ^n preserve sifted colimits, and are polynomial functors of degree $\leq n$ in the sense of [BGMN21, Definition 2.11].

(3) These functors are natural in morphisms of the underlying stacks. That is, the functors S^n, Γ^n, Λ^n can be enhanced to endomorphisms of the functor $D(-) : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends X to $D(X)$ and a morphism $f : X \to Y$ to the pullback functor $f^* : D(Y) \to D(X)$.

These derived functors are first constructed on affine schemes using the following:

Lemma 2.1 ([BGMN21])**.** *For a commutative ring R, the restriction functor*

$$\mathrm{Func}_{\leq n}(D(R), D(R)) \to \mathrm{Func}_{\leq n}(\mathrm{Proj}_R^{\mathrm{f.g}}, D(R)) \tag{2.1}$$

is an equivalence. Here the source of the functor is the category of polynomial of degree $\leq n$ endofunctors of the stable ∞-category $D(R)$ that preserve sifted colimits. The target category is the category of polynomial functors of degree $\leq n$ from the abelian category of finitely generated projective R-modules to the ∞-category $D(R)$.

Proof. Statement (2) is a combination of [BGMN21, Theorem 2.19] applied to $\mathcal{A} = \mathrm{Proj}(R)^{\mathrm{f.g}}$ with the fact that $D(R)$ is the ind-completion of $\mathrm{Stab}(\mathcal{A}) = \mathrm{Perf}(R)$. □

Denote by $\mathrm{Mod} : \mathrm{Ring} \to \mathrm{Cat}$ the functor from the ordinary category of commutative rings to the 2-category of ordinary categories that sends a commutative ring R to the ordinary category Mod_R or R modules, and a morphism of rings $f : R \to R'$ to the functor $R' \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$. Similarly, denote by $\mathrm{Proj}^{\mathrm{f.g.}}$ the subfunctor of Mod that sends a ring R to the category $\mathrm{Proj}^{\mathrm{f.g.}}_R$ of finitely generated projective R-modules.

We denote by $\mathrm{Mor}_{\leq n}(\mathrm{Proj}^{\mathrm{f.g.}}, \mathrm{Mod}) \subset \mathrm{Mor}(\mathrm{Proj}^{\mathrm{f.g.}}, \mathrm{Mod})$ the full subcategory of the ordinary 1-category of natural transformations $\mathrm{Proj}^{\mathrm{f.g.}} \to \mathrm{Mod}$ spanned by transformations that are given by polynomial functors $\mathrm{Proj}^{\mathrm{f.g.}}_R \to \mathrm{Mod}_R$ of degree $\leq n$, for every ring R. We have the following construction principle for polynomial functors on arbitrary prestacks

Lemma 2.2. *For every prestack X there is a functor*

$$\Sigma_X : \mathrm{Mor}_{\leq n}(\mathrm{Proj}^{\mathrm{f.g.}}, \mathrm{Mod}) \to \mathrm{Func}(D(X), D(X)) \tag{2.2}$$

satisfying the following property:

For any morphism $f : \operatorname{Spec} R \to X$ from an affine scheme, and a polynomial functor $T \in \mathrm{Mor}_{\leq n}(\mathrm{Proj}^{\mathrm{f.g.}}, \mathrm{Mod})$ there is an equivalence of functors $f^* \circ \Sigma_X(T) \simeq T_R^{\mathrm{derived}} \circ f^*$ from $D(X)$ to $D(R)$, where T_R^{derived} is the result of applying the inverse of the equivalence (2.1) to $T_R : \mathrm{Proj}^{\mathrm{f.g.}}_R \to \mathrm{Mod}_R$.

Moreover, if X is an algebraic stack flat over a ring A, then Σ_X factors through the category $\mathrm{Mor}_{\leq n}(\mathrm{Proj}^{\mathrm{f.g.}}|_{A-flat}, \mathrm{Mod}|_{A-flat})$ of morphisms between these functors restricted to the category of flat A-algebras.

Proof. For a general X the category $D(X)$ is, by definition, equivalent to $\lim\limits_{\operatorname{Spec} R \to X} D(R)$, where the limit is taken over all affine schemes mapping to X. Therefore $\mathrm{Func}(D(X), D(X)) \simeq \lim\limits_{\operatorname{Spec} R \to X} \mathrm{Func}(D(X), D(R))$ and we define $\Sigma_X(T)$ as the object of this limit given by $T_R^{\mathrm{derived}} \circ f^* \in \mathrm{Func}(D(X), D(R))$ for every map $f : \operatorname{Spec} R \to X$.

The last assertion follows from the fact that if X is a flat algebraic stack over A, then in the above limit we may restrict to morphisms $\operatorname{Spec} R \to X$ for which R is flat over A, by [GR17, Proposition 3.1.4.2]. □

Definition 2.3. For a prestack X, we define functors $S^n, \Gamma^n, \Lambda^n : D(X) \to D(X)$ as images under Σ_X of the corresponding polynomial functors on projective modules over rings. We will denote these functors by $S^n_X, \Gamma^n_X, \Lambda^n_X$ if the base is not clear from the context.

Remark 2.4. We can give an explicit recipe for computing the derived functors S^n, Γ^n, Λ^n that we state in a special case, for simplicity. Suppose that X is a quasi-compact separated scheme and $M \in D^-(X)$ is an object with cohomology bounded from above, that can be represented by a complex $M^0 \to M^1 \to \dots$ concentrated in degrees ≥ 0

of flat quasicoherent sheaves. Then $S^n(M)$ (and likewise for the other functors Λ^n, Γ^n) is the totalization of the cosimplicial diagram

$$S^n(\mathrm{DK}(M)^0) \rightrightarrows S^n(\mathrm{DK}(M)^1) \ldots \tag{2.3}$$

where $\mathrm{DK}(M)$ is the cosimplicial sheaf associated to the complex M by the Dold-Kan equivalence. The miracle is that the resulting object $S^n(M)$ does not depend on the choice of a resolution, up to a quasi-isomorphism.

We will use Lemma 2.2 frequently for homotopy-coherent computations with polynomial functors. Note that specifying an object of $\mathrm{Mor}_{\leq n}(\mathrm{Proj}^{\mathrm{f.g.}}, \mathrm{Mod})$ amounts to a manageable collection of data: we need to give a functor $T_R : \mathrm{Proj}^{\mathrm{f.g.}}_R \to \mathrm{Mod}_R$ for every ring R, construct equivalences $R' \otimes_R T_R(-) \simeq T_{R'}(R' \otimes_R -)$ for all maps of rings $R \to R'$, and then check (rather than provide any additional constructions) that these equivalences are associative with respect to the composition of maps of rings. Let us illustrate this technique by constructing certain maps between the symmetric and divided power functors.

For a projective module M over a ring R there are natural maps between $S^nM = (M^{\otimes n})_{S_n}$ and $\Gamma^n M = (M^{\otimes n})^{S_n}$:

$$r_n : \Gamma^n M \hookrightarrow M^{\otimes n} \twoheadrightarrow S^n M \qquad N_n = \sum_{\sigma \in S_n} \sigma : S^n M \to \Gamma^n M \tag{2.4}$$

The compositions $N_n \circ r_n$ and $r_n \circ N_n$ are equal to $n! \cdot \mathrm{Id}_{\Gamma^n M}$ and $n! \cdot \mathrm{Id}_{S^n M}$, respectively. Since maps r_n and N_n are compatible with base change along arbitrary maps of rings $R \to R'$, they define morphism between functors S^n and Γ^n in the category $\mathrm{Mor}_{\leq n}(\mathrm{Proj}^{\mathrm{f.g.}}, \mathrm{Mod})$. Therefore Lemma 2.2 provides us with maps

$$r_n : \Gamma^n \to S^n \qquad N_n : S^n \to \Gamma^n \tag{2.5}$$

of endofucntors of the category $D(X)$ for every prestack X.

Lemma 2.5. *For an $\mathbb{F}_p$-algebra R_0 and a projective R_0-module M the map $r_p : \Gamma^p M \to S^p M$ naturally factors as*

$$\Gamma^p M \xrightarrow{\psi_M} F^*_{R_0} M \xrightarrow{\Delta_M} S^p M \tag{2.6}$$

where the first arrow is a surjection and the second one is an injection.

Proof. We start by constructing the map $\Delta_M : M \otimes_{R_0, F_{R_0}} R_0 = F^*_{R_0} M \to S^p M$. Define $\Delta_M(m \otimes r) = r \cdot m^p$, this is a well-defined additive R_0-linear map. To check that r_p factors through Δ_M we may work locally on $\mathrm{Spec}\, R_0$ and thus assume that M is a free R_0-module. Denote a basis for M by $e_1, \ldots, e_n$. Then a basis for the module $\Gamma^p M \subset M^{\otimes p}$ is given by the elements $e_I := \sum\limits_{\sigma \in S_p / \mathrm{Stab}_I} e_{i_1} \otimes \ldots \otimes e_{i_p}$ where $I = (i_1, \ldots, i_p)$ runs through S_p-orbits on $\{1, \ldots, n\}^{\times p}$. The map r_p sends e_I to $[S_p : \mathrm{Stab}_I] \cdot e_{i_1} \cdot \ldots \cdot e_{i_p}$. If the stabilizer of I inside S_p has index coprime to p, then Stab_I contains a long cycle of length p which forces all $i_1, \ldots, i_p$ to be equal. Hence only basis elements of the form $e_i^{\otimes p} \in \Gamma^p M$ are not killed by r_p, which gives the desired factorization. $\square$

By Lemma 2.2, having constructed maps ψ_M, Δ_M for projective modules in a functorial fashion, we get maps

$$\Delta_M : F^*_{X_0} M \to S^p M \qquad \psi_M : \Gamma^p M \to F^*_{X_0} M \tag{2.7}$$

for all objects $M \in D(X_0)$ for an arbitrary $\mathbb{F}_p$-prestack X_0. The composition $\Delta_M \circ \psi_M$ is naturally homotopic to $r_p : \Gamma^p M \to S^p M$, by construction.

A useful computation tool for us will be the following 'décalage' isomorphisms:

Lemma 2.6 ([Ill71, Proposition 4.3.2.1], [KP21, Proposition A.2.49])**.** *There are natural equivalences for $M \in D(X)$*

$$S^n(M[1]) \simeq (\Lambda^n M)[n] \quad \Gamma^n(M[-1]) \simeq (\Lambda^n M)[-n] \tag{2.8}$$

2.2. **Derived commutative algebras.** The precise version of the notion of a commutative algebra that we will need is derived commutative algebras, in the sense of Mathew. In this section we collect the necessary material about this notion, largely following [Rak20, Section 4]. Let us stress again that this section is purely expository and contains no original results.

Recall the following point of view on ordinary commutative algebras over a field k, afforded by the Barr-Beck theorem. The endofunctor $S^\bullet = \bigoplus_{n\geq 0} S^n$ of the category of k-vector spaces has a structure of a monad, and the category of k-algebras is equivalent to the category of modules over the monad $S^\bullet$ in the category of k-vector spaces. The idea of derived commutative algebras is to replicate this definition in the derived world by using the appropriate derived version of the symmetric algebra monad.

Construction 2.7 ([Rak20, Construction 4.2.19])**.** For a ring R consider the endofunctor $S^\bullet := \bigoplus_{n\geq 0} S^n : \mathrm{Proj}_R \to \mathrm{Proj}_R$ of the ordinary category of projective R-modules. It has a monad structure induced by the maps $S^i(S^j M) \twoheadrightarrow S^{i\cdot j} M$ for every projective module M. It naturally extends to a monad, in the sense of [Lur17, Definition 4.7.0.1], on the category $D(X)$ for every prestack X which we denote as

$$S^\bullet := \bigoplus_{n\geq 0} S^n : D(X) \to D(X) \tag{2.9}$$

and call it the derived symmetric algebra monad.

We can now define derived commutative algebras on X:

Definition 2.8. The category $\mathrm{DAlg}(X)$ of derived commutative algebras on X is the category of modules over the derived symmetric algebra monad $S^\bullet$ in $D(X)$.

By construction, there is a functor $S^\bullet : D(X) \to \mathrm{DAlg}(X)$ left adjoint to the forgetful functor $\mathrm{DAlg}(X) \to D(X)$. For a derived commutative algebra $A \in \mathrm{DAlg}(X)$ we will usually denote the underlying object in $D(X)$ by the same symbol A. The structure of a derived commutative algebra, in particular, gives a map

$$m : A \otimes_{\mathcal{O}_X} A \to S^2 A \to A \tag{2.10}$$

which induces a graded commutative product operation $H^i(A) \otimes_{\mathcal{O}_X} H^j(A) \to H^{i+j}(A)$ on cohomology sheaves.

Definition 2.9. For an object M, we call $S^\bullet(M) \in \mathrm{DAlg}(X)$ the free derived commutative algebra on M.

Remark 2.10. For every $M \in D(X)$ there are natural maps $(M^{\otimes n})_{hS_n} \to S^n(M)$ which give rise to a map from the monad $M \mapsto \bigoplus_{n\geq 0}(M^{\otimes n})_{hS_n}$ to $S^\bullet$. It is in general far from equivalence, we make a precise comparison between their values in a special case in Lemma 3.8. Since modules over the the former monad are the E_∞-algebras on X, we get a functor $\mathrm{DAlg}(X) \to \mathrm{Alg}_{E_\infty} D(X)$.

Derived commutative algebras in characteristic p have natural Frobenius endomorphisms:

Lemma 2.11. *Suppose that X is a prestack over $\mathbb{F}_p$. For all $A \in \mathrm{DAlg}(X)$ there is a natural morphism $\varphi_A : F_X^* A \to A$ in $D(X)$ that is equal to the linearization of the usual Frobenius endomorphism when A is a flat ordinary commutative algebra on an affine scheme X.*

Remark 2.12. It should be possible to enhance φ_A to a morphism of derived commutative algebras, but we do not pursue this here.

Proof. We define φ_A as the composition $F_X^* A \xrightarrow{\Delta_A} S^p A \xrightarrow{m_A} A$ where m_A is a part of the $S^\bullet$-module structure on A, and Δ_A is the morphism defined in (2.7). Compatibility with the usual Frobenius follows from the defining formula of Δ_A given in the proof of Lemma 2.5. □

One of the main motivations for introducing the notion of derived commutative algebras is that many cohomological invariants arising in geometry are naturally equipped with the structure of a derived commutative algebra.

Proposition 2.13. *If $f : X \to Y$ is a morphism of prestacks then the functor $Rf_* : D(X) \to D(Y)$ can be naturally enhanced to a functor $Rf_*^{\mathrm{alg}} : \mathrm{DAlg}(X) \to \mathrm{DAlg}(Y)$.*

Proof. Recall that $Rf_* : D(X) \to D(Y)$ is defined as the right adjoint functor to the pullback functor $f^* : D(Y) \to D(X)$. The equivalences $f^* \circ S_Y^n \simeq S_X^n \circ f^*$ induce a colimit-preserving functor $f^*_{\mathrm{alg}} : \mathrm{DAlg}(Y) \to \mathrm{DAlg}(X)$ given by f^* on the underlying objects of $D(-)$. Since $\mathrm{DAlg}(Y)$ is presentable (e.g. by [Rak20, Proposition 4.1.10]), this functor admits a right adjoint $Rf_*^{\mathrm{alg}} : \mathrm{DAlg}(X) \to \mathrm{DAlg}(Y)$, by the adjoint functor theorem [Lur09, Corollary 5.5.2.9(1)].

It remains to check that Rf_*^{alg} defined this way induces Rf_* on the underlying sheaves. By construction, the compositions $D(Y) \xrightarrow{f^*} D(X) \xrightarrow{S_X^\bullet} \mathrm{DAlg}(X)$ and $D(Y) \xrightarrow{S_Y^\bullet} \mathrm{DAlg}(Y) \xrightarrow{f^*_{\mathrm{alg}}} \mathrm{DAlg}(X)$ are equivalent. Their right adjoints are $\mathrm{DAlg}(X) \to D(X) \xrightarrow{Rf_*} D(Y)$ and $\mathrm{DAlg}(X) \xrightarrow{Rf_*^{\mathrm{alg}}} \mathrm{DAlg}(Y) \to D(Y)$, and their equivalence is precisely the desired compatibility between Rf_* and Rf_*^{alg}. □

In particular, $Rf_*\mathcal{O}_X \in D(Y)$ is naturally equipped with a derived commutative algebra structure. The functor Rf_*^{alg} is compatible with Frobenius endomorphisms:

Lemma 2.14. *Suppose that Y is a prestack over $\mathbb{F}_p$ and $f : X \to Y$ is a morphism of prestacks. For $A \in \mathrm{DAlg}(X)$ there is a natural equivalence between the composition $F_Y^* Rf_*A \to Rf_*F_X^*A \xrightarrow{Rf_*\varphi_A} Rf_*A$ and the map $F_Y^*Rf_*A \xrightarrow{\varphi_{Rf_*^{\mathrm{alg}}A}} Rf_*A$ in $D(Y)$. Here $F_Y^*Rf_*A \to Rf_*F_X^*A$ is the base change map corresponding to the equality $F_Y \circ f = f \circ F_X$.*

Proof. By adjunction between f^* and Rf_*, and the definition of φ_A and $\varphi_{Rf_*^{\mathrm{alg}}A}$, our task is equivalent to identifying the composition

$$f^*F_Y^*Rf_*A \to f^*Rf_*F_X^*A \xrightarrow{f^*Rf_*\Delta_A} f^*Rf_*S_X^pA \xrightarrow{f^*Rf_*m_A} f^*Rf_*A \to A \tag{2.11}$$

with the composition

$$f^*F_Y^*Rf_*A \xrightarrow{f^*\Delta_{Rf_*A}} f^*S_Y^pRf_*A \xrightarrow{f^*m_{Rf_*^{\mathrm{alg}}A}} f^*Rf_*A \to A \tag{2.12}$$

The map $S_Y^pRf_*A \xrightarrow{m_{Rf_*^{\mathrm{alg}}A}} Rf_*A$ is adjoint to the map $f^*S_Y^pRf_*A \simeq S_X^pf^*Rf_*A \to S_X^pA \xrightarrow{m_A} A$, which allows us to rewrite (2.12) as

$$f^*F_Y^*Rf_*A \to f^*S_Y^pRf_*A \simeq S_X^pf^*Rf_*A \to S_X^pA \xrightarrow{m_A} A \tag{2.13}$$

For any object $M \in D(Y)$ the map $f^*F_Y^*M \xrightarrow{f^*\Delta_M} f^*S_Y^pM \simeq S_X^pf^*M$ can be identified with the composition $f^*F_Y^*M \simeq F_X^*f^*M \xrightarrow{\Delta_{f^*M}} S_X^pf^*M$ where the first equivalence arises from the fact that f intertwines the Frobenius endomorphisms of X and Y. Applying this to $M = Rf_*A$ allows us to identify (2.11) with (2.13), as desired. □

We can use Lemma 2.13 to construct examples of derived commutative algebras:

Definition 2.15. For a finite locally free sheaf M (concentrated in degree 0) on a scheme X we define the *free divided power algebra* on $M[-1]$ denote by $\Gamma^\bullet(M[-1]) \in \mathrm{DAlg}(X)$ as $R\pi_*^{\mathrm{alg}}\mathcal{O}_{B_XM^{\vee\#}}$ where $\pi : B_XM^{\vee\#} \to X$ is the relative classifying stack of the divided power group scheme $M^{\vee\#}$ on X associated to $M^\vee$.

This terminology is justified by the fact that the underlying E_∞-algebra of $\Gamma^\bullet(M[-1])$ is identified with $\bigoplus\limits_{i\geq 0}\Lambda^iM[-i]$, e.g. by [BL22b, Lemma 7.8]. If X is an $\mathbb{F}_p$-scheme, by Lemma 2.14 the Frobenius map $\varphi^*_{\Gamma^\bullet(M[-1])} : F_X^*\Gamma^\bullet(M[-1]) \to \Gamma^\bullet(M[-1])$ factors as $F_X^*\Gamma^\bullet(M[-1]) \to \mathcal{O}_X \to \Gamma^\bullet(M[-1])$ because the Frobenius endomorphism of $M^{\vee\#}$ factors through the identity section.

Cohomology of a sheaf of rings on a site can be equipped with a derived commutative algebra structure:

Lemma 2.16. *Let $\mathcal{C}$ be a site and $\mathcal{F}$ be a sheaf of (ordinary) commutative algebras over a ring R on $\mathcal{C}$. Then for any object $X \in \mathcal{C}$ the complex $\mathrm{R}\Gamma(X, \mathcal{F}) \in D(R)$ is naturally endowed with a structure of a derived commutative R-algebra.*

Proof. By [Sta23, 01GZ] we can compute $\mathrm{R}\Gamma(X, \mathcal{F})$ as a filtered colimit over all hypercovers $U_\bullet \to X$ of Čech cohomology with respect to $U_\bullet$:

$$\mathrm{R}\Gamma(X, \mathcal{F}) \simeq \operatorname*{colim}_{U_\bullet \to X} \lim_n \mathcal{F}(U_n) \tag{2.14}$$

Each $\mathcal{F}(U_n)$ is a commutative R-algebra, which we view as an object of $\mathrm{DAlg}(R)$. Since the forgetful functor $\mathrm{DAlg}(R) \to D(R)$ commutes with limits and filtered colimits, this endows $\mathrm{R}\Gamma(X, \mathcal{F})$ with the structure of a derived commutative R-algebra. □

Applying this construction to étale cohomology with coefficients in $\mathbb{F}_p$ produces derived commutative algebras with the special property that the Frobenius endomorphism is homotopic to identity:

Lemma 2.17. *If X is a scheme then the Frobenius endomorphism $\varphi_{\mathrm{R}\Gamma_{et}(X,\mathbb{F}_p)}$ of the derived commutative $\mathbb{F}_p$-algebra $\mathrm{R}\Gamma_{et}(X,\mathbb{F}_p)$ is naturally homotopic to the identity morphism.*

Proof. In the formula (2.14) each $\mathbb{F}_p(U_n) = \mathrm{Func}(\pi_0(U_n), \mathbb{F}_p)$ is the algebra of $\mathbb{F}_p$-valued functions on a set, and its Frobenius endomorphism is the identity, so the lemma follows. □

When studying the Sen operator, we will use that it is compatible with the derived commutative algebra structure on the diffracted Hodge cohomology (to be defined in Section 5). Specifically, we need the following notion

Definition 2.18. For a prestack X and a derived commutative algebra $A \in \mathrm{DAlg}(X)$, a *derivation* $f : A \to A$ is a map in $D(X)$ such that the map $\mathrm{Id}_A + \varepsilon \cdot f : A \otimes \mathbb{Z}[\varepsilon]/\varepsilon^2 \to A \otimes \mathbb{Z}[\varepsilon]/\varepsilon^2$ in $D(X \times \mathbb{Z}[\varepsilon]/\varepsilon^2)$ is equipped with the structure of a map in $\mathrm{DAlg}(X \times \mathbb{Z}[\varepsilon]/\varepsilon^2)$.

2.3. **Cosimplicial commutative algebras.** For the duration of this subsection assume that X is a scheme. In all of our main applications we will in fact be presented with a cosimplicial commutative algebra in the ordinary category $\mathrm{QCoh}(X)$. We denote by CAlg_X^{Δ} the ordinary category of cosimplicial commutative algebras in the abelian symmetric monoidal category $\mathrm{QCoh}(X)$. In this subsection we make some remarks on the relation between CAlg_X^{Δ} and $\mathrm{DAlg}(X)$. These facts are not used in any of our main results, but the reader who feels more comfortable with CAlg_X^{Δ} than with $\mathrm{DAlg}(X)$ is encouraged to specialize the results in Section 4 to a situation where the algebra A is a cosimplicial commutative algbera, using this subsection as a dictionary.

In general, a cosimplicial commutative algebra gives rise to a derived commutative algebra:

Lemma 2.19. *Let X be a scheme. There is a natural functor $\mathcal{R}_X : \mathrm{CAlg}^{\Delta}(X) \to \mathrm{DAlg}(X)$ from the ordinary category of cosimplicial commutative algebras in quasi-coherent sheaves on X to the category of derived commutative algebras on X. This functor is compatible with the cosimplicial totalization functor on the level of underlying complexes.*

Remark 2.20. For all of our main results we will work with derived commutative algebras concentrated in degrees ≥ 0. In a forthcoming work, Mathew and Mondal will show that for any commutative base ring R the ∞-category of derived commutative R-algebras concentrated in degrees ≥ 0 is equivalent to the ∞-category of cosimplicial commutative R-algebras, cf. [MR23, Remark 2.1.8].

Proof of Lemma 2.19. First of all, there is a functor from ordinary commutative algebras in $\mathrm{QCoh}(X)$ to $\mathrm{DAlg}(X)$. Indeed, for an ordinary algebra A there is a natural map $S^n A \to (A^{\otimes_{\mathcal{O}_X} n})_{S_n}$ where the tensor product and coinvariants are taken in the non-derived sense. Hence the commutative multiplication on $A \in \mathrm{QCoh}(X)$ endows it with a structure of a monad over $S^\bullet$ in $D(X)$.

Now, if $A = \left[\ A^0 \rightrightarrows A^1 \dots\ \right]$ is a cosimplicial commutative algebra in $\mathrm{QCoh}(X)$, we define $\mathcal{R}_X(A)$ as $\lim\limits_{[n]\in\Delta} \mathcal{R}_X(A^n)$, where each $\mathcal{R}_X(A^n)$ was defined in the previous paragraph. This indeed induces the totalization on underlying sheaves because the forgetful functor $\mathrm{DAlg}(X) \to D(X)$ commutes with limits. □

Example 2.21. If $f : X \to \operatorname{Spec} R$ is a morphism from a separated scheme to the spectrum of a ring R, the object $\mathrm{R}\Gamma(X, \mathcal{O}_X) = f_*(\mathcal{O}_X) \in D(R)$ is endowed with a structure of a cosimplicial commutative algebra using the Čech construction associated to a cover $X = \bigcup\limits_i U_i$:

$$\mathrm{R}\Gamma(X, \mathcal{O}_X) \simeq \left[\ \prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i<j} \mathcal{O}(U_i \cap U_j) \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \dots\ \right] \tag{2.15}$$

The result of applying $\mathcal{R}_R$ to this cosimplicial algebra is naturally equivalent to $f_*^{\mathrm{alg}}\mathcal{O}_X$ from Lemma 2.13.

If X is a scheme over $\mathbb{F}_p$, for every cosimplicial commutative algebra $A \in \mathrm{CAlg}_X^\Delta$ there is a natural map $\varphi_A^{\mathrm{cosimp}} : F_X^* A \to A$ of algebras induced by the term-wise Frobenius endomorphism. It follows from the construction of the functor $\mathcal{R}_X$ that $\varphi_A^{\mathrm{cosimp}}$ coincides with the Frobenius map arising from the derived commutative algebra structure:

Lemma 2.22. *For an $\mathbb{F}_p$-scheme X, and a term-wise flat cosimplicial commutative algebra $A \in \mathrm{CAlg}_X^\Delta$ there is a natural homotopy between $\mathcal{R}_X(\varphi_A^{\mathrm{cosimp}})$ and φ_A in $D(X)$.*

Proof. If A is an ordinary commutative algebra, this was established along with defining φ_A in Lemma 2.11. The case of an arbitrary A is obtained by passing to the limit over the cosimplicial category Δ. □

For a locally free sheaf M on a scheme X we can represent the free divided power algebra $\Gamma^\bullet(M[-1]) \in \mathrm{DAlg}(X)$ of Definition 2.15 by a cosimplicial commutative algebra. Denote by $\mathrm{DK}(M[-1])$ the cosimplicial object in the category of locally free sheaves on X, obtained by applying the Dold-Kan correspondence to the complex $M[-1]$.

Lemma 2.23. *The derived commutative algebra $\Gamma^\bullet(M[-1])$ is equivalent to $\mathcal{R}_X(\Gamma^\bullet_{\mathrm{naive}}(\mathrm{DK}(M[-1])))$ where $\Gamma^\bullet_{\mathrm{naive}}(\mathrm{DK}(M[-1]))$ is the result of applying term-wise the free divided power algebra functor to the cosimplicial sheaf $\mathrm{DK}(M[-1])$.*

Proof. The derived pushforward of the structure sheaf along the map $B_X M^{\vee\#} \to X$ is equivalent, as a derived commutative algebra, to the totalization of the cosimplicial diagram

$$\mathcal{O}_X \rightrightarrows \pi_*\mathcal{O}_{M^{\vee\#}} \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \pi_*\mathcal{O}_{M^{\vee\#}\times_X M^{\vee\#}} \dots \tag{2.16}$$

obtained by applying the functor $\pi_*\mathcal{O}_{(-)}$ to the bar-resolution associated to the group scheme $\pi : M^{\vee\#} \to X$ over X. The nth term of the diagram (2.16) is the commutative algebra $\pi_*\mathcal{O}_{M^{\vee\#\times_X n}} \simeq \Gamma_X^\bullet(M^{\oplus n})$ concentrated in degree 0, so the cosimplicial commutative algebra defined by (2.16) is indeed equivalent to the result of applying $\Gamma^\bullet_{\mathrm{naive}}$ to $\mathrm{DK}(M[-1])$. □

2.4. **Bockstein morphisms.** For a stable $\mathbb{Z}$-linear ∞-category $\mathcal{C}$, any object $M \in \mathcal{C}$ gives rise to a natural fiber sequence

$$M \xrightarrow{p} M \to M/p \tag{2.17}$$

We will denote the corresponding connecting map by $\mathrm{Bock}_M : M/p \to M[1]$ and refer to it as the *Bockstein morphism* corresponding to M. Note that $\mathrm{Bock}_{M[1]}$ is naturally homotopic to $(-\mathrm{Bock}_M[1])$.

Similarly, for a $\mathbb{Z}/p^n$-linear stable ∞-category $\mathcal{C}$ for any object $M \in \mathcal{C}$ we have a fiber sequence

$$M \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^{n-1} \to M \to M \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p \tag{2.18}$$

inducing the connecting map $\mathrm{Bock}_M : M \otimes \mathbb{Z}/p \to M \otimes \mathbb{Z}/p^{n-1}[1]$. These constructions over $\mathbb{Z}$ and $\mathbb{Z}/p^n$ are compatible in the sense that Bock_{M/p^n} is the composition $M/p \xrightarrow{\mathrm{Bock}_M} M[1] \to M/p^{n-1}[1]$.

3. Symmetric power S^p, class α, and Steenrod operations

In this section we study in detail the derived functor S^p of pth symmetric power by comparing it with the divided power functor Γ^p. For complexes of vector spaces over a field k the values of S^n can be described non-canonically using the computations of $S^n(k[-i])$ done by Priddy [Pri73], but we crucially need to understand S^pM as a complex of sheaves, rather that its separate cohomology sheaves.

3.1. **Symmetric powers vs. divided powers.** Let X be an arbitrary prestack. Recall that in the previous section for an object $M \in D(X)$ we defined natural morphisms

$$N_n : S^nM \to \Gamma^nM \quad r_n : \Gamma^nM \to S^nM \tag{3.1}$$

Denote by $T_n(M)$ the cofiber $\mathrm{cofib}(N_n : S^nM \to \Gamma^nM)$ of the norm map. The functor T_n, especially for $n = p$, has been extensively studied in the literature. References close to our point of view are works of Friedlander-Suslin [FS97, Section 4], and Kaledin [Kal18, 6.3]. We have the following classical results about T_n:

Lemma 3.1 ([FS97, Lemma 4.12], [Kal18, Lemma 6.9])**.** *(1) If X is a prestack over $\mathbb{Z}[\frac{1}{n!}]$ then $T_n(M) \simeq 0$ for every $M \in D(X)$.*

(2) If R is a flat $\mathbb{Z}_p$-algebra and M is a flat R-module, then $N_p : S^pM \to \Gamma^pM$ is an injection and its cokernel $T_p(M)$ is naturally isomorphic to $F^*_{R/p}(M/p)$ as an R-module.

(3) If X is a flat algebraic stack over $\mathbb{Z}_p$ then there is a natural equivalence $T_p(M) \simeq i_*F^*_{X_0}i^*M$, where $i : X_0 = X \times_{\mathbb{Z}_p} \mathbb{F}_p \to X$ is the closed immersion of the special fiber, and $F_{X_0} : X_0 \to X_0$ is the absolute Frobenius morphism.

(4) If X_0 is a prestack over $\mathbb{F}_p$ then $T_p(M_0)$ fits into a natural fiber sequence

$$F^*_{X_0}M_0[1] \to T_p(M_0) \to F^*_{X_0}M_0. \tag{3.2}$$

Remark 3.2. (1) In the setting of (3), if $M_0 \simeq i^*M \in D(X_0)$ is the reduction of an object $M \in D(X)$ then $T_p(M_0)$ can be naturally (in M) described as

$$T_p(M_0) \simeq i^*T_p(M) \simeq i^*i_*F^*_{X_0}M_0 \tag{3.3}$$

Under this identification, the fiber sequence in (3.2) is identified with the sequence induced by the fiber sequence of functors $\mathrm{Id}[1]\to i^*i_*\to \mathrm{Id}$ from $D(X_0)$ to $D(X_0)$.

(2) This will not be used in any of the proofs but let us remark that one can describe the extension (3.2) completely, even in the absence of a lift of X_0 to $\mathbb{Z}_p$ together with the object M. It is proven in [PVV18, Theorem 6] that in the setting of (4), at least if X is a scheme, $T_p(M)$ can be upgraded to an object of the derived category of crystals of quasi-coherent crystals of $\mathcal{O}$-modules $D(\mathrm{Cris}(X_0/\mathbb{F}_p))$ on the scheme X_0 such that (3.2) is an extension of crystals where $F^*_{X_0}M$ is endowed with a crystal structure using the canonical connection. Moreover, homotopy classes of splittings of (3.2) in $D(\mathrm{Cris}(X_0/\mathbb{F}_p))$ are in bijection with lifts of $F^*_{X_0}M$ to an object in $D(\mathrm{Cris}(X_0/(\mathbb{Z}/p^2)))$. If X_0 is equipped with a lift X_1 over $\mathbb{Z}/p^2$ then splittings of (3.2) in $D(X_0)$ are in bijection with lifts of $F^*_{X_0}M$ to an object of $D(X_1)$.

Proof. 1) We have $r_n\circ N_n = n!\cdot\ \mathrm{Id}_{S^nM}$, $N_n\circ r_n = n!\cdot \mathrm{Id}_{\Gamma^nM}$. Therefore N_n is an isomorphism if $n!$ is invertible, so T^n vanishes on prestacks over $\mathbb{Z}[\frac{1}{n!}]$.

2) By the previous part, the map $N_p : S^pM\to\Gamma^pM$ becomes an isomorphism after inverting p. Since the module S^pM is p-torsion free, the map N_p is injective.

We will now construct a natural isomorphism α between the module $F^*_{R/p}(M/p) = M/p\otimes_{R/p,F_{R/p}} R/p$ and the cokernel of N_p. To an element $m\otimes r\in M/p\otimes_{R/p,F_{R/p}} R/p$ assign the element $\alpha(m\otimes r):=\widetilde{r}\cdot\widetilde{m}^{\otimes p}\in\operatorname{coker} N_p$ where $\widetilde{r}\in R$ and $\widetilde{m}\in M$ are arbitrary lifts of r and m, respectively.

To see that α gives a well-defined map $F^*_{R/p}(M/p)\to\operatorname{coker} N_p$ we need to check that $\widetilde{r}\cdot\widetilde{m}^{\otimes p}\in\operatorname{coker} N_p$ does not depend on the choices of the lifts $\widetilde{r},\widetilde{m}$ and that $(\widetilde{m}_1+\widetilde{m}_2)^{\otimes p} = \widetilde{m}_1^{\otimes p}+\widetilde{m}_2^{\otimes p}\in\operatorname{coker} N_p$. The first claim follows from the fact that $p\cdot\Gamma^p_R(M)\subset\operatorname{Im} N_p$, and the additivity is demonstrated by the formula

$$(\widetilde{m}_1+\widetilde{m}_2)^{\otimes p}-\widetilde{m}_1^{\otimes p}-\widetilde{m}_2^{\otimes p} = N_p\left(\sum_{i=1}^{p-1}\frac{1}{i!(p-i)!}\widetilde{m}_1^i\widetilde{m}_2^{p-i}\right)\tag{3.4}$$

Finally, to show that $\alpha : F^*_{R/p}(M/p)\to\operatorname{coker}(N_p)$ is an isomorphism, we may assume that M is a free R-module, because both functors $M\mapsto\operatorname{coker}(N_p : S^pM\to\Gamma^pM)$ and $M\mapsto F^*_{R/p}(M/p)$ commute with filtered colimits and every flat module can be represented as a filtered colimit of free modules. Let $\{e_i\}_{i\in I}$ be a basis of M over R. Then $\{N_p(e_{i_1}\dots e_{i_p})\}_{(i_1,\dots,i_p)\in I^p\setminus I}\cup\{e_i^{\otimes p}\}_{i\in I}$ is an R-basis for $\Gamma^p_R(M)$ so $\{e_i^{\otimes p}\}_{i\in I}$ is an R/p-basis for $\operatorname{coker} N_p$, as desired.

3) Part (2) produced a natural short exact sequence $S^pM\to\Gamma^pM\to F^*_{R/p}(M/p)$ of R-modules for every projective module M over a flat $\mathbb{Z}_p$-algebra R. The functor $M\mapsto F^*_{R/p}(M/p)$ from $\mathrm{Proj}^{\mathrm{f.g.}}_R$ to Mod_R is polynomial (in fact, linear), and the formation of this short exact sequence is compatible with base change along arbitrary maps $R\to R'$ of flat $\mathbb{Z}_p$-algebras, so Lemma 2.2 produces a fiber sequence $S^pM\to\Gamma^pM\to i_*F^*_{X_0}i^*M$ for every $M\in D(X)$ which gives the desired identification.

4) We will establish such a fiber sequence when M_0 is a flat sheaf on an affine scheme $X_0=\operatorname{Spec} R$ and the general case will follow formally as in (3). If M_0 is a flat R-module where R is an $\mathbb{F}_p$-algebra, then $T_p(M_0)$ is represented by the complex $S^pM_0\xrightarrow{N_p}\Gamma^pM_0$ concentrated in degrees $[-1,0]$. In Lemma 2.5 we defined maps $\psi_{M_0}:\Gamma^pM_0\to F^*_{R_0}M_0$

and $\Delta_{M_0} : F^*_{R_0}M_0 \to S^p M_0$ that give rise to a sequence

$$0 \to F^*_{R_0}M_0 \xrightarrow{\Delta_{M_0}} S^p_{R_0}M_0 \xrightarrow{N_p} \Gamma^p M_0 \xrightarrow{\psi_{M_0}} F^*_{R_0}M_0 \to 0 \tag{3.5}$$

that one checks to be exact by a direct calculation with a basis of M_0 in case it is free, as in part (2). This exact sequence gives rise to the desired fiber sequence (3.2) by Lemma 2.2. ☐

Non-decomposability of the de Rham complex will arise from the fact that the map $N_p : S^p M \to \Gamma^p M$ does not have a section in general. By definition, for an object $M \in D(X)$ on an arbitrary prestack X there is a natural fiber sequence

$$T_p(M)[-1] \xrightarrow{\gamma_M} S^p M \xrightarrow{N_p} \Gamma^p M \tag{3.6}$$

It will be slightly more natural for us to start with an arbitrary object $E \in D(X)$, and apply (3.6) to $M = E[-1]$ to get a fiber sequence

$$T_p(E[-1])[-1] \to S^p(E[-1]) \to (\Lambda^p E)[-p] \tag{3.7}$$

where we used the décalage identification $\Gamma^p(E[-1]) \simeq (\Lambda^p E)[-p]$ from Lemma 2.8 to rewrite the third term. If X is an algebraic stack flat over $\mathbb{Z}_p$ then, by Lemma 3.1(3) this fiber sequence takes the form

$$F^*_{X_0}(E/p)[-2] \to S^p(E[-1]) \to (\Lambda^p E)[-p] \tag{3.8}$$

Here $F^*_{X_0}(E/p)$ is an abbreviation for $i_* F^*_{X_0} i^* E$, we will prefer using this notation in what follows. Let us record the description of the cohomology sheaves of $S^p(E[-1])$ in the case E is a locally free sheaf, for $p > 2$, the analogous result for $p = 2$ will be established in Corollary 3.7:

Lemma 3.3. *Suppose that $p > 2$.*

(1) If E is a locally free sheaf on a scheme X flat over $\mathbb{Z}_p$, then

$$H^2(S^p(E[-1])) \simeq F^*_{X_0}(E/p), \quad H^p(S^p(E[-1])) \simeq \Lambda^p E, \tag{3.9}$$

and all other cohomology sheaves of $S^p(E[-1])$ are zero.

(2) If E is a locally free sheaf on a scheme X_0 over $\mathbb{F}_p$, then

$$H^1(S^p(E[-1])) \simeq H^2(S^p(E[-1])) \simeq F^*_{X_0}E, \quad H^p(S^p(E[-1])) \simeq \Lambda^p E \tag{3.10}$$

and all other cohomology sheaves of $S^p(E[-1])$ are zero.

Proof. This is immediate from (3.7) and Lemma 3.1(3),(4). ☐

Let now X_0 be an arbitrary prestack over $\mathbb{F}_p$. For an object $E \in D(X_0)$ the pushout of the fiber sequence (3.7) along the map $T_p(E[-1])[-1] \to F^*_{X_0}E[-2]$ from (3.2) defines a fiber sequences

$$F^*_{X_0}E[-2] \to S^p(E[-1]) \coprod_{T_p(E)[-2]} F^*_{X_0}E[-2] \to (\Lambda^p E)[-p] \tag{3.11}$$

In case where E is a locally free sheaf on an $\mathbb{F}_p$-scheme scheme X_0, the sequence (3.11) is simply the truncation of (3.7) in degrees ≥ 2.

Definition 3.4. We will denote by $\alpha(E) : \Lambda^p E \to F^*_{X_0}E[p-1]$ the shift by $[p]$ of the connecting morphism corresponding to the fiber sequence (3.11).

Suppose now that X is a flat algebraic stack over $\mathbb{Z}_p$. Denote by $i : X_0 \simeq X \times_{\mathbb{Z}_p} \mathbb{F}_p \hookrightarrow X$ the closed embedding of the special fiber. Let us remark that the information contained in the extension (3.7) for an object $E \in D(X)$ is completely captured by $\alpha(i^*E)$:

Lemma 3.5. *For an object $E \in D(X)$ denote i^*E by E_0. The image of the map $\alpha(E_0)$ under the adjunction identification $\mathrm{RHom}_{X_0}(\Lambda^p E_0, F^*_{X_0}E_0[p-1]) = \mathrm{RHom}_X(\Lambda^p E, i_*F^*_{X_0}i^*E[p-1])$ is the connecting morphism corresponding to the fiber sequence*

$$T_p(E[-1])[-1] \to S^p(E[-1]) \to (\Lambda^p E)[-p] \tag{3.12}$$

where we identify $T_p(E[-1])[-1]$ with $i_*F^*_{X_0}i^*E[-2]$ via Lemma 3.1(3).

Proof. In general, for two objects $M \in D(X), N \in D(X_0)$ the adjunction identification $\mathrm{RHom}_{D(X)}(M, i_*N) \simeq \mathrm{RHom}_{D(X_0)}(i^*M, N)$ can be described as sending a map $f : M \to i_*N$ to the composition $i^*M \xrightarrow{i^*f} i^*i_*N \to N$ where the second map is the counit of the adjunction. The fiber sequence $T_p(E_0[-1])[-1] \to S^p(E_0[-1]) \to (\Lambda^p E_0)[-p]$ is the result of applying i^* to the sequence (3.12), and the map $T_p(E_0[-1])[-1] \to F^*_{X_0}E_0[-2]$ used to form the pushout sequence (3.11) is precisely the counit map $i^*i_* \to \mathrm{Id}$ evaluated on $F^*_{X_0}E_0[-2]$ by Remark 3.2, so the lemma follows. □

The extension defining $\alpha(E)$ can be described in more classical terms for $p = 2$. For a projective module M over an $\mathbb{F}_2$-algebra R_0 we have a natural short exact sequence

$$0 \to F^*_{R_0}M \xrightarrow{\Delta_M} S^2M \xrightarrow{j} \Lambda^2 M \to 0 \tag{3.13}$$

Here Δ_M is the map defined in (2.7), it sends an element $m \otimes 1 \in M \otimes_{R_0, F_{R_0}} R_0$ to $m \cdot m \in S^2M$, and j sends $m_1 \cdot m_2 \in S^2M$ to $m_1 \wedge m_2 \in \Lambda^2 M$. By Lemma 2.2 this gives rise to a fiber sequence

$$F^*_{X_0}E \xrightarrow{\Delta_E} S^2E \xrightarrow{j} \Lambda^2 E \tag{3.14}$$

for every object $E \in D(X_0)$ on any $\mathbb{F}_2$-prestack X_0.

Lemma 3.6. *When $p = 2$, and E is an object of $D(X_0)$, the fiber sequence (3.11) is naturally equivalent to the shift by $[-2]$ of the sequence (3.14).*

Proof. Consider the shift of the fiber sequence (3.11) by [2]:

$$F^*_{X_0}E \to S^2(E[-1])[2] \bigsqcup_{T_2(E)} F^*_{X_0}E \to \Lambda^2 E \tag{3.15}$$

When $X_0 = \operatorname{Spec} R_0$ is an affine scheme and E corresponds to a projective R_0-module, first and third terms of (3.15) are concentrated in degree 0, so this fiber sequence is an exact sequence of R_0-modules. We will functorially identify it with the sequence (3.13), which will prove the Lemma in general thanks to Lemma 2.2.

First, note that the middle term of (3.15) is isomorphic to the cohomology module $H^2(S^2(E[-1]))$. On the other hand, the fiber sequence

$$F^*_{X_0}E[-1] \xrightarrow{\Delta_{E[-1]}} S^2(E[-1]) \xrightarrow{j_{E[-1]}} \Lambda^2(E[-1])$$

identifies $H^2(S^2(E[-1]))$ with $H^2(\Lambda^2 E[-1]) \simeq S^2E$. This already identifies the middle terms of exact sequences (3.14) and (3.15), so it remains to check that this identification fits into an identification of fiber sequences.

To this end, note that for any projective R_0-module M the norm map $N_2 : S^2M \to \Gamma^2M$ factors as $S^2M \xrightarrow{j} \Lambda^2M \xrightarrow{j'} \Gamma^2M$ where j' sends $m_1 \wedge m_2$ to $m_1 \otimes m_2 + m_2 \otimes m_1 \in \Gamma^2M \subset M^{\otimes 2}$. Therefore we get such a factorization for any object $M \in D(X_0)$. Applying this to $M = E[-1]$ we learn that the map $S^2(E[-1]) \to \Gamma^2(E[-1]) \simeq \Lambda^2E$ factors through $j_{E[-1]}$ which proves the lemma. □

We can deduce the computation of cohomology sheaves of $S^p(E[-1])$ when $p = 2$, complementing Lemma 3.3(2):

Corollary 3.7. *For a locally free sheaf E on an $\mathbb{F}_2$-scheme X_0 we have*

$$H^1(S^2(E[-1])) \simeq F^*_{X_0}E \quad H^2(S^2(E[-1])) \simeq S^2E \tag{3.16}$$

and all other cohomology sheaves are zero.

For an arbitrary p, we can relate $S^p(E[-1])$ to the homotopy coinvariants $(E[-1]^{\otimes p})_{hS_p}$ of the symmetric group S_p acting on $E[-1]^{\otimes p}$ by permutations. This identification will be used in our proof of non-vanishing of the class α in Section 12.

Lemma 3.8. *For a locally free sheaf E on an $\mathbb{F}_p$-scheme X_0 there is a natural equivalence $S^p(E[-1]) \simeq \tau^{\geq 1}(E[-1]^{\otimes p})_{hS_p}$*

Proof. First, let us recall the values of the cohomology sheaves of $(E[-1]^{\otimes p})_{hS_p}$:

Lemma 3.9. *If $p > 2$ then*

$$H^i((E[-1]^{\otimes p})_{hS_p}) \simeq \begin{cases} \Lambda^pE, i = p \\ F^*_{X_0}E, i \equiv 1 \text{ or } 2 \bmod p-1, \text{ and } i \leq 2 \\ 0 \text{ otherwise} \end{cases} \tag{3.17}$$

If $p = 2$ then

$$H^i((E[-1]^{\otimes 2})_{hS_2}) \simeq \begin{cases} S^2E, i = 2 \\ F^*_{X_0}E, i < 2 \\ 0, i > 2 \end{cases} \tag{3.18}$$

Proof. We will give the argument in the case $p > 2$ and the case $p = 2$ is proven analogously. Note that the S_p-equivariant object $E[-1]^{\otimes p}$ is equivalent to $\mathrm{sgn}\otimes E^{\otimes p}[-p]$ where S_p acts on $E^{\otimes p}$ via permutation of factors, and sgn is the sign character. In particular, $H^p((E[-1]^{\otimes p})_{hS_p})$ is isomorphic to Λ^pE by definition of the exterior power.

To identify other cohomology sheaves, we will compare homology of the symmetric group with that of a cyclic group. Denote by $C_p \subset S_p$ a cyclic subgroup of order p. There is a natural map $(E[-1]^{\otimes p})_{hC_p} \to (E[-1]^{\otimes p})_{hS_p}$. The coinvariants $(E[-1]^{\otimes p})_{hC_p}$ for the cyclic group can be represented by the following two-periodic complex

$$\dots \xrightarrow{1-\sigma} E^{\otimes p} \xrightarrow{N_{C_p}} E^{\otimes p} \xrightarrow{1-\sigma} E^{\otimes p} \tag{3.19}$$

where σ is the endomorphism of $E^{\otimes p}$ given by cyclic permutation of the factors, and $N_{C_p} = \sum\limits_{i=0}^{p-1} \sigma^i$. For all $i < p$ we get a map $F^*_{X_0}E \to H^i((E[-1]^{\otimes p})_{hC_p})$ given by sending a section $e \otimes 1 \in E(U) \otimes_{\mathcal{O}_{X_0}(U), F_{X_0}} \mathcal{O}_{X_0}(U)$ on an affine open $U \subset X_0$ to $e^{\otimes p} \in E^{\otimes p}(U)$. One checks on stalks that this map is an isomorphism for all $i < p$. We get

$$H^i((E[-1]^{\otimes p})_{hC_p}) \simeq \begin{cases} (E^{\otimes p})_{C_p}, i = p \\ F^*_{X_0}E, i < p \\ 0 \text{ otherwise} \end{cases} \tag{3.20}$$

Since $C_p \subset S_p$ is a p-Sylow subgroup, the map $(E[-1]^{\otimes p})_{hC_p} \to (E[-1]^{\otimes p})_{hS_p}$ establishes $(E[-1]^{\otimes p})_{hS_p}$ as a direct summand of $(E[-1]^{\otimes p})_{hC_p}$. Hence to prove the lemma it remains to check that the surjective map $H^i((E[-1]^{\otimes p})_{hC_p}) \to H^i((E[-1]^{\otimes p})_{hS_p})$ is an isomorphism for $i \equiv 1, 2 \bmod p$ and is the zero map for all other $i < p$. This can be checked on stalks, and $F^*_{X_0}$ is an additive functor, so we may assume that $X_0 = \operatorname{Spec} \mathbb{F}_p$ and E is a 1-dimensional $\mathbb{F}_p$-vector space. The statement is now a consequence of classical computations of homology of the symmetric group with coefficients in the sign character, cf. [Ste62, Chapter V, Proposition 7.8]. □

There is a natural map $E[-1]^{\otimes p} \to S^p(E[-1])$ which is S_p-equivariant with respect to the permutation action on the source and the trivial action on the target. Hence it induces a natural map $(E[-1]^{\otimes p})_{hS_p} \to S^p(E[-1])$. Moreover, this map factors through $(E[-1]^{\otimes p})_{hS_p} \to \tau^{\geq 1}(E[-1]^{\otimes p})_{hS_p}$ because $S^p(E[-1])$ is concentrated in degrees ≥ 1. We thus obtain a map

$$\xi_E : \tau^{\geq 1}(E[-1]^{\otimes p})_{hS_p} \to S^p(E[-1]) \tag{3.21}$$

between objects of $D(X_0)$ whose cohomology sheaves are isomorphic, by (3.17) and Lemma 3.3(2). It remains to check that this particular map induces isomorphisms on cohomology sheaves.

For cohomology in degree p, this is a consequence of the fact that the composition $E[-1]^{\otimes p} \to S^p(E[-1])$ induces (up to a sign) the surjection $E^{\otimes p} \twoheadrightarrow \Lambda^p E$ on H^p when $p > 2$, and the surjection $E^{\otimes 2} \twoheadrightarrow S^2 E$ when $p = 2$. To show that the map ξ_E induces isomorphisms on H^1 and H^2 it is enough to consider the case $X_0 = \operatorname{Spec} \mathbb{F}_p$ and E being a 1-dimensional vector space. This follows from the fact that cohomology classes of $(\mathbb{F}_p[-1]^{\otimes p})_{hS_p}$ are responsible for Steenrod operations in the cohomology of E_∞-algebras, as proven in Lemma 3.16 below. □

3.2. T_p **and commutative algebras.** We will now compute how the defect between S^p and Γ^p, given by T_p, interacts with the structure of a derived commutative algebra. This computation plays a central role in our proof of Theorem 4.1.

Lemma 3.10. *Let X be an algebraic stack or a formal scheme flat over $\mathbb{Z}_p$ with the special fiber $X_0 := X \times_{\mathbb{Z}_p} \mathbb{F}_p$. For a derived commutative algebra $A \in \operatorname{DAlg}(X)$ the composition $F^*_{X_0}(A/p)[-1] \xrightarrow{\gamma_A} S^p A \xrightarrow{m_A} A$ is naturally homotopic in $D(X)$ to*

$$F^*_{X_0}(A/p)[-1] \xrightarrow{\varphi_{A/p}} A/p[-1] \xrightarrow{\operatorname{Bock}_A[-1]} A \tag{3.22}$$

Proof. For a projective module M over a flat $\mathbb{Z}_p$-algebra R there is a diagram of R-modules

$$
\begin{tikzcd}
S^pM \arrow[r, "N_p"] \arrow[d, "\mathrm{id}_{S^pM}"] & \Gamma^pM \arrow[r] \arrow[d, "r_p"] & F^*_{R/p}(M/p) \arrow[d, "\Delta_{M/p}"] \\
S^pM \arrow[r, "p!"] & S^pM \arrow[r] & S^p_{R/p}(M/p)
\end{tikzcd} \tag{3.23}
$$

where both rows are short exact sequences of R-modules. It evidently gives rise to a diagram of polynomial functors that is moreover natural in arbitrary maps $R \to R'$. We can construct from this the following diagram in $D(X)$ where rows are fiber sequences and vertical maps organize into morphisms of fiber sequences:

$$
\begin{tikzcd}
S^pA \arrow[r, "N_p"] \arrow[d, equal] & \Gamma^pA \arrow[r] \arrow[d] & F^*_{X_0}(A/p) \arrow[d, "\Delta_{A/p}"] \\
S^pA \arrow[r, "p!"] \arrow[d, "m_A"] & S^pA \arrow[r] \arrow[d, "m"] & S^p_{X_0}(A/p) \arrow[d, "m_A"] \\
A \arrow[r, "p!"] & A \arrow[r] & A/p
\end{tikzcd} \tag{3.24}
$$

The maps between the first two rows are obtained by applying the functor Σ_X of Lemma 2.2 to the diagram (3.23), and evaluating the resulting maps on the object $A \in D(X)$. The maps between the second and third rows are obtained by applying the functor $\mathrm{cofib}(p!)$ to the map $m_A : S^pA \to A$.

The connecting morphism $F^*_{X_0}(A/p) \to (S^pA)[1]$ corresponding to the top row of (3.24) is equivalent to $-\gamma_A[1]$, by definition of γ_A given in (3.6). Commutativity of the diagram implies the lemma because the connecting morphism $A/p \to A[1]$ of the bottom row is the negative $(-\mathrm{Bock}_A)$ of the Bockstein morphism by the identity $(p-1)! = -1$ in $\mathbb{F}_p$, and the composition of the right vertical column is the map $\varphi_{A/p} : F^*_{X_0}(A/p) \to A/p$, by the definition of Frobenius endomorphisms of derived commutative algebras. $\square$

Remark 3.11. If A happens to be represented by a term-wise flat cosimplicial commutative algebra $A^\bullet$ in $\mathrm{QCoh}(X)$, for a flat $\mathbb{Z}_p$-scheme X, then the diagram (3.24) can be obtained from the strictly commutative diagram in the ordinary category of cosimplicial sheaves on X, where the rows are term-wise exact:

$$
\begin{tikzcd}
S^p_{\mathrm{naive}}A^\bullet \arrow[r, "N"] \arrow[d, "m"] & \Gamma^p_{\mathrm{naive}}A^\bullet \arrow[r] \arrow[d, "m"] & F^*(A/p) \arrow[d, "\varphi^{\mathrm{cosimp}}_{A/p}"] \\
A \arrow[r, "p!"] & A \arrow[r] & A/p
\end{tikzcd} \tag{3.25}
$$

where S^p_{naive} and $\Gamma^p_{\mathrm{naive}}$ denote the endofunctors of the ordinary category of cosimplicial objects in $\mathrm{QCoh}(X)$ induced by the (non-derived) functors S^p and Γ^p.

3.3. **Steenrod operations on cohomology of cosimplicial algebras.** This subsection is not used in the rest of the paper and contains classical material, if only presented somewhat differently, but we include it, as Proposition 3.14 was the original motivation for our approach and Theorem 4.1 should be viewed as its generalization. Let us also mention the result of Scavia [Sca21, Theorem 1.1(iv),(v)] which is related to and implied by Proposition 3.14.

Recall that cohomology of derived symmetric powers is related to natural operations on cohomology of cosimplicial commutative algebras:

Lemma 3.12. *Fix a commutative base ring R. The R-module of natural transformations between functors $H^i, H^j : \mathrm{CAlg}_R^{\Delta} \to \mathrm{Mod}_R$ can be described as*

$$\mathrm{Hom}(H^i, H^j) \simeq H^j(S^\bullet(R[-i])) \tag{3.26}$$

where the isomorphism takes a natural transformation $\alpha : H^i \to H^j$ to the image of the class $1 \in R = H^i(S^1(R[-i])) \subset H^i(S^\bullet(R[-i]))$ under α evaluated on the free algebra $S^\bullet(R[-i])$.

Remark 3.13. The resulting module of natural transformations can be computed completely, cf. [Pri73] for the case $R = \mathbb{F}_p$.

Proof. We will describe the inverse map. Given a class $c \in H^j(S^\bullet(R[-i]))$, for every cosimplicial commutative algebra A we define a morphism $H^i(A) \to H^j(A)$ as follows. Let $x : R[-i] \to A$ be a map representing a cohomology class $[x] \in H^i(A)$. Applying the functor $S^\bullet$ to this map of cosimplicial R-modules and composing it with the multiplication on A gives a map $S^\bullet(R[-i]) \xrightarrow{S^\bullet x} S^\bullet A \xrightarrow{m_A} A$. We declare the image of $[x]$ under the natural transformation to be the image of the class c under $m_A \circ S^\bullet x$. This produces a map $H^j(S^\bullet(R[-i])) \to \mathrm{Func}(H^i, H^j)$ inverse to the map (3.26). $\square$

We will now deduce from Lemma 3.10 that certain cohomology operations of degree 0 and 1 are related to Frobenius endomorphism, and Witt vectors Bockstein homomorphism introduced by Serre in [Ser58, §3]. Recall that for any object $M \in D(R)$ of the derived category of modules over a ring R we defined a map $T_p(M)[-1] \xrightarrow{\gamma_M} S^p_R M$ in (3.6).

In the case $M = \mathbb{Z}_p[-i]$ over $R = \mathbb{Z}_p$ it takes form $\mathbb{F}_p[-i-1] \to S^p(\mathbb{Z}_p[-i])$, and we denote by $P^1_{\mathbb{Z}_p} \in H^{i+1}(S^p(\mathbb{Z}_p[-i]))$ the resulting p-torsion class. For $M = \mathbb{F}_p[-i]$ over $R = \mathbb{F}_p$ the map $\gamma_{\mathbb{F}_p}$ has the form $\mathbb{F}_p[-i] \oplus \mathbb{F}_p[-i-1] \to S^p_{\mathbb{F}_p}(\mathbb{F}_p[-i])$, and we denote by $P^0_{\mathbb{F}_p} \in H^i(S^p(\mathbb{F}_p[-i])), P^1_{\mathbb{F}_p} \in H^{i+1}(S^p_{\mathbb{F}_p}(\mathbb{F}_p[-i]))$ the resulting classes. Note that $P^1_{\mathbb{F}_p}$ is the image of $P^1_{\mathbb{Z}_p}$ under the reduction map.

By Lemma 3.12 the class $P^1_{\mathbb{Z}_p}$ gives rise to a natural homomorphism

$$P^1_{\mathbb{Z}_p} : H^i(A) \to H^{i+1}(A) \tag{3.27}$$

for every cosimplicial commutative $\mathbb{Z}_p$-algebra A. Similarly, classes $P^0_{\mathbb{F}_p}, P^1_{\mathbb{F}_p}$ define operations

$$P^0_{\mathbb{F}_p} : H^i(A_0) \to H^i(A_0) \quad P^1_{\mathbb{F}_p} : H^i(A_0) \to H^{i+1}(A_0) \tag{3.28}$$

for every cosimplicial commutative $\mathbb{F}_p$-algebra A_0. In the following result by $W_2(A_0)$ we mean the cosimplicial commutative $\mathbb{Z}/p^2$-algebra obtained by applying term-wise the length 2 Witt vectors functor to A_0.

Proposition 3.14. *(1) For a cosimplicial commutative algebra A over $\mathbb{Z}_p$ the operation $P^1_{\mathbb{Z}_p} : H^i(A) \to H^{i+1}(A/p)$ is equal to the composition*

$$H^i(A) \to H^i(A/p) \xrightarrow{\varphi_{A/p}} H^i(A/p) \xrightarrow{\mathrm{Bock}^i_A} H^{i+1}(A)$$

(2) *For a cosimplicial commutative algebra A_0 over $\mathbb{F}_p$ the operation $P^0_{\mathbb{F}_p}: H^i(A_0) \to H^i(A_0)$ is equal to the Frobenius endomorphism of A_0.*

(3) *For a cosimplicial commutative algebra A_0 over $\mathbb{F}_p$ the operation $P^1_{\mathbb{F}_p}: H^i(A_0) \to H^{i+1}(A_0)$ is equal to the connecting homomorphism induced by the exact sequence $A_0 \xrightarrow{V} W_2(A_0) \to A_0$.*

Proof. Given a class $[x] \in H^i(A)$ represented by a map $x : \mathbb{Z}_p[-i] \to A$ we get a commutative diagram

$$\begin{tikzcd} T_p(\mathbb{Z}_p[-i])[-1] \arrow[r, "\gamma_{\mathbb{Z}_p[-i]}"] \arrow[d, "T_p(x)[-1]"] & S^p\mathbb{Z}_p[-i] \arrow[d, "S^p x"] & \\ T_p(A)[-1] \arrow[r, "\gamma_A"] & S^pA \arrow[r, "m_A"] & A \end{tikzcd} \tag{3.29}$$

By definition, the value $P^1_{\mathbb{Z}_p}([x]) \in H^{i+1}(A)$ is equal to the image of the class $1 \in \mathbb{F}_p \simeq H^{i+1}(T_p(\mathbb{Z}_p[-i])[-1])$ under the clockwise composition in the diagram (3.29). On the other hand, the counter-clockwise composition is homotopic to $\mathbb{F}_p[-i-1] \xrightarrow{x[-1] \bmod p} A/p[-1] \xrightarrow{\varphi_{A/p}[-1]} A/p[-1] \xrightarrow{\mathrm{Bock}_A} A$ by Lemma 3.10, and this implies part (1).

For a class $[x] \in H^i(A_0)$ represented by a map $x : \mathbb{F}_p[-i] \to A_0$ the value $P^0_{\mathbb{F}_p}([x]) \in H^i(A_0)$ is defined as the image of the unit in $\mathbb{F}_p = H^i(\mathbb{F}_p[-i])$ under the composition $\mathbb{F}_p[-i] \xrightarrow{\Delta_{\mathbb{F}_p[-i]}} S^p\mathbb{F}_p[-i] \xrightarrow{S^p x} S^pA_0 \xrightarrow{m_{A_0}} A_0$. This composition is homotopic to $\mathbb{F}_p[-i] \xrightarrow{x} A_0 \xrightarrow{\Delta_{A_0}} S^pA_0 \xrightarrow{m_{A_0}} A_0$ which implies statement (2) by the definition of Frobenius on A_0.

By Lemma 3.12, in (3) it is enough to check the claimed formula for $P^1_{\mathbb{F}_p}$ on the universal class $1 \in H^i(S^\bullet\mathbb{F}_p[-i])$ for $A_0 = S^\bullet\mathbb{F}_p[-i]$. This cosimplicial $\mathbb{F}_p$-algebra lifts to $A = S^\bullet\mathbb{Z}_p[-i]$ and the universal class lifts along the map $H^i(A) \to H^i(A_0)$. Therefore we can apply part (1) to get that $P^1_{\mathbb{F}_p}(1)$ is equal to the image of 1 under the composition $H^i(A_0) \xrightarrow{\varphi_{A_0}} H^i(A_0) \xrightarrow{\mathrm{Bock}_{A/p^2}} H^{i+1}(A_0)$.

To see that this composition is equal to the Witt vector Bockstein homomorphism, recall that there is a map of cosimplicial rings $\phi : W_2(A_0) \to A/p^2$ term-wise given by $[a_0] + V[a_1] \mapsto \widetilde{a}_0^p + p\widetilde{a}_1$ where $\widetilde{a}_0, \widetilde{a}_1 \in A^i/p^2$ are arbitrary lifts of $a_0, a_1 \in A_0^i$. This map fits into a commutative diagram where rows are term-wise exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r, "V"] \arrow[d, equal] & W_2(A_0) \arrow[r] \arrow[d, "\phi"] & A_0 \arrow[r] \arrow[d, "\varphi_{A_0}"] & 0 \\ 0 \arrow[r] & A_0 \arrow[r, "p"] & A/p^2 \arrow[r] & A_0 \arrow[r] & 0 \end{tikzcd} \tag{3.30}$$

The induced map between the associated long exact sequence of cohomology proves that the connecting homomorphism induced by the top row is equal to $\mathrm{Bock}_{A/p^2} \circ \varphi_{A_0}$ which finishes the proof of (3). $\square$

3.4. **Steenrod operations on cohomology of E_∞-algebras.** In this expository subsection we recall how power operations on cohomology of an E_∞-algebra is defined, and relate them to the operations on the cohomology of cosimplicial commutative algebras. All of this material is contained in [Ste62], [May70], [Pri73].

Let A be an E_∞-algebra over a (ordinary) commutative ring R. This structure, in particular, gives a multiplication map $A^{\otimes p} \to A$ in $D(R)$ which is S_p-equivant where S_p acts via permutations of the factors in $A^{\otimes p}$ and acts trivially on A. The multiplication map therefore factors through a map

$$m_A : (A^{\otimes p})_{hS_p} \to A \tag{3.31}$$

This map is used to define operations on cohomology of A. Given a cohomology class $[x] \in H^i(A)$ we can represent it by a map $x : R[-i] \to A$ in $D(R)$ and form the composition

$$(R[-i]^{\otimes p})_{hS_p} \xrightarrow{x^{\otimes p}} (A^{\otimes p})_{hS_p} \xrightarrow{m_A} A \tag{3.32}$$

We now specialize to the case $R = \mathbb{F}_p$. The cohomology groups of $(\mathbb{F}_p[-i]^{\otimes p})_{hS_p}$ are given by

(3.33)

$$H^j((\mathbb{F}_p[-i]^{\otimes p})_{hS_p}) = \begin{cases} \mathbb{F}_p, & \text{for } j = pi-(2k+1)(p-1) \text{ or } pi-(2k+1)(p-1)+1 \text{ with } k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

if i is odd, and by

(3.34)

$$H^j((\mathbb{F}_p[-i]^{\otimes p})_{hS_p}) = \begin{cases} \mathbb{F}_p, & \text{for } j = pi-2k(p-1) \text{ or } pi-2k(p-1)+1 \text{ with } k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

if i is even, see [Ste62, Chapter V, Lemmas 6.1, 6.2 + Proposition 7.8]. Thus for every m of the form $2k(p-1)$ or $2k(p-1)+1$ we can define $P^m([x]) \in H^{i+m}(A)$ as the image of a fixed generator of $H^{i+m}((\mathbb{F}_p[-i]^{\otimes p})_{hS_p})$ under the map (3.32). Applying this construction to the singular cohomology of a topological space gives rise to natural cohomology operations which satisfy special properties that are false for general E_∞-algebras:

Theorem 3.15. *For a topological space X and an integer $i \geq 0$ the operations $P^0 : H^i_{\mathrm{sing}}(X, \mathbb{F}_p) \to H^i_{\mathrm{sing}}(X, \mathbb{F}_p), P^1 : H^i_{\mathrm{sing}}(X, \mathbb{F}_p) \to H^{i+1}_{\mathrm{sing}}(X, \mathbb{F}_p)$ arising from the E_∞-algebra structure on $C^\bullet_{\mathrm{sing}}(X, \mathbb{F}_p)$ are described as*

(1) $P^0 = \mathrm{Id}$
(2) P^1 is the Bockstein homomorphism corresponding to the term-wise exact sequence of complexes $C^\bullet_{\mathrm{sing}}(X, \mathbb{F}_p) \to C^\bullet_{\mathrm{sing}}(X, \mathbb{Z}/p^2) \to C^\bullet_{\mathrm{sing}}(X, \mathbb{F}_p)$.

Moreover, all operations P^m with $m < 0$ are zero.

We can reconcile these operations with the operations on cohomology of cosimplicial commutative algebras, which will justify denoting these by the same symbols P^i. Let $A \in \mathrm{CAlg}^{\Delta}_{\mathbb{F}_p}$ be a cosimplicial commutative $\mathbb{F}_p$-algebra. We can view it as an E_∞-algebra with the symmetric multiplication factoring through the cosimplicial symmetric multiplication

$$(A^{\otimes p})_{hS_p} \to S^p A \xrightarrow{m_A} A \tag{3.35}$$

In particular, for a class $x \in H^i(A)$ the map (3.32) factors through the natural map $(\mathbb{F}_p[-i]^{\otimes p})_{hS_p} \to S^p(\mathbb{F}_p[-i])$.

Lemma 3.16. *For every i the map $(\mathbb{F}_p[-i]^{\otimes p})_{hS_p} \to S^p(\mathbb{F}_p[-i])$ factors through an equivalence $\tau^{\geq i}(\mathbb{F}_p[-i]^{\otimes p})_{hS_p} \simeq S^p(\mathbb{F}_p[-i])$.*

Proof. The cohomology groups of the object $S^p(\mathbb{F}_p[-i])$ are abstractly isomorphic to that of $\tau^{\geq i}(\mathbb{F}_p[-i]^{\otimes p})_{hS_p}$ by [Pri73, Theorems 4.1.1,4.2.1] so it is enough to show that the map $\tau^{\geq i}(\mathbb{F}_p[-i]^{\otimes p})_{hS_p} \to S^p\mathbb{F}_p[-i]$ induces an injection on cohomology in degrees $\geq i$.

For this it suffices to provide, for every $m \geq 0$, a topological space X such that the operation $P^m : H^i_{\mathrm{sing}}(X, \mathbb{F}_p) \to H^{i+m}_{\mathrm{sing}}(X, \mathbb{F}_p)$ is non-zero. The Eilenberg-MacLane space $X = K(\mathbb{F}_p, i)$ satisfies this condition, because the operations on cohomology of topological spaces defined from cohomology classes of Eilenberg-MacLane spaces coincide with those defined using the E_∞-algebra structure, e.g. by the uniqueness of functorial operations [Ste62, §VIII.3]. □

This discussion can be applied to the cosimplicial commutative algebra $A = C^\bullet_{\mathrm{sing}}(X, \mathbb{F}_p)$ of singular cochains on a topological space X whose underlying E_∞-algebra was referenced in Theorem 3.15. The fact that operations P^m vanish for $m < 0$ can thus be explained by the fact that $S^p(\mathbb{F}_p[-i])$ is concentrated in degrees $\geq i$. The algebras of the form $C^\bullet_{\mathrm{sing}}(X, \mathbb{F}_p)$ are special among general cosimplicial commutative $\mathbb{F}_p$-algebras in that their Frobenius endomorphism is equal to the identity, because $C^{\mathrm{sing}}(X, \mathbb{F}_p)$ is defined as the algebra of $\mathbb{F}_p$-valued functions on the simplicial singular set of X, cf. [Pri73, 6.1]. In particular, the relations $P^0 = \mathrm{Id}$ and $P^1 = \mathrm{Bock}$ in Theorem 3.15 can be deduced from our Proposition 3.14.

4. Extensions in complexes underlying derived commutative algebras

In this section we prove our main algebraic result on extensions in the canonical filtration on the complexes underlying certain derived commutative algebras. In this section X will be an algebraic stack flat over $\mathbb{Z}_p$, or a formal scheme flat over $\operatorname{Spf} \mathbb{Z}_p$. In both regimes we denote by $X_0 = X \times_{\mathbb{Z}_p} \mathbb{F}_p$ the special fiber of X.

Theorem 4.1. *Let $A \in \mathrm{DAlg}^{\geq 0}(X)$ be a derived commutative algebra on X concentrated in degrees ≥ 0 such that $H^0(A) = \mathcal{O}_X$, $H^1(A)$ a locally free $\mathcal{O}_X$-module, and multiplication on cohomology induces an isomorphism $H^\bullet(A) \simeq \Lambda^\bullet H^1(A)$. Assume further that there is a morphism $s : H^1(A)[-1] \to A$ in $D(X)$ splitting the canonical filtration on $\tau^{\leq 1}A$.*

(1) *There exists a natural equivalence $\bigoplus\limits_{i=0}^{p-1} H^i(A)[-i] \simeq \tau^{\leq p-1}A$ in $D(X)$.*
(2) *There is a natural homotopy between the map $H^p(A) \simeq \mathrm{cofib}(\tau^{\leq p-1}A \to \tau^{\leq p}A)[p] \to (\tau^{\leq p-1}A)[p+1]$ corresponding to the extension $\tau^{\leq p-1}A \to \tau^{\leq p}A \to H^p(A)[-p]$ and the composition*

(4.1)
$$H^p(A) = \Lambda^p H^1(A) \xrightarrow{\alpha(H^1(A)/p)} F^*_{X_0} H^1(A/p)[p-1] \xrightarrow{F^*_{X_0} s[p]} (\tau^{\leq 1} F^*_{X_0}(A/p))[p] \xrightarrow{\varphi_{A/p}}$$
$$(\tau^{\leq 1}A/p)[p] \xrightarrow{\mathrm{Bock}_A} (\tau^{\leq 1}A)[p+1] \to (\tau^{\leq p-1}A)[p+1]$$

Proof. For each $i \geq 0$, consider the map

$$s_i : S^i(H^1(A)[-1]) \xrightarrow{S^i s} S^i A \xrightarrow{m_A} A \tag{4.2}$$

The composition $H^1(A)[-1]^{\otimes i} \to S^i(H^1(A)[-1]) \xrightarrow{s_i} A$ induces the multiplication map $m : H^1(A)^{\otimes i} \to H^i(A)$ on i-th cohomology and 0 on all other cohomology groups. By the assumption that $H^\bullet(A)$ is freely generated by $H^1(A)$, the map $m : H^1(A)^{\otimes i} \to H^i(A)$ identifies $H^i(A)$ with $\Lambda^i H^1(A)$.

For $i \le p-1$ we have $S^i(H^1(A)[-1]) \simeq \Gamma^i(H^1(A)[-1]) \simeq (\Lambda^i H^1(A))[-i]$, and the map $H^1(A)[-1]^{\otimes i} \to S^i(H^1(A)[-1])$ is the shift by $[-i]$ of the natural surjection $H^1(A)^{\otimes i} \to \Lambda^i H^1(A)$. Therefore s_i induces an isomorphism on H^i, which proves the first part of the theorem.

For $i = p$ the natural map $S^p(H^1(A)[-1]) \xrightarrow{N_p} \Gamma^p(H^1(A)[-1]) \simeq (\Lambda^p H^1(A))[-p]$ is not an equivalence anymore. Using the map s_p, we will relate the extension $\tau^{\le p-1}A \to \tau^{\le p}A \to H^p(A)[-p]$ to the extension

$$F^*_{X_0}(H^1(A)/p)[-2] \to S^p(H^1(A)[-1]) \to \Lambda^p H^1(A)[-p] \tag{4.3}$$

constructed in (3.8). To begin, let us compute the map induced by s_p on the cohomology in degree p.

Since $S^p(H^1(A)[-1])$ is concentrated in degrees $\le p$, the map s_p naturally factors through $\tau^{\le p}A$. We claim that the composition

$$S^p(H^1(A)[-1]) \xrightarrow{s_p} \tau^{\le p}A \to H^p(A)[-p] \tag{4.4}$$

is naturally homotopic to the norm map $N_p : S^p(H^1(A)[-1]) \to (\Lambda^p H^1(A))[-p]$. This composition factors uniquely through the norm map, because the mapping space $\mathrm{Map}_{D(X)}(\mathrm{fib}(N_p), H^p(A)[-p])$ is contractible as $H^p(A)[-p]$ is a locally free sheaf placed in degree p, and $\mathrm{fib}(N_p) \simeq F^*_{X_0}H^1(A)/p[-2]$ is a p-torsion object concentrated in degrees $\le 2 \le p$. Therefore the composition (4.4) has the form $S^p(H^1(A)[-1]) \xrightarrow{N_p} \Lambda^p H^1(A)[-p] \xrightarrow{\psi} H^p(A)[-p]$ for some map ψ. To check that ψ is equal to the cup-product map, we may precompose this composition with the map $H^1(A)[-1]^{\otimes p} \to S^p H^1(A)[-1]$, and the claim follows from:

Lemma 4.2. *Let E be a locally free sheaf on X. Under the décalage identification $\Gamma^p(E[-1]) \simeq \Lambda^p E[-p]$ the composition $E[-1]^{\otimes p} \to S^p(E[-1]) \xrightarrow{N_p} \Gamma^p(E[-1])$ is identified with the shift by $[-p]$ of the map $E^{\otimes p} \to \Lambda^p E$.*

Proof. [Ill71, Proposition I.4.3.2.1] shows that décalage equivalences are compatible with graded algebra structures on symmetric, divided power, and exterior algebras. Our assertion is a special case of this. □

This allows us to fit s_p into the following map of fiber sequences for some map $\beta_p^{\le p-1}$:

$$\begin{tikzcd} F^*_{X_0}(H^1(A)/p)[-2] \arrow[r, "\gamma_{H^1(A)[-1]}"] \arrow[d, "\beta_p^{\le p-1}"] & S^p(H^1(A)[-1]) \arrow[r, "N_p"] \arrow[d, "s_p"] & (\Lambda^p H^1(A))[-p] \arrow[d, "\sim"] \\ \tau^{\le p-1}A \arrow[r] & \tau^{\le p}A \arrow[r] & H^p(A)[-p] \end{tikzcd} \tag{4.5}$$

This diagram implies that the extension class $H^p(A) \to (\tau^{\le p-1}A)[p+1]$ corresponding to the bottom row can be described as the composition

$$H^p(A) \simeq \Lambda^p H^1(A) \xrightarrow{\alpha(H^1(A)/p)} F^*_{X_0}(H^1(A)/p)[p-1] \xrightarrow{\beta_p^{\le p-1}[p+1]} (\tau^{\le p-1}A)[p+1] \tag{4.6}$$

Here $\alpha(H^1(A)/p)$ is the natural class attached to the vector bundle $H^1(A)/p$ on X_0 by Definition 3.4, we view it as a map from $\Lambda^p H^1(A)$ to $F^*_{X_0}(H^1(A)/p)[p-1]$ via adjunction as in Lemma 3.5.

To finish the proof of Theorem 4.1, it remains to compute $\beta_p^{\leq p-1}$. Note that $\beta_p^{\leq p-1}$ can be naturally recovered from the composition $\beta_p : F^*_{X_0}H^1(A)/p[-2] \xrightarrow{\beta_p^{\leq p-1}} \tau^{\leq p-1}A \to A$ as the truncation $\tau^{\leq p-1}\beta_p$, because the mapping space $\mathrm{Map}_{D(X)}(F^*_{X_0}H^1(A)/p[-2], \tau^{\geq p}A)$ is contractible. To identify β_p, consider the following commutative diagram

$$\begin{tikzcd}
F^*_{X_0}(H^1(A)/p)[-2] \arrow[r, "\gamma_{H^1(A)[-1]}"] \arrow[d, "F^*_{X_0}s[-1]"] \arrow[rrd, bend left, "\beta_p"] & S^p(H^1(A)[-1]) \arrow[d, "S^p s"] & \\
F^*_{X_0}(A/p)[-1] \arrow[r] & S^pA \arrow[r, "m"] & A
\end{tikzcd} \tag{4.7}$$

The composition of the bottom row was proven in Lemma 3.10 to be homotopic to $F^*_{X_0}(A/p)[-1] \xrightarrow{\varphi_{A/p}} A/p[-1] \xrightarrow{\mathrm{Bock}_A} A$, hence $\beta_p : F^*_{X_0}(H^1(A)/p)[-2] \to A$ is homotopic to the composition

$$F^*_{X_0}(H^1(A)/p)[-2] \xrightarrow{F^*_{X_0}s[-1]} F^*_{X_0}(A/p)[-1] \xrightarrow{\varphi_{A/p}} A/p[-1] \xrightarrow{\mathrm{Bock}_A[-1]} A \tag{4.8}$$

that can be factored as

$$F^*_{X_0}(H^1(A)/p)[-2] \xrightarrow{F^*_{X_0}s[-1]} (\tau^{\leq 1}F^*_{X_0}(A/p))[-1] \xrightarrow{\varphi_{A/p}} \tau^{\leq 1}(A/p)[-1] \xrightarrow{\mathrm{Bock}_A[-1]} \tau^{\leq 1}A \to A \tag{4.9}$$

Therefore $\beta_p^{\leq p-1}$ is the composition of the first 3 arrows in (4.9) followed by the map $\tau^{\leq 1}A \to \tau^{\leq p-1}A$. Plugging this expression for $\beta_p^{\leq p-1}$ into (4.6) finishes the proof of the second part of the theorem. □

Let us record the special form that Theorem 4.1 takes in the case of augmented algebras.

Corollary 4.3. *Let $A \in \mathrm{DAlg}^{\geq 0}(X)$ be a derived commutative algebra such that $H^0(A) \simeq \mathcal{O}_X$ and the multiplication on cohomology induces an isomorphism $H^\bullet(A) = \Lambda^\bullet H^1(A)$. Assume also that A is equipped with a map $\varepsilon : A \to \mathcal{O}_X$ of derived commutative algebras that induces an isomorphism on H^0. Then*

(1) $\tau^{\leq p-1}A$ naturally decomposes in $D(X)$ as $\bigoplus\limits_{i=0}^{p-1} H^i(A)[-i]$

(2) The extension class $H^p(A) \to \tau^{\leq p-1}A[p+1]$ corresponding to $\tau^{\leq p}A$ can be described as the composition

$$\begin{aligned} H^p(A) \xrightarrow{\alpha(H^1(A)/p)} F^*(H^1(A)/p)[p-1] \xrightarrow{\varphi_{A/p}} (H^1(A)/p)[p-1] \xrightarrow{\mathrm{Bock}_{H^1(A)}[p-1]} \\ \to H^1(A)[p] \xrightarrow{\oplus} \tau^{\leq p-1}A[p+1] \end{aligned} \tag{4.10}$$

Proof. The augmentation map ε induces a splitting of the fiber sequence $H^0(A) \to \tau^{\leq 1}A \to H^1(A)[-1]$. In particular, there exists a map $s : H^1(A)[-1] \to A$ inducing an isomorphism on H^1, so we are in a position to apply Theorem 4.1. The formula (4.1) specializes to (4.10) because under the decompositions $\tau^{\leq 1}A \simeq H^0(A) \oplus H^1(A)[-1]$ and $\tau^{\leq 1}(A/p) \simeq H^0(A/p) \oplus H^1(A/p)[-1]$ the Bockstein and Frobenius morphisms are diagonalized. □

4.1. **Equivariant situation.** Suppose that X is a flat $\mathbb{Z}_p$-scheme equipped with an action of a discrete group G. Let us explicitly record that Theorem 4.1 applied to the global quotient $[X/G]$ can be rephrased as a statement about G-equivariant algebras on X, where we take the definition of the category of G-equivariant derived commutative algebras on X to be $\mathrm{DAlg}([X/G])$. We state the result in the augmented setting because this is the version that will be used in all the applications.

Theorem 4.4. *Given a G-equivariant augmented derived commutative algebra A on X, such that $H^0(A) = \mathcal{O}_X$, the sheaf $H^1(A)$is a locally free sheaf of $\mathcal{O}_X$-modules, and the multiplication induces isomorphisms $\Lambda^\bullet H^1(A) \simeq H^\bullet(A)$, we have*

(1) *There is an equivalence $\tau^{\leq p-1}A \simeq \bigoplus_{i=0}^{p-1} H^i(A)[-i]$ in $D_G(X)$.*
(2) *The map $H^p(A) \to \tau^{\leq p-1}A[p+1]$ corresponding to the fiber sequence $\tau^{\leq p-1}A \to \tau^{\leq p}A \to H^p(A)[-p]$ in $D_G(X)$ can be described as*

(4.11)
$$H^p(A) \xrightarrow{\alpha(H^1(A)/p)} F^*(H^1(A)/p)[p-1] \xrightarrow{\varphi_{A/p}} (H^1(A)/p)[p-1] \xrightarrow{\mathrm{Bock}_{H^1(A)}[p-1]} H^1(A)[p] \xrightarrow{\oplus} \tau^{\leq p-1}A[p+1].$$

5. Applications to de Rham and Hodge-Tate cohomology

In this section we apply Theorem 4.1 to de Rham and diffracted Hodge complexes. Let k be a perfect field of characteristic $p > 0$. We now work in a setup where X is a formally smooth formal scheme over $W(k)$, and as before denote by X_0 its special fiber $X \times_{W(k)} k$. We denote by $F_{X_0/k} : X_0 \to X_0^{(1)} := X_0 \times_{k,\mathrm{Fr}_p} k$ the relative Frobenius morphism, and as before $F_{X_0} : X_0 \to X_0$ denotes the absolute Frobenius morphism. We have the following cohomological invariants associated to X and X_0, each equipped with a derived commutative algebra structure.

- The diffracted Hodge complex $\Omega^{\not{D}}_X \in D(X)$ defined in [BL22a, Notation 4.7.12] whose cohomology algebra $H^\bullet(\Omega^{\not{D}}_X)$ is isomorphic to the algebra $\Omega^\bullet_X$ of differential forms on X. By [BL22b, Theorem 7.20(2)] it can be identified with the derived pushforward $R\pi^{\mathrm{HT}}_*\mathcal{O}_{X^{\not{D}}}$ of the structure sheaf along the map $\pi^{\mathrm{HT}} : X^{\not{D}} \to X$, hence Lemma 2.13 equips $\Omega^{\not{D}}_X$ with a structure of a derived commutative algebra in $D(X)$. The Sen operator on $\Omega^{\not{D}}_X$ induces a decomposition $\tau^{\leq p-1}\Omega^{\not{D}}_X \simeq \bigoplus_{i=0}^{p-1} \Omega^i_X[-i]$ in $D(X)$, and, in particular, gives rise to a map $s : \Omega^1_X[-1] \to \Omega^{\not{D}}_X$ that induces an isomorphism on H^1.
- The de Rham complex $\mathrm{dR}_{X_0/k} = F_{X_0/k*}\mathcal{O}_{X_0} \xrightarrow{d} F_{X_0/k*}\Omega^1_{X_0/k} \xrightarrow{d} \dots$ viewed as an object of $D(X_0^{(1)})$. The Cartier isomorphism provides an identification $H^\bullet(\mathrm{dR}_{X_0/k}) \simeq \Omega^\bullet_{X_0^{(1)}/k}$ of graded algebras. By de Rham comparison, $\mathrm{dR}_{X_0/k}$ is naturally identified with $\Omega^{\not{D}}_{X^{(1)}}/p$, where $X^{(1)} := X \times_{W(k),\mathrm{Fr}_p} W(k)$ is the

Frobenius-twist of the formal $W(k)$-scheme X. We give $\mathrm{dR}_{X_0/k}$ the structure of an object of $\mathrm{DAlg}(X_0^{(1)})$ by identifying it with the derived pushforward of the structure sheaf along the map $\pi^{\mathrm{HT}} : (X_0^{(1)})^{\not{D}} \to X_0^{(1)}$ using Lemma 2.13.

Remark 5.1. The diffracted Hodge complex can be identified with the Hodge-Tate cohomology of X relative to an appropriate prism, by [BL22a, Example 4.7.8]. Theorem 4.1 also applies to Hodge-Tate cohomology of smooth formal schemes over arbitrary prisms, as well as to the decompleted version of the diffracted Hodge cohomology [BL22a, Construction 4.9.1].

We will apply Theorem 4.1 to $\Omega^{\not{D}}_{X^{(1)}}$ and will then compute the extension in the canonical filtration on $\mathrm{dR}_{X_0/k}$ by reducing modulo p. To begin, we will relate the Frobenius endomorphism of the de Rham complex to the obstruction to lifting Frobenius onto $X \times_{W(k)} W_2(k)$.

5.1. **Obstruction to lifting Frobenius over $W_2(k)$.** We prove the results in this subsection without the smoothness assumption, and only assuming the existence of a flat lift over $W_2(k)$, for a future application in Section 7. For a scheme Y_0 over k equipped with a lift Y_1 over $W_2(k)$ we denote by $\mathrm{ob}_{F,Y_1} : F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \to \mathcal{O}_{Y_0}[1]$ the obstruction to lifting $F_{Y_0/k} : Y_0 \to Y_0^{(1)}$ to a morphism $Y_1 \to Y_1^{(1)}$, as defined by Illusie [Ill71]. We also denote by $\mathrm{dR}_{Y_0/k}$ the derived de Rham complex of Y_0 relative to k, viewed as an object of $D(Y_0^{(1)})$, cf. [Bha12, §3]. It is equipped with a filtration $\mathrm{Fil}^\bullet_{\mathrm{conj}}$ whose graded pieces are equivalent to the shifted exterior powers of the cotangent complex: $\mathrm{gr}^i_{\mathrm{conj}} \mathrm{dR}_{Y_0/k} \simeq L\Omega^i_{Y_0^{(1)}/k}[-i]$. Note also that the natural map induces an equivalence $\mathrm{dR}_{Y_0/\mathbb{F}_p} \simeq \mathrm{dR}_{Y_0/k}$.

Using the relation between the cotangent complex of Y_0 over $W(k)$ and the de Rham complex of Y_0, due to [BS22] and [Ill20] we will prove:

Proposition 5.2. *Let Y_0 be a quasisyntomic scheme over k equipped with a flat lift Y_1 over $W_2(k)$. Denote by $s : L\Omega^1_{Y_0^{(1)}/k}[-1] \to \mathrm{dR}_{Y_0/k}$ the splitting of the conjugate filtration in degree 1 arising from Y_1. The composition*

$$F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k}[-1] \xrightarrow{F^*_{Y_0/k} s} F^*_{Y_0/k} \mathrm{Fil}_1^{\mathrm{conj}} \mathrm{dR}_{Y_0/k} \xrightarrow{dF_{Y_0^{(-1)}/k}} \mathrm{Fil}_1^{\mathrm{conj}} \mathrm{dR}_{Y_0^{(-1)}/k} \tag{5.1}$$

is homotopic to the composition $F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}}[-1] \xrightarrow{\mathrm{ob}_{F,Y_1}} \mathcal{O}_{Y_0} \to \mathrm{Fil}_1^{\mathrm{conj}} \mathrm{dR}_{Y_0^{(-1)}}$. Here $Y_0^{(-1)}$ is the twist $Y_0 \times_{k,\mathrm{Fr}_p^{-1}} k$ by the inverse of Frobenius and $dF_{Y_0^{(-1)}/k}$ is the map induced by the functoriality of the de Rham complex.

Remark 5.3. For Y_0 smooth over k this result also follows from the explicit model for the map $s : \Omega^1_{Y_0^{(1)}/k} \to \mathrm{dR}_{Y_0/k}$ using local Frobenius lifts, provided by [DI87], cf. [Sri90].

Remark 5.4. Using the isomorphism $Y_0^{(1)} \simeq Y_0$ of $\mathbb{F}_p$-schemes we may view ob_{F,Y_1} as a map $F^*_{Y_0} L\Omega^1_{Y_0}[-1] \to \mathcal{O}_{Y_0}[1]$ that coincides with the obstruction to lifting the absolute Frobenius morphism $F_{Y_0} : Y_0 \to Y_0$ to an endomorphism of Y_1, we will denote this map by the same symbol ob_{F,Y_1}.

Convention 5.5. We take ob_{F,Y_1} to mean the *negative* of the obstruction class defined in [Ill71], to avoid a trailing sign in all of the subsequent expressions.

We start by recalling from [Ill20] how a lift of Y_0 over $W_2(k)$ provides a decomposition of $L\Omega^1_{Y_0/W(k)}$. In general, if Y_0 is a quasisyntomic scheme over k we have the fundamental triangle corresponding to the morphisms $Y_0 \to \operatorname{Spec} k \to \operatorname{Spec} W(k)$

$$\mathcal{O}_{Y_0}[1] \simeq \mathcal{O}_{Y_0} \otimes_k L\Omega^1_{k/W(k)} \to L\Omega^1_{Y_0/W(k)} \to L\Omega^1_{Y_0/k} \tag{5.2}$$

The natural map $\operatorname{fib}(L\Omega^1_{Y_0/W(k)} \to L\Omega^1_{Y_0/k}) \to \operatorname{fib}(L\Omega^1_{Y_0/W_2(k)} \to L\Omega^1_{Y_0/k})$ establishes the source as a direct summand of the target, hence a flat scheme Y_1 over $W_2(k)$ lifting Y_0 induces a splitting of this fiber sequence via the map $di : L\Omega^1_{Y_0/k} \simeq i^* L\Omega^1_{Y_1/W_2(k)} \to L\Omega^1_{Y_0/W_2(k)}$, where $i : Y_0 \hookrightarrow Y_1$ is the inclusion of the special fiber, cf. [Ill20, §4]. We denote by $s'_{Y_1} : L\Omega^1_{Y_0/k} \to L\Omega^1_{Y_0/W(k)}$ the resulting section of (5.2).

Lemma 5.6. *For a flat scheme Y_1 over $W_2(k)$ the obstruction $\mathrm{ob}_{F,Y_1} : F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \to \mathcal{O}_{Y_0}[1]$ to lifting $F_{Y_0/k} : Y_0 \to Y_0^{(1)}$ to a morphism from Y_1 to $Y_1^{(1)}$ is homotopic to the composition*

$$F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \xrightarrow{F^*_{Y_0/k} di^{(1)}} F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/W_2(k)} \xrightarrow{dF_{Y_0/k}} L\Omega^1_{Y_0/W_2(k)} \to \mathcal{O}_{Y_0}[1] \tag{5.3}$$

Proof. Let us start by recalling the definition of this obstruction given in [Ill71, Proposition III.2.2.4]. The scheme Y_1 viewed as a lift of Y_0 produces a map $L\Omega^1_{Y_0/W_2(k)} \to \mathcal{O}_{Y_0}[1]$, and composing it with $dF_{Y_0/k}$ we get a map

$$\gamma_{Y_0} : F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/W_2(k)} \xrightarrow{dF_{Y_0/k}} L\Omega^1_{Y_0/W_2(k)} \to \mathcal{O}_{Y_0}[1] \tag{5.4}$$

On the other hand, the lift $Y_1^{(1)}$ of $Y_0^{(1)}$ gives rise to a map $L\Omega^1_{Y_0^{(1)}/W_2(k)} \to \mathcal{O}_{Y_0^{(1)}}[1]$, and pulling it back along $F_{Y_0/k}$ we obtain a map

$$\gamma_{Y_0^{(1)}} : F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/W_2(k)} \to F^*_{Y_0/k} \mathcal{O}_{Y_0^{(1)}}[1] \simeq \mathcal{O}_{Y_0}[1] \tag{5.5}$$

By definition, $\mathrm{ob}_{F,Y_1} : F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \to \mathcal{O}_{Y_0}[1]$ is the unique (up to a contractible space of choices) map such that the composition $F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/W_2(k)} \to F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \xrightarrow{\mathrm{ob}_{F,Y_1}} \mathcal{O}_{Y_0}[1]$ is homotopic to the difference $\gamma_{Y_0} - \gamma_{Y_0^{(1)}}$ (we have incorporated Convention 5.5 at this point). Equivalently, ob_{F,Y_1} is the composition

$$F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \xrightarrow{F^*_{Y_0/k} di^{(1)}} F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/W_2(k)} \xrightarrow{\gamma_{Y_0} - \gamma_{Y_0^{(1)}}} \mathcal{O}_{Y_0}[1] \tag{5.6}$$

However, the composition $F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/k} \xrightarrow{F^*_{Y_0/k} di^{(1)}} F^*_{Y_0/k} L\Omega^1_{Y_0^{(1)}/W_2(k)} \xrightarrow{\gamma_{Y_0^{(1)}}} \mathcal{O}_{Y_0}[1]$ is zero by construction, so the lemma follows from the definition of γ_{Y_0}. □

Proof of Proposition 5.2. [BS22, Proposition 4.15] in the smooth case, and [Ill20, Corollary 3.3] in general identifies $\mathrm{Fil}_1^{\mathrm{conj}} \mathrm{dR}_{Y_0/k}$ with the shifted cotangent complex $L\Omega^1_{Y_0^{(1)}/W(k)}[-1]$. Moreover, the fiber sequence $\mathcal{O}_{Y_0^{(1)}} \to \mathrm{Fil}_1^{\mathrm{conj}} \mathrm{dR}_{Y_0/k} \to L\Omega^1_{Y_0^{(1)}/k}[-1]$ induced by the conjugate filtration is identified with the shift of the fundamental triangle

$$L\Omega^1_{k/W(k)} \otimes_k \mathcal{O}_{Y_0^{(1)}} \to L\Omega^1_{Y_0^{(1)}/W(k)} \to L\Omega^1_{Y_0^{(1)}/k} \tag{5.7}$$

corresponding to the sequence of morphisms $Y_0^{(1)} \to \operatorname{Spec} k \to \operatorname{Spec} W(k)$. Denote by $i : Y_0^{(1)} \hookrightarrow Y_1^{(1)}$ the usual inclusion. We have a map $s'_{Y_0^{(1)}} : L\Omega^1_{Y_0^{(1)}/k} \to L\Omega^1_{Y_0^{(1)}/W_2(k)}$ that splits this fundamental triangle, hence defining a map $s'_{Y_1^{(1)}} : L\Omega^1_{Y_0^{(1)}/k}[-1] \to \mathrm{Fil}_1^{\mathrm{conj}}\mathrm{dR}_{Y_0/k}$. By the uniqueness of functorial decompositions of the de Rham complex [LM21, Theorem 5.10], $s'_{Y_1^{(1)}}$ is naturally equivalent to the section $s_{Y_1^{(1)}}$ constructed using the Sen operator.

Since the identification $\mathrm{Fil}_1^{\mathrm{conj}}\mathrm{dR}_{Y_0/k} \simeq L\Omega^1_{Y_0/W(k)}[-1]$ is functorial in Y_0, the map $F^*_{Y_0/k}\mathrm{Fil}_1^{\mathrm{conj}}\mathrm{dR}_{Y_0/k} \to \mathrm{Fil}_1^{\mathrm{conj}}\mathrm{dR}_{Y_0^{(-1)}/k}$ induced by $F_{Y_0/k}$ is identified with the shift of $dF_{Y_0/k} : F^*_{Y_0/k}L\Omega^1_{Y_0^{(1)}/W(k)} \to L\Omega^1_{Y_0/W(k)}$. The latter map factors as

$$F^*_{Y_0/k}L\Omega^1_{Y_0^{(1)}/W(k)} \to \mathcal{O}_{Y_0} \to L\Omega^1_{Y_0/W(k)} \tag{5.8}$$

because $dF_{Y_0/k} : F^*_{Y_0/k}L\Omega^1_{Y_0^{(1)}/k} \to L\Omega^1_{Y_0/k}$ is zero, and precomposing it with $F^*_{Y_0/k}\Omega^1_{Y_0^{(1)}} \xrightarrow{F^*_{Y_0/k}s'_{Y_1^{(1)}}} F^*_{Y_0/k}L\Omega^1_{Y_0^{(1)}/W(k)}$ we complete the proof of Proposition 5.2 by the formula for the obstruction to lifting Frobenius given by Lemma 5.6. □

The Frobenius arising from the derived commutative structure on the de Rham complex coincides with the map induced by the geometric Frobenius morphism:

Lemma 5.7. *For a smooth scheme X_0 over k the Frobenius map $\varphi_{\mathrm{dR}_{X_0/k}} : F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k} \to \mathrm{dR}_{X_0/k}$ of the derived commutative algebra $\mathrm{dR}_{X_0/k} \in \mathrm{DAlg}(X_0^{(1)})$ is naturally identified with the map $dF_{X_0^{(1)}} : F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k} \to \mathrm{dR}_{X_0/k}$ induced by the functoriality of the de Rham complex under the absolute Frobenius endomorphism.*

Explicitly, this morphism is the composition

$$F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k} \to F^*_{X_0^{(1)}}F_{X_0/k*}\mathcal{O}_{X_0} \xrightarrow{\varphi_{X_0^{(1)}}} \mathcal{O}_{X_0^{(1)}} \to \mathrm{dR}_{X_0/k} \tag{5.9}$$

where the first map is induced by the map from the de Rham complex to its 0th term.

Proof. We endowed $\mathrm{dR}_{X_0/k}$ with the structure of a derived commutative algebra by identifying it with the pushforward $R\pi^{\not{D}}_*\mathcal{O}_{(X_0^{(1)})^{\not{D}}}$ of the structure sheaf along the morphism of stacks $\pi^{\not{D}} : (X_0^{(1)})^{\not{D}} \to X_0^{(1)}$. By Lemma 2.14, the map $\varphi_{\mathrm{dR}_{X_0/k}} : F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k} \to \mathrm{dR}_{X_0/k}$ is the composition $F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k} = F^*_{X_0^{(1)}}R\pi^{\mathrm{HT}}_*\mathcal{O}_{(X_0^{(1)})^{\not{D}}} \to R\pi^{\mathrm{HT}}_*F^*_{(X_0^{(1)})^{\not{D}}}\mathcal{O}_{(X_0^{(1)})^{\not{D}}} \simeq R\pi^{\mathrm{HT}}_*\mathcal{O}_{(X_0^{(1)})^{\not{D}}} = \mathrm{dR}_{X_0/k}$. The Frobenius endomorphism of the stack $X_0^{\not{D}}$ coincides with the endomorphism induced from $F_{X_0^{(1)}}$ by functoriality, cf. [BL22b, Remark 3.6]. Hence $\varphi_{\mathrm{dR}_{X_0/k}}$ is equivalent to the morphism induced by F_{X_0} by functoriality of the de Rham complex. The last assertion follows from the fact that the maps $dF_{X_0^{(1)}} : F^*_{X_0^{(1)}}F_{X_0/k*}\Omega^i_{X_0} \to \Omega^i_{X_0^{(1)}}$ are zero for all $i \geq 1$. □

Let us record the compatibility between two Frobenius morphisms on the level of cohomology arising from Lemma 5.7:

Lemma 5.8. *For a smooth k-scheme X_0, the Frobenius morphism $\varphi_{\mathrm{R}\Gamma_{\mathrm{dR}}(X_0/k)}$ of the derived commutative k-algebra $\mathrm{R}\Gamma_{\mathrm{dR}}(X_0/k)$ is naturally homotopic to the morphism induced by the relative Frobenius morphism $F_{X_0/k} : X_0 \to X_0^{(1)}$.*

5.2. **Proof of Theorem 5.9.** We can now prove the main result of this section:

Theorem 5.9. *For a smooth formal scheme X over $\operatorname{Spf} W(k)$ there is a natural decomposition $\tau^{\leq p-1}\Omega^{\not{D}}_X \simeq \bigoplus_{i=0}^{p-1} \Omega^i_X[-i]$ in $D(X)$. The class of the extension $\tau^{\leq p-1}\Omega^{\not{D}}_X \to \tau^{\leq p}\Omega^{\not{D}}_X \to \Omega^p_X[-p]$ in $D(X)$ is naturally equivalent to the composition*

(5.10)
$$\Omega^p_X \xrightarrow{\alpha(\Omega^1_{X_0})} F^*_{X_0}\Omega^1_{X_0}[p-1] \xrightarrow{\mathrm{ob}_{F,X\times_{W(k)}W_2(k)}} \mathcal{O}_{X_0}[p] \xrightarrow{\mathrm{Bock}_{\mathcal{O}_X}} \mathcal{O}_X[p+1] \to \Omega^{\not{D}}_X[p+1]$$

Proof. Applying Theorem 4.1 to the derived commutative algebra $\Omega^{\not{D}}_X$ on X equipped with the splitting $s : \Omega^1_X[-1] \to \Omega^{\not{D}}_X$ gives the first assertion (which by now we knew anyway thanks to the existence of the Sen operator), as well as the following formula for the extension class of $\tau^{\leq p}\Omega^{\not{D}}_X$:

(5.11)
$$\Omega^p_X \xrightarrow{\alpha(\Omega^1_{X_0})} F^*_{X_0}\Omega^1_{X_0}[p-1] \xrightarrow{F^*_{X_0}s[p]} (\tau^{\leq 1}F^*_{X_0}\mathrm{dR}_{X_0^{(-1)}/k})[p] \xrightarrow{\varphi_{\mathrm{dR}_{X_0^{(-1)}/k}}} (\tau^{\leq 1}\mathrm{dR}_{X_0^{(-1)}/k})[p]$$
$$\xrightarrow{\mathrm{Bock}_{\Omega^{\not{D}}_X}} (\tau^{\leq 1}\Omega^{\not{D}}_X)[p+1] \to (\tau^{\leq p-1})\Omega^{\not{D}}_X[p+1]$$

Using Proposition 5.2 together with Remark 5.4, and Lemma 5.7 we can rewrite this composition as

$$\Omega^p_X \xrightarrow{\alpha(\Omega^1_{X_0})} F^*_{X_0}\Omega^1_{X_0}[p-1] \xrightarrow{\mathrm{ob}_{F,X\times_{W(k)}W_2(k)}} \mathcal{O}_{X_0}[p] \to (\tau^{\leq 1}\mathrm{dR}_{X_0^{(-1)}/k})[p]$$
$$\xrightarrow{\mathrm{Bock}_{\Omega^{\not{D}}_X}} (\tau^{\leq 1}\Omega^{\not{D}}_X)[p+1] \to (\tau^{\leq p-1}\Omega^{\not{D}}_X)[p+1] \tag{5.12}$$

Converting this into (5.10) amounts to the observation that the Bockstein maps for $\Omega^{\not{D}}_X$ and $\mathcal{O}_X$ are related by the commutative square

$$\begin{tikzcd} \mathcal{O}_{X_0} \arrow[r] \arrow[d, "\mathrm{Bock}_{\mathcal{O}_X}"] & \mathrm{dR}_{X_0^{(-1)}/k} \arrow[d, "\mathrm{Bock}_{\Omega^{\not{D}}_X}"] \\ \mathcal{O}_X[1] \arrow[r] & \Omega^{\not{D}}_X[1] \end{tikzcd} \tag{5.13}$$

∎

5.3. **Cohomological consequences.** We can rewrite the answer provided by (5.10) as a formula for the extension class in $H^{p+1}(X, \Lambda^p T_X)$. Getting to this statement from Theorem 5.9 amounts to a general piece of bookkeeping concerning the Bockstein homomorphisms, given by Lemma 5.11 below.

Theorem 5.10. *The class of the extension* $\bigoplus\limits_{i=0}^{p-1}\Omega^i_X[-i]\to\tau^{\leq p}\Omega^{\not{D}}_X\to\Omega^p_X[-p]$ *in* $H^{p+1}(X,\Lambda^pT_X)$ *is equal to*

$$\mathrm{Bock}_X(\mathrm{ob}_{F,X\times_{W(k)}W_2(k)}\cup\alpha(\Omega^1_{X_0})),\tag{5.14}$$

that is the result of applying the Bockstein homomorphism $\mathrm{Bock}_X:H^p(X_0,\Lambda^pT_{X_0})\to H^{p+1}(X,\Lambda^pT_X)$ *to the product of classes* $\alpha(\Omega^1_{X_0})\in\mathrm{Ext}^{p-1}_{X_0}(\Omega^p_{X_0},F^*_{X_0}\Omega^1_{X_0})=H^{p-1}(X_0,\Lambda^pT_{X_0}\otimes F^*_{X_0}\Omega^1_{X_0})$ *and* $\mathrm{ob}_{F,X\times_{W(k)}W_2(k)}\in H^1(X_0,F^*T_{X_0})$.

Lemma 5.11. *For two objects* $M,N\in D(X)$ *denote by* $\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N)\in D(X)$ *their internal* Hom *object. The Bockstein morphism*

$$\mathrm{RHom}_{X_0}(i^*M,i^*N)=\mathrm{R}\Gamma(X_0,i^*\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N))\to\mathrm{R}\Gamma(X,\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N)[1])=\mathrm{RHom}_X(M,N[1])\tag{5.15}$$

can be described as sending $f:i^*M\to i^*N$ *to the composition*

$$M\to i_*i^*M\xrightarrow{i_*f}i_*i^*N\to N[1]\tag{5.16}$$

Proof. For an object $K\in D(X)$ there is a natural fiber sequence $K\to i_*i^*K\to K[1]$ coming from the identification $i_*i^*K\simeq\mathrm{cofib}(K\xrightarrow{p}K)$. Applying $\mathrm{R}\Gamma(X,-)$ to the second map $i^*i_*K\to K[1]$ induces the Bockstein homomorphoism $\mathrm{R}\Gamma(X_0,i^*K)\to\mathrm{R}\Gamma(X,K[1])$ on cohomology. The lemma aims to compute this map in the case $K=\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N)$. By adjunction, we have $i_*i^*\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N)\simeq i_*\underline{\mathrm{RHom}}_{\mathcal{O}_{X_0}}(i^*M,i^*N)\simeq\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,i_*i^*N)$, and the Bockstein map $i_*i^*\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N)\to\underline{\mathrm{RHom}}_{\mathcal{O}_X}(M,N)[1]$ is induced by composing with the map $i_*i^*N\to N[1]$. This proves the lemma by the observation that the adjunction equivalence $\mathrm{RHom}_{X_0}(i^*M,i^*N)\simeq\mathrm{RHom}_X(M,i_*i^*N)$ takes a map $f:i^*M\to i^*N$ to the composition $M\to i_*i^*M\xrightarrow{i_*f}i_*i^*N$. □

Reducing modulo p we also get a description of the extension class of $\tau^{\leq p}\mathrm{dR}_{X_0/k}$:

Theorem 5.12. *If* X_0 *is a smooth scheme over* k *equipped with a smooth lift* X *over* $W(k)$ *then the class of the extension* $\bigoplus\limits_{i=0}^{p-1}\Omega^i_{X_0^{(1)}}[-i]\to\tau^{\leq p}\mathrm{dR}_{X_0/k}\to\Omega^p_{X_0^{(1)}}[-p]$ *in* $H^{p+1}(X_0^{(1)},\Lambda^pT_{X_0^{(1)}})$ *is equal to*

$$\mathrm{Bock}_{X^{(1)}_{W_2(k)}}(\mathrm{ob}_{F,X^{(1)}_{W_2(k)}}\cup\alpha(\Omega^1_{X_0^{(1)}})),\tag{5.17}$$

the result of applying the Bockstein homomorphism $\mathrm{Bock}_{X^{(1)}_{W_2(k)}}:H^p(X_0^{(1)},\Lambda^pT_{X_0^{(1)}})\to H^{p+1}(X_0^{(1)},\Lambda^pT_{X_0^{(1)}})$ *to the product of classes* $\alpha(\Omega^1_{X_0^{(1)}})\in\mathrm{Ext}^{p-1}_{X_0^{(1)}}(\Omega^p_{X_0^{(1)}},F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}})=H^{p-1}(X_0^{(1)},\Lambda^pT_{X_0^{(1)}}\otimes F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}})$ *and* $\mathrm{ob}_{F,X^{(1)}_{W_2(k)}}\in H^1(X_0^{(1)},F^*_{X_0^{(1)}}T_{X_0^{(1)}})$.

For the convenience of applications, let us state explicitly what Theorem 5.10 tells us about the differentials in the Hodge-Tate spectral sequence

Corollary 5.13. *For a smooth* $W(k)$*-scheme* X *the Hodge-Tate spectral sequence* $E_2^{ij}=H^i(X,\Omega^j_{X/W(k)})\Rightarrow H^{i+j}_{\not{D}}(X)$ *has no non-zero differentials on pages* $E_2,\dots,E_p$ *and for every* $i\geq 0$ *the differential* $d^{i,p}_{p+1}:H^i(X,\Omega^p_{X/W(k)})\to H^{i+p+1}(X,\mathcal{O}_X)$ *on page* E_{p+1} *can be described as the composition*

$$H^i(X,\Omega^p_{X/W(k)}) \to H^i(X_0,\Omega^p_{X_0/k}) \xrightarrow{\alpha(\Omega^1_{X_0})} H^{i+p-1}(X_0, F^*_{X_0}\Omega^1_{X_0/k}) \xrightarrow{\mathrm{ob}_{F,X}} H^{i+p}(X_0,\mathcal{O}_{X_0}) \xrightarrow{\mathrm{Bock}_X} H^{i+p+1}(X,\mathcal{O}_X). \tag{5.18}$$

5.4. **Decomposing de Rham complex compatibly with the algebra structure.** Another application of the results of Section 3 is an obstruction to formality of the de Rham complex as a commutative algebra. In what follows, by the derived commutative algebra $\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]$ we mean the free divided power algebra on the object $\Omega^1_{X_0^{(1)}}[-1]\in D(X_0^{(1)})$, see Definition 2.15.

Proposition 5.14. *Let X_0 be an arbitrary smooth scheme over k. The following are equivalent*

(1) $\mathrm{dR}_{X_0/k}$ is equivalent to $\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]$ as a derived commutative algebra in $D(X_0^{(1)})$

(2) $\mathrm{dR}_{X_0/k}$ is equivalent to $\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]$ as an E_∞-algebra in $D(X_0^{(1)})$

(3) there exists a map $\mathrm{dR}_{X_0/k}\to\mathcal{O}_{X_0^{(1)}}$ of E_∞-algebras in $D(X_0^{(1)})$ that induces an isomorphism $H^0(\mathrm{dR}_{X_0/k})\simeq\mathcal{O}_{X_0^{(1)}}$

(4) X_0 together with its Frobenius endomorphism admits a lift over $W_2(k)$.

Proof. Formally, (1) implies (2), and (2) implies (3). Let us first show that (4) implies (1). It is well-known that (4) implies the formality of $\mathrm{dR}_{X_0/k}$ as a commutative DG algebra [DI87, Remarque 2.2(ii)], or as an E_∞-algebra [Bha12, Proposition 3.17]. We have not discussed the relation between commutative DG algebras and derived commutative algebras, so let us give an independent argument for the equivalence $\mathrm{dR}_{X_0/k}\simeq\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}$ in $\mathrm{DAlg}(X_0)$, in the presence of a Frobenius lift over $W_2(k)$.

For this equivalence, we will represent $\mathrm{dR}_{X_0/k}$ by the Čech complex associated to the flat cover $X_0^{\mathrm{perf}}\to X_0$ where $X_0^{\mathrm{perf}}:=\varprojlim X_0^{(-n)}$ is the perfection of X_0. By fpqc descent for derived de Rham cohomology we have that $\mathrm{dR}_{X_0/k}$ is equivalent to the cosimplicial totalization of the following diagram of derived commutative algebras in $D(X_0^{(1)})$, cf. [BMS19, Remark 8.15]:

$$\mathrm{dR}_{X_0^{\mathrm{perf}}/k} \rightrightarrows \mathrm{dR}_{X_0^{\mathrm{perf}}\times_{X_0}X_0^{\mathrm{perf}}/k} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots \tag{5.19}$$

Since each $(X_0^{\mathrm{perf}})^{\times_{X_0}n}$ is a quasiregular semiperfect scheme, all terms of this diagram are classical commutative algebras in $\mathrm{QCoh}(X_0^{(1)})$ placed in degree 0. The lift X_1 of X_0 together with a Frobenius endomorphism induces lifts of the scheme X_0^{perf}, the morphism $X_0^{\mathrm{perf}}\to X_0$, and the Frobenius endomorphism of X_0^{perf}. Therefore each algebra $\mathrm{dR}_{(X_0^{\mathrm{perf}})^{\times_{X_0}n}/k}$ decomposes as the Frobenius-twist of $\Gamma^\bullet(L\Omega^1_{(X_0^{\mathrm{perf}})^{\times_{X_0}n}/k}[-1])$, by [Bha12, Proposition 3.17]. Moreover all the maps in (5.19) are compatible with these decompositions, so the cosimplicial commutative algebra defined by this diagram is quasi-isomorphic to $\Gamma^\bullet_{\mathrm{naive}}(\mathrm{DK}(\Omega^1_{X_0^{(1)}/k}[-1]))$. This cosimplicial commutative algebra is a model for the derived commutative algebra $\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]$ by Lemma 2.23, so the implication (4)⇒(1) is proven.

This is not logically necessary, but we will first give the proof of the implication (1)⇒(4) to illustrate the idea of the implication (3)⇒(4) in a simpler case. Suppose that such an equivalence $\alpha:\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]\simeq \mathrm{dR}_{X_0/k}$ of derived commutative algebras exists. In particular, $\tau^{\leq 1}\mathrm{dR}_{X_0/k}$ is decomposed as $\mathcal{O}_{X_0^{(1)}}\oplus\Omega^1_{X_0^{(1)}/k}[-1]$. By [DI87, Théorème 3.5] any such decomposition is induced, up to a homotopy, by a lift of X_0 over $W_2(k)$. Denote the lift corresponding to $\tau^{\leq 1}\alpha$ by X_1.

The Frobenius endomorphism $\varphi_{\bigoplus\Omega^i_{X_0^{(1)}/k}[-i]}:F^*_{X_0^{(1)}}\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]\to\bigoplus\limits_{i\geq 0}\Omega^i_{X_0^{(1)}/k}[-i]$ is zero on all components with $i\geq 1$ and is the usual adjunction map $F^*_{X_0^{(1)}}\mathcal{O}_{X_0^{(1)}}\xrightarrow{\sim}\mathcal{O}_{X_0^{(1)}}$ on the structure sheaf. Hence composing the section $\Omega^1_{X_0^{(1)}}[-1]\to\mathrm{dR}_{X_0/k}$ induced by α with the Frobenius map $\varphi_{\mathrm{dR}_{X_0/k}}:F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k}\to\mathrm{dR}_{X_0/k}$ is homotopic to zero. By Proposition 5.2 and Lemma 5.7 this implies that the composition $F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}}[-1]\xrightarrow{\mathrm{ob}_{F,X_1}}\mathcal{O}_{X_0^{(1)}}\to\mathrm{dR}_{X_0/k}$ is homotopic to zero. Since the second map admits a section, it follows that $\mathrm{ob}_{F,X_1}=0$ and X_1 is a lift of X_0 that admits a lift of Frobenius.

Finally, we prove that (3) implies (4). The augmentation map $\varepsilon:\mathrm{dR}_{X_0/k}\to\mathcal{O}_{X_0^{(1)}}$ induces, in particular, a map $s:\Omega^1_{X_0^{(1)}}[-1]$ in $D(X_0^{(1)})$ such that the composition $\varepsilon\circ s$ is zero. Denote by X_1 the lift of X_0 corresponding to this section s. The map $s:\Omega^1_{X_0^{(1)}}[-1]\to\mathrm{dR}_{X_0/k}$ also induces a map $s_p:S^p(\Omega^1_{X_0^{(1)}}[-1])\to\mathrm{dR}_{X_0/k}$. The assumption that ε is a map of E_∞-algebras implies that the composition

$$(\Omega^1_{X_0^{(1)}}[-1]^{\otimes p})_{hS_p}\to S^p(\Omega^1_{X_0^{(1)}}[-1])\xrightarrow{s_p}\mathrm{dR}_{X_0/k}\xrightarrow{\varepsilon}\mathcal{O}_{X_0^{(1)}}\tag{5.20}$$

is homotopic to zero. We will deduce from this that the composition $T_p(\Omega^1_{X_0^{(1)}/k}[-1])[-1]\to S^p\Omega^1_{X_0^{(1)}}[-1]\to\mathrm{dR}_{X_0/k}\xrightarrow{\varepsilon}\mathcal{O}_{X_0^{(1)}}$ is homotopic to zero.

Denote by K the fiber of the map $(\Omega^1_{X_0^{(1)}}[-1]^{\otimes p})_{hS_p}\to\Omega^p_{X_0^{(1)}}[-p]$, it is equipped with a map $K\to T_p(\Omega^1_{X_0^{(1)}}[-1])[-1]$, and the object $\mathrm{fib}(K\to T_p(\Omega^1_{X_0^{(1)}}[-1])[-1])\simeq\mathrm{fib}((\Omega^1_{X_0^{(1)}}[-1]^{\otimes p})_{hS_p}\to S^p\Omega^1_{X_0^{(1)}}[-1])$ is concentrated in degrees ≤ 0 by Lemma 3.8. Our assumption implies that the map $T_p(\Omega^1_{X_0^{(1)}}[-1])[-1]\to\mathrm{dR}_{X_0/k}\xrightarrow{\varepsilon}\mathcal{O}_{X_0^{(1)}}$ factors through the map $T_p(\Omega^1_{X_0^{(1)}}[-1])[-1]\to\mathrm{fib}(K[1]\to T_p(\Omega^1_{X_0^{(1)}}[-1]))$ which forces it to be homotopic to zero because this fiber object is concentrated in degrees ≥ -1.

In Lemma 3.1(4) we constructed a map $F^*_{X_0}M\to T_p(M)[-1]$ for any $M\in D(X_0)$ such that the composition $F^*_{X_0}M\to T_p(M)[-1]\xrightarrow{\gamma_M}S^pM$ is the map $\Delta_M:F^*_{X_0}M\to S^pM$. By definition, Frobenius map is the composition $F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k}\xrightarrow{\Delta_{\mathrm{dR}_{X_0/k}}}S^p\mathrm{dR}_{X_0/k}\xrightarrow{m}\mathrm{dR}_{X_0/k}$ which allows us to conclude that the composition

$$F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}}[-1]\to T_p(\Omega^1_{X_0^{(1)}}[-1])[-1]\to S^p\Omega^1_{X_0^{(1)}}[-1]\xrightarrow{s_p}\mathrm{dR}_{X_0/k}\tag{5.21}$$

is homotopic to the composition

$$F^*_{X_0^{(1)}}\Omega^1_{X_0^{(1)}}[-1] \xrightarrow{F^*_{X_0^{(1)}}s} F^*_{X_0^{(1)}}\mathrm{dR}_{X_0/k} \xrightarrow{\varphi_{\mathrm{dR}_{X_0/k}}} \mathrm{dR}_{X_0/k} \tag{5.22}$$

We are given that the composition of (5.22) with $\varepsilon : \mathrm{dR}_{X_0/k} \to \mathcal{O}_{X_0^{(1)}}$ is homotopic to zero, which is equivalent to the vanishing of the obstruction to lifting F_{X_0} on X_1, by Proposition 5.2. □

6. Preliminaries on the Sen operator

In this section we collect preliminary material on the Sen operator, and more generally on the derived category of sheaves equipped with an endomorphism. We first discuss the Sen operator on diffracted Hodge cohomology of a scheme over $\mathbb{Z}_p$ and relate its non-semisimplicity to non-decomposability of the de Rham complex, and then give a parallel discussion over $\mathbb{Z}/p^n, n \geq 2$ for which we, in particular, describe the Hodge-Tate locus of the Cartier-Witt stack of $\mathbb{Z}/p^n$, generalizing [BL22b, Example 5.15]. For the purposes of the application to the de Rham complex in Section 7, the case of $\mathbb{Z}/p^n$ subsumes that of $\mathbb{Z}_p$, but we invite the reader to consider the case of schemes over $\mathbb{Z}_p$ first.

6.1. **Category of objects with an endomorphism.** Let X be a flat formal scheme over $W(k)$ and as before $D(X)$ is the ∞-category of quasicoherent sheaves on X. In this section we will work in the ∞-category $D_{\mathbb{N}}(X) := \mathrm{Func}(B\mathbb{N}, D(X))$ of objects of $D(X)$ equipped with an additional endomorphism. Objects of this category are pairs (M, f_M) where M is an object of $D(X)$ and $f_M : M \to M$ is an endomorphism of M in $D(X)$. The morphisms between objects (M, f_M) and (N, f_N) are given by

(6.1)
$$\mathrm{RHom}_{D_{\mathbb{N}}(X)}((M, f_M), (N, f_N)) = \mathrm{fib}(\mathrm{RHom}_{D(X)}(M, N) \xrightarrow{f \mapsto f\circ f_M - f_N \circ f} \mathrm{RHom}_{D(X)}(M, N))$$

For a scalar $\lambda \in \mathbb{Z}_p$ we have the functor $D(X) \to D_{\mathbb{N}}(X)$ sending an object $M \in D(X)$ to itself equipped with the endomorphism $\lambda \cdot \mathrm{Id}_M$. We will often use the following special cases of the formula (6.1):

Lemma 6.1. *For an arbitrary object $(M, f_M) \in D_{\mathbb{N}}(X)$ the morphisms between it and an object of the form $(N, \lambda \cdot \mathrm{Id}_N)$ can be described as*

$$\mathrm{RHom}_{D_{\mathbb{N}}(X)}((M, f_M), (N, \lambda \cdot \mathrm{Id}_N)) = \mathrm{RHom}_{D(X)}(\mathrm{cofib}(f_M - \lambda \cdot \mathrm{Id}_M : M \to M), N) \tag{6.2}$$

$$\mathrm{RHom}_{D_{\mathbb{N}}(X)}((N, \lambda \cdot \mathrm{Id}_N), (M, f_M)) = \mathrm{RHom}_{D(X)}(N, M^{f_M = \lambda}) \tag{6.3}$$

6.2. **Diffracted Hodge cohomology and Sen operator over $\mathbb{Z}_p$.** The works of Drinfeld [Dri21] and Bhatt-Lurie imply that the diffracted Hodge cohomology is equipped with a natural endomorphism, referred to by [BL22a] as 'Sen operator'. We work with arbitrary (not necessarily smooth over $W(k)$) flat formal schemes because our proof will proceed through a computation with diffracted Hodge cohomology of quasiregular semiperfectoid rings.

Theorem 6.2 ([BL22a, Construction 4.7.1])**.** *For a bounded p-adic formal scheme X there is a natural object $(\Omega^{\not D}_X, \Theta_X) \in D_{\mathbb{N}}(X)$ whose underlying object is the diffracted*

Hodge complex, equipped with a filtration $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ *such that the graded quotients* $\mathrm{gr}^{\mathrm{conj}}_i$ *of this filtration are equivalent to* $(L\Omega^i_X[-i], -i)$.

From now on we assume that X is a flat formal $\mathbb{Z}_p$-scheme. When X is formally smooth over $W(k)$ for some perfect field k, the conjugate filtration on $\Omega^{\not D}_X$ coincides with the canonical filtration, and Θ_X acts on $H^i(\Omega^{\not D}_X) \simeq \Omega^i_X$ by $(-i)\cdot \mathrm{Id}_{\Omega^i_X}$. The interesting information contained in this new cohomological invariant of X is the data of extensions between the graded quotients of the conjugate filtration. One of our main goals, achieved in Theorem 7.1, is to explicate what this information is for the smallest potentially non-split step $(\mathrm{Fil}^p_{\mathrm{conj}}\Omega^{\not D}_X, \Theta_X)$ of the conjugate filtration.

Given that Θ_X acts via multiplication by $-i$ on $\mathrm{gr}^{\mathrm{conj}}_i\,\Omega^{\not D}_X$, the product $\Theta_X(\Theta_X + 1)\dots(\Theta_X+i)$ is naturally homotopic to zero as an endomorphism of $\mathrm{Fil}^{\mathrm{conj}}_i\Omega^{\not D}_X$. Therefore for each i the object $\mathrm{Fil}^{\mathrm{conj}}_i\Omega^{\not D}_X \in D_{\mathbb{N}}(X)$ is naturally equipped with the structure of a $\mathbb{Z}_p[t]/t(t+1)\dots(t+i)$-module, and the decomposition of the spectrum of this ring into a union of connected components induces a decomposition

$$(\Omega^{\not D}_X, \Theta_X) \simeq \bigoplus_{i=0}^{p-1} (\Omega^{\not D}_{X,i}, \Theta_X) \tag{6.4}$$

such that for every $n \in \mathbb{Z}_p$ the endomorphism $\Theta_X + n$ of $\mathrm{Fil}_i\Omega^{\not D}_{X, n \bmod p}$ is topologically nilpotent for every i, cf. [BL22a, Remark 4.7.20]. Restricting to the special fiber $X_0 := X \times_{\mathbb{Z}_p} \mathbb{F}_p \overset{i}{\hookrightarrow} X$ we similarly get an endomorphism Θ_X of dR_{X_0}, and a decomposition

$$(\mathrm{dR}_{X_0}, \Theta_X) \simeq \bigoplus_{i=0}^{p-1} (\mathrm{dR}_{X_0,i}, \Theta_X) \tag{6.5}$$

We will now set up a way to package the information about the extensions in the conjugate filtration on $(\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_X, \Theta_X)$. There is a decomposition $(\mathrm{Fil}^{\mathrm{conj}}_{p-1}\Omega^{\not D}_X, \Theta_X) \simeq \bigoplus_{i=0}^{p-1}(L\Omega^i_X[-i], -i)$ and the fiber sequence $(\mathrm{Fil}^{\mathrm{conj}}_{p-1}\Omega^{\not D}_X, \Theta_X) \to (\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_X, \Theta_X) \to (L\Omega^p_X[-p], -p)$ gives rise to the map in $D_{\mathbb{N}}(X)$:

$$(L\Omega^p_X[-p], -p) \to \bigoplus_{i=0}^{p-1} (L\Omega^i_X[-i], -i)[1] \tag{6.6}$$

In the remainder of this section by $L\Omega^i_{X_0}$ we will mean the ith exterior power of the cotangent complex of X_0 relative to $\mathbb{F}_p$, so that $L\Omega^i_{X_0} \simeq i^* L\Omega^i_X$.

Lemma 6.3. *There is a natural equivalence*

$$\mathrm{Map}_{D_{\mathbb{N}}(X)}\Big((L\Omega^p_X[-p], -p), \bigoplus_{i=0}^{p-1}(L\Omega^i_X[-i], -i)[1]\Big) \simeq \mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p]) \tag{6.7}$$

Under this identification the map from the LHS to $\mathrm{Map}_{D(X)}(L\Omega^p_X[-p], \bigoplus_{i=0}^{p-1} L\Omega^i_X[-i])$ *induced by the forgetful functor* $D_{\mathbb{N}}(X) \to D(X)$ *sends* $f \in \mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p])$ *to the composition* $L\Omega^p_X \to L\Omega^p_{X_0} \xrightarrow{f} \mathcal{O}_{X_0}[p] \xrightarrow{\mathrm{Bock}_{\mathcal{O}_X}[p]} \mathcal{O}_X[p+1]$.

Proof. By Lemma 6.1 the left hand side of (6.7) is equivalent to $\mathrm{Map}_{D(X)}(L\Omega^p_X, \mathcal{O}_X^{p=0}[p+1])$, because the summands $(L\Omega^i_X[-i], -i)$ with $0 < i \leq p-1$ do not contribute to this mapping space. The fiber of multiplication by p on $\mathcal{O}_X$ is identified with $\mathcal{O}_X/p[-1] = i_*i^*\mathcal{O}_X[-1]$ hence this mapping space can be further described as $\mathrm{Map}_{D(X)}(L\Omega^p_X, i_*i^*\mathcal{O}_X[p]) \simeq \mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p])$.

For the second assertion note that, firstly, under the identification $\mathrm{Map}_{D_{\mathbb{N}}(X)}((L\Omega^p_X, -p), (\mathcal{O}_X[p], 0)) \simeq \mathrm{Map}_{D(X)}(L\Omega^p_X, \mathcal{O}_X^{p=0}[p+1])$ the map induced by the forgetful functor $D_{\mathbb{N}}(X) \to D(X)$ is given by composing with the natural map $\mathcal{O}_X^{p=0} \to \mathcal{O}_X$ which is nothing but the Bockstein map $\mathrm{Bock}_{\mathcal{O}_X}[-1] : i_*\mathcal{O}_{X_0}[-1] \to \mathcal{O}_X$. Secondly, the adjunction identification $\mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p]) \simeq \mathrm{Map}_{D(X)}(L\Omega^p_X, i_*i^*\mathcal{O}_X[p])$ sends $f \in \mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p])$ to $L\Omega^p_X \to i_*L\Omega^p_{X_0/\mathbb{F}_p} \xrightarrow{i_*f} i_*\mathcal{O}_{X_0}[p]$, which implies the claim. $\square$

Notation 6.4. We denote by $c_{X,p} \in \mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p])$ the result of transporting the map (6.6) along the equivalence (6.7), and by $e_{X,p} \in \mathrm{Map}_{D(X)}(L\Omega^p_X, \mathcal{O}_X[p+1])$ we denote the extension class in the conjugate filtration on $\mathrm{Fil}_p^{\mathrm{conj}}\Omega^{\not{D}}_{X,0}$.

Let us explicitly record how $e_{X,p}$ can be recovered from $c_{X,p}$, which is immediate from Lemma 6.3:

Lemma 6.5. *The element* $e_{X,p} \in \mathrm{Map}_{D(X)}(L\Omega^p_X, \mathcal{O}_X[p+1])$ *corresponding to the extension* $\mathcal{O}_X \to \mathrm{Fil}_p\Omega^{\not{D}}_{X,0} \to L\Omega^p_X[-p]$ *is naturally homotopic to the composition*

$$L\Omega^p_X \to L\Omega^p_{X_0} \xrightarrow{c_{X,p}} \mathcal{O}_{X_0}[p] \xrightarrow{\mathrm{Bock}_{\mathcal{O}_X}} \mathcal{O}_X[p+1] \tag{6.8}$$

Lemma 5.11 can be used to restate Lemma 6.5 as follows

Corollary 6.6. *The map* $e_{X,p}$ *is naturally homotopic to the image of* $c_{X,p}$ *under the Bockstein morphism*

$$\mathrm{Bock} : \mathrm{RHom}_{X_0}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p]) \to \mathrm{RHom}_X(L\Omega^p_X, \mathcal{O}_X[p+1]) \tag{6.9}$$

arising from the object $\mathrm{RHom}_X(L\Omega^p_X, \mathcal{O}_X[p+1]) \in D(\mathbb{Z}_p)$.

The map $c_{X,p}$ can also be read off from the action of the Sen operator on dR_{X_0}, as we will now show. In general, if $(M_1, 0) \to (M, f_M) \to (M_2, 0)$ is a fiber sequence in $D_{\mathbb{N}}(X_0)$ then $f_M : M \to M$ naturally factors as $M \to M_2 \xrightarrow{h} M_1 \to M$ for a morphism $h \in \mathrm{Map}_{D(X_0)}(M_2, M_1)$. The element $\delta \in \mathrm{Map}_{D_{\mathbb{N}}(X_0)}((M_2, 0), (M_1, 0)[1]) = \mathrm{Map}_{D(X_0)}(M_2, M_1[1]) \oplus \mathrm{Map}_{D(X_0)}(M_2, M_1)$ corresponding to this fiber sequence is then given by $\underline{\delta} \oplus h$ where $\underline{\delta}$ is the image of δ under the forgetful map $D_{\mathbb{N}}(X_0) \to D(X_0)$. Applying this discussion to $\mathrm{Fil}_p^{\mathrm{conj}}\mathrm{dR}_{X_0}$ we obtain:

Lemma 6.7. *The endomorphism* Θ_X *of* $\mathrm{Fil}_p\mathrm{dR}_{X_0,0}$ *naturally factors as*

$$\mathrm{Fil}_p\mathrm{dR}_{X_0,0} \to L\Omega^p_{X_0}[-p] \xrightarrow{c_{X,p}[-p]} \mathcal{O}_{X_0} = \mathrm{Fil}_0^{\mathrm{conj}}\mathrm{dR}_{X_0,0} \to \mathrm{Fil}_p^{\mathrm{conj}}\mathrm{dR}_{X_0,0} \tag{6.10}$$

Proof. Given the paragraph preceding the statement of the lemma, this amounts to observing that the map $\mathrm{RHom}_{D_{\mathbb{N}}(X)}((L\Omega^p_X[-p],-p),(\mathcal{O}_X,0)) \to \mathrm{RHom}_{D_{\mathbb{N}}(X_0)}((L\Omega^p_{X_0}[-p],-p),(\mathcal{O}_{X_0},0))$ induced by the restriction to the special fiber X_0 can be described as $\mathrm{RHom}_{D(X_0)}(L\Omega^p_{X_0},\mathcal{O}_{X_0}[p]) \xrightarrow{(\mathrm{Bock},\mathrm{id})} \mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0},\mathcal{O}_{X_0}[p+1]) \oplus \mathrm{RHom}_{D(X_0)}(L\Omega^p_{X_0},\mathcal{O}_{X_0}[p])$ under the identification (6.7). ☐

For our computation of the Sen operator in Theorem 7.1 it will be convenient to directly relate $c_{X,p}$ to the action of Θ_X on all of the object $\mathrm{Fil}_p\mathrm{dR}_{X_0}$, rather than its weight zero part:

Lemma 6.8. *The endomorphism $\Theta_X - \Theta_X^p$ of $\mathrm{Fil}_p\mathrm{dR}_{X_0}$ naturally factors as*

$$\mathrm{Fil}_p\mathrm{dR}_{X_0} \to L\Omega^p_{X_0}[-p] \xrightarrow{c_{X,p}[-p]} \mathcal{O}_{X_0} = \mathrm{Fil}_0\mathrm{dR}_{X_0} \to \mathrm{Fil}_p\mathrm{dR}_{X_0} \tag{6.11}$$

Proof. The endomorphism $\mathrm{Id} - \Theta_X^{p-1} : \mathrm{Fil}_p\mathrm{dR}_{X_0} \to \mathrm{Fil}_p\mathrm{dR}_{X_0}$ is an idempotent corresponding to the direct summand $\mathrm{Fil}_p\mathrm{dR}_{X_0,0}$ of $\mathrm{Fil}_p\mathrm{dR}_{X_0}$. Therefore $\Theta_X - \Theta_X^p = \Theta_X\cdot(\mathrm{Id}-\Theta_X^{p-1})$ can be factored as $\mathrm{Fil}_p\mathrm{dR}_{X_0} \to \mathrm{Fil}_p\mathrm{dR}_{X_0,0} \xrightarrow{\Theta_X} \mathrm{Fil}_p\mathrm{dR}_{X_0,0} \to \mathrm{Fil}_p\mathrm{dR}_{X_0}$ where the first and last maps establish $\mathrm{Fil}_p\mathrm{dR}_{X_0,0}$ as a direct summand of $\mathrm{Fil}_p\mathrm{dR}_{X_0}$. Hence the claim follows from Lemma 6.7. ☐

Since $\tau^{\le p}\mathrm{dR}_{X_0}$ decomposes as $\bigoplus\limits_{i=0}^{p-1}\tau^{\le p}\mathrm{dR}_{X_0,i} \simeq \tau^{\le p}\mathrm{dR}_{X_0,0}\oplus\bigoplus\limits_{i=0}^{p-1} L\Omega^i_X[-i]$ compatibly with the Sen operator, Θ_X on $\tau^{\le p}\mathrm{dR}_{X_0}$ is semi-simple if and only if $c_{X,p}\sim 0$. In particular, if $c_{X,p}$ vanishes then the conjugate filtration on the diffracted Hodge complex splits in degrees $\le p$:

Corollary 6.9. *If the Sen operator on $\mathrm{Fil}^p_{\mathrm{conj}}\mathrm{dR}_{X_0}$ is semi-simple then there exists a decomposition*

$$(\mathrm{Fil}^p_{\mathrm{conj}}\Omega^{⅁}_X,\Theta_X) \simeq \bigoplus_{i=0}^{p}(L\Omega^i_X[-i],-i) \tag{6.12}$$

6.3. **Diffracted Hodge cohomology and Sen operator over $\mathbb{Z}/p^n, n\ge 2$.** In this subsection we give a discussion parallel to the above for a flat scheme X_{n-1} over $\mathbb{Z}/p^n$ for $n\ge 2$. We will define the diffracted Hodge complex $\Omega^{⅁}_{X_{n-1}/\mathbb{Z}/p^n}$ of X_{n-1} relative to $\mathbb{Z}/p^n$, equipped with a Sen operator with properties analogous to Theorem 6.2.

Theorem 6.10. *For a scheme X_{n-1} quasisyntomic over $\mathbb{Z}/p^n$ there is a natural object $(\Omega^{⅁}_{X_{n-1}/\mathbb{Z}/p^n},\Theta_{X_{n-1}})\in D_{\mathbb{N}}(X_{n-1})$ equipped with a filtration $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ with graded quotients equivalent to $(L\Omega^i_{X_{n-1}/\mathbb{Z}/p^n}[-i],-i)$. The object $\Omega^{⅁}_{X_{n-1}/\mathbb{Z}/p^n}\in D(X_{n-1})$ has a natural structure of a filtered derived commutative algebra such that $\Theta_{X_{n-1}}$ is a derivation.*

The base change $\Omega^{⅁}_{X_{n-1}/\mathbb{Z}/p^n}\otimes_{\mathcal{O}_{X_{n-1}}}\mathcal{O}_{X_0}$ is identified with dR_{X_0}, and the base change of the conjugate filtration matches the conjugate filtration on dR_{X_0}. If X is a quasisyntomic formal scheme over $\mathbb{Z}_p$ then for $X_{n-1} := X\times_{\mathbb{Z}_p}\mathbb{Z}_p/p^n$ we have $\Omega^{⅁}_{X_{n-1}/\mathbb{Z}/p^n}\simeq \Omega^{⅁}_X\otimes_{\mathcal{O}_X}\mathcal{O}_{X_{n-1}}$ compatibly with the conjugate filtrations and the Sen operators.

Given a quasisyntomic $\mathbb{Z}/p^2$-scheme X_1 with the special fiber $X_0 = X_1\times_{\mathbb{Z}/p^2}\mathbb{F}_p \overset{i}{\hookrightarrow} X_1$, the de Rham complex $\mathrm{dR}_{X_0}\simeq i^*\Omega^{⅁}_{X_1/\mathbb{Z}/p^2}$ therefore gets equipped with a Sen operator

Θ_{X_1}. Generalized eigenspaces for Θ_{X_1} give a decomposition

$$\mathrm{dR}_{X_0} \simeq \bigoplus_{i=0}^{p-1} \mathrm{dR}_{X_0,i} \tag{6.13}$$

The object $\mathrm{Fil}^p_{\mathrm{conj}}\mathrm{dR}_{X_0,0}$ fits into a fiber sequence

$$\mathcal{O}_{X_0} \to \mathrm{Fil}^p_{\mathrm{conj}}\mathrm{dR}_{X_0,0} \to L\Omega^p_{X_0}[-p] \tag{6.14}$$

on which Θ_{X_1} naturally acts, inducing zero on the first and third terms.

Notation 6.11. We denote by $e_{X_1,p} : L\Omega^p_{X_0} \to \mathcal{O}_{X_0}[p+1]$ the connecting map corresponding to the fiber sequence (6.14), and by $c_{X_1,p} : L\Omega^p_{X_0} \to \mathcal{O}_{X_0}[p]$ the map induced by the nilpotent operator Θ_{X_1} on $\mathrm{Fil}_p^{\mathrm{conj}}\mathrm{dR}_{X_0,0}$.

This is consistent with Notation 6.4 in the sense that for a quasisyntomic formal $\mathbb{Z}_p$-scheme X with the mod p^2 reduction $X_1 = X \times_{\mathbb{Z}_p} \mathbb{Z}/p^2$ the map $c_{X_1,p}$ is naturally homotopic to $c_{X,p}$, and $e_{X_1,p}$ is the mod p reduction of $e_{X,p}$.

We can also consider the endomorphism $\Theta_{X_1} - \Theta^p_{X_1}$ of $\mathrm{Fil}_p^{\mathrm{conj}}\mathrm{dR}_{X_0}$ which acts by zero on $\mathrm{Fil}_{p-1}^{\mathrm{conj}}$ and hence naturally induces a map $\Theta_{X_1} - \Theta^p_{X_1} : L\Omega^p_{X_0}[-p] \to \mathrm{Fil}_{p-1}^{\mathrm{conj}}\mathrm{dR}_{X_0}$. As in the previous discussion in the presence of a lift of X_0 over $W(k)$, this map, $c_{X_1,p}$, and $e_{X_1,p}$ are related as follows:

Lemma 6.12. *There is a natural homotopy*

$$e_{X_1,p} \sim \mathrm{Bock}_{\mathrm{RHom}_{D(X_1)}(L\Omega^p_{X_1},\mathcal{O}_{X_1}[p])}(c_{X_1,p}) \tag{6.15}$$

where

(6.16)

$$\mathrm{Bock}_{\mathrm{RHom}_{D(X_1)}(L\Omega^p_{X_1},\mathcal{O}_{X_1}[p])} : \mathrm{RHom}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p]) \to \mathrm{RHom}_{D(X_0)}(L\Omega^p_{X_0}, \mathcal{O}_{X_0}[p+1])$$

is the Bockstein homomorphism induced by $\mathrm{RHom}_{D(X_1)}(L\Omega^p_{X_1}, \mathcal{O}_{X_1}[p]) \in D(W_2(k))$. *The map* $\Theta_{X_1} - \Theta^p_{X_1} : L\Omega^p_{X_0}[-p] \to \mathrm{Fil}_{p-1}^{\mathrm{conj}}\mathrm{dR}_{X_0}$ *is naturally homotopic to the composition* $L\Omega^p_{X_0}[-p] \xrightarrow{c_{X_1,p}} \mathcal{O}_{X_0} \to \mathrm{Fil}_{p-1}^{\mathrm{conj}}\mathrm{dR}_{X_0}$

We will now define diffracted Hodge cohomology relative to $\mathbb{Z}/p^n$, proving Theorem 6.10, and will then prove Lemma 6.12. As in [BL22a], the construction of diffracted Hodge cohomology over $\mathbb{Z}/p^n$ together with its Sen operator arises from the computation of the Cartier-Witt stack of the base ring $\mathbb{Z}/p^n$. I learned the following fact from Sanath Devalapurkar:

Lemma 6.13. *There is an isomorphism of stacks* $\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n} \simeq \mathbb{G}^{\#}_{a,\mathbb{Z}/p^n}/\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n}$ *where the quotient is taken with respect to the scaling action. Moreover, the natural map* $\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n} \to \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}_p}$ *is identified with* $\mathbb{G}^{\#}_{a,\mathbb{Z}/p^n}/\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n} \to B\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n} \to B\mathbb{G}^{\#}_{m,\mathbb{Z}_p}$.

Proof. We denote by $\mathbb{G}_a^{\#}$ the divided power envelope of 0 in $\mathbb{G}_a$, and by $\mathbb{G}_m^{\#}$ the divided power envelope of 1 in $\mathbb{G}_m$, both viewed as group schemes over $\mathbb{Z}_p$. In this proof we will use repeatedly the identifications of group schemes $\mathbb{G}_m^{\#} \simeq W^{\times}[F] := W^{F=1}$ and $\mathbb{G}_a^{\#} \simeq W[F] := W^{F=0}$ proven in [BL22a, Lemmma 3.4.11, Variant 3.4.12] and [Dri21, Lemma 3.2.6], together with the fact that the multiplication action of $W^{\times}[F]$ on $W[F]$ corresponds to the usual scaling action of $\mathbb{G}_m^{\#}$ on $\mathbb{G}_a^{\#}$ under these identifications.

By [BL22b, Construction 3.8] the stack $\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n}$ is the quotient of $(\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}}$ by $\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n}$ where for a test algebra S in which p is nilpotent we have $(\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}}(S) = \mathrm{Map}(\mathbb{Z}/p^n, W(S)/V(1))$ where the mapping space is taken in the category of animated commutative rings. As in [BL22b, Example 5.15] we can rewrite this more explicitly as $(\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}}(S) \simeq \{x \in W(S) | V(1)x = p^n\}$ with the $\mathbb{G}^{\#}_m(S) = W(S)^{F=1}$-action given by multiplication on x. The key computation that will allow us to identify this set with $\mathbb{G}^{\#}_a(S)$ in a $\mathbb{G}^{\#}_m(S)$-equivariant fashion is the following:

Lemma 6.14. *We have $p^n = V(p^{n-1})$ in $W(\mathbb{Z}/p^n)$.*

Proof. Recall that for any ring R there is the ghost map $W(R) \to R^{\mathbb{N}}$ sending a Witt vector $[x_0] + V[x_1] + V^2[x_2] + \ldots$ to $(x_0, x_0^p + px_1, x_0^{p^2} + px_1^p + p^2x_2, \ldots)$. This is a map of rings and it is injective if R is p-torsion-free. For an element $a \in W(R)$ with ghost coordinates $(a_0, a_1, \ldots)$ the ghost coordinates of $F(a)$ are given by $(a_1, a_2, \ldots)$, and the ghost coordinates of $V(a)$ are $(0, pa_0, pa_1, \ldots)$.

Therefore the ghost coordinates of $V(p^{n-1}) \in W(\mathbb{Z}_p)$ are $(0, p^n, p^n, \ldots)$ and the ghost coordinates of $p^n - V(p^{n-1})$ are equal to $(p^n, 0, 0, \ldots)$. The isomorphism $\mathbb{G}^{\#}_a \simeq W[F]$ composed with the ghost map sends $r \in \mathbb{G}^{\#}_a$ to $(r, 0, 0, \ldots)$, hence $p^n - V(p^{n-1}) \in W(\mathbb{Z}_p)$ is the image of $p^n \in \mathbb{G}^{\#}_a(\mathbb{Z}_p)$ under the natural map $\mathbb{G}^{\#}_a \simeq W[F] \subset W$. Since p^n is annihilated by the map $\mathbb{G}^{\#}_a(\mathbb{Z}_p) \to \mathbb{G}^{\#}_a(\mathbb{Z}/p^n)$ induced by $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^n$, the element $p^n - V(p^{n-1})$ is zero in $W(\mathbb{Z}/p^n)$, as desired. $\square$

We now define an isomorphism $\mathbb{G}^{\#}_a \simeq (\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}}$ by sending $y \in \mathbb{G}^{\#}_a(S) \simeq W(S)^{F=0}$ to $y + V(p^{n-2}) \in (\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}}(S)$. The element $y + V(p^{n-2})$ indeed satisfies the equation $V(1)(y + V(p^{n-2})) = p^n$ because

$$V(1) \cdot (y + V(p^{n-2})) = VF(y + V(p^{n-2})) = VFV(p^{n-2}) = V(p^{n-1}) = p^n \tag{6.17}$$

This isomorphism intertwines the $\mathbb{G}^{\#}_m$-action on $(\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}}$ with the usual scaling action on $\mathbb{G}^{\#}_a$ because for $a \in \mathbb{G}^{\#}_m(S) = W(S)^{F=1}$ we have $a \cdot (y + V(p^{n-2})) = a \cdot y + V(F(a)p^{n-2}) = a \cdot y + V(p^{n-2})$. $\square$

We can now define diffracted Hodge cohomology relative to $\mathbb{Z}/p^n$. Given a flat scheme X_{n-1} over $\mathbb{Z}/p^n$, the stack $\mathrm{WCart}^{\mathrm{HT}}_{X_{n-1}}$ lives over $\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n}$ and we define $(X_{n-1}/\mathbb{Z}/p^n)^{\not{D}}$ as the fiber product

$$\begin{tikzcd} (X_{n-1}/\mathbb{Z}/p^n)^{\not{D}} \arrow[r] \arrow[d] & \mathrm{WCart}^{\mathrm{HT}}_{X_{n-1}} \arrow[d] \\ \mathrm{Spec}(\mathbb{Z}/p^n) \arrow[r, "\eta"] & \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n} \end{tikzcd} \tag{6.18}$$

where $\eta : \mathrm{Spec}(\mathbb{Z}/p^n) \to \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n} \simeq \mathbb{G}^{\#}_{a,\mathbb{Z}/p^n}/\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n}$ is the composition $\mathrm{Spec}(\mathbb{Z}/p^n) \xrightarrow{0} \mathbb{G}^{\#}_{a,\mathbb{Z}/p^n} \to \mathbb{G}^{\#}_{a,\mathbb{Z}/p^n}/\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n}$. The stack $(X_{n-1}/\mathbb{Z}/p^n)^{\not{D}}$ is equipped with a map $\pi^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} : (X_{n-1}/\mathbb{Z}/p^n)^{\not{D}} \to X_{n-1}$ obtained as the composition $(X_{n-1}/\mathbb{Z}/p^n)^{\not{D}} \to \mathrm{WCart}^{\mathrm{HT}}_{X_{n-1}} \to X_{n-1}$. Define the diffracted Hodge cohomology of X_{n-1} realtive to $\mathbb{Z}/p^n$ as the pushforward of the structure sheaf along the map $\pi^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$:

$$\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} := R\pi_{X_{n-1}/\mathbb{Z}/p^n *}\mathcal{O}_{(X_{n-1}/\mathbb{Z}/p^n)^{\not{D}}} \in D(X_{n-1}) \tag{6.19}$$

We will now prove that this object shares the basic properties of diffracted Hodge cohomology of $\mathbb{Z}_p$-schemes:

Proof of Theorem 6.10. Since, by construction, the stack $(X_{n-1}/\mathbb{Z}/p^n)^{\not{D}}$ is equipped with an action of $\mathbb{G}^{\#}_{m,\mathbb{Z}/p^n}$ such that the map $\pi^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} : (X_{n-1}/\mathbb{Z}/p^n)^{\not{D}} \to X_{n-1}$ is $\mathbb{G}^{\#}_{m,\mathbb{Z}_p}$-equivariant for the trivial action of $\mathbb{G}^{\#}_{m,\mathbb{Z}_p}$ on the target, the object $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} \in D(X_{n-1})$ is naturally equipped with a $\mathbb{G}^{\#}_{m,\mathbb{Z}_p}$-action. As in [BL22a, Theorem 3.5.8], this gives rise to an endomorphism $\Theta_{X_{n-1}}$ of $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$. We endow $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$ with the structure of a derived commutative algebra via Lemma 2.13, and $\Theta_{X_{n-1}}$ is seen to be a derivation (Definition 2.18) by following [BL22a, Construction 3.5.4].

We will now construct the conjugate filtration on $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$ and identify its graded quotients. To do this we will compare $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$ with the 'absolute' diffracted Hodge cohomology $\Omega^{\not{D}}_{X_{n-1}}$. By [BL22a, Construction 4.7.1] the object $\Omega^{\not{D}}_{X_{n-1}}$ is equipped with a filtration $\mathrm{Fil}^{\mathrm{conj}}_n \Omega^{\not{D}}_{X_{n-1}}$ with graded quotients $\mathrm{gr}^{\mathrm{conj}}_n \simeq L\Omega^n_{X_{n-1}/\mathbb{Z}_p}[-n]$. In the case $X_{n-1} = \mathrm{Spec}(\mathbb{Z}/p^n)$ we have an equivalence $\Omega^{\not{D}}_{\mathbb{Z}/p^n} \simeq \mathbb{Z}/p^n[\mathbb{G}^{\#}_{a,\mathbb{Z}/p^n}]$ of derived commutative $\mathbb{Z}/p^n$-algebras by Lemma 6.13.

By base change for the diagram

$$\begin{tikzcd}
(X_{n-1}/\mathbb{Z}/p^n)^{\not{D}} \arrow[r] \arrow[d] & (X_{n-1})^{\not{D}} \arrow[d] \\
X_{n-1} \arrow[r, "\eta"] & (\mathrm{Spec}\,\mathbb{Z}/p^n)^{\not{D}} \times_{\mathbb{Z}/p^n} X_{n-1}
\end{tikzcd} \tag{6.20}$$

we have

$$\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} \simeq \mathbb{Z}/p^n \otimes_{\mathbb{Z}/p^n[\mathbb{G}^{\#}_a]} \Omega^{\not{D}}_{X_{n-1}}. \tag{6.21}$$

The ordinary commutative algebra $\mathbb{Z}/p^n[\mathbb{G}^{\#}_a]$ is endowed with a conjugate filtration via the identification $\mathbb{Z}/p^n[\mathbb{G}^{\#}_a] \simeq \Omega^{\not{D}}_{\mathbb{Z}/p^n}$, and the associated graded algebra is the free divided power algebra on the $\mathbb{Z}/p^n$-module $L\Omega^1_{\mathbb{Z}/p^n/\mathbb{Z}_p}[-1] \simeq \mathbb{Z}/p^n$. Moreover, the conjugate filtration on $\Omega^{\not{D}}_{X_{n-1}}$ makes it into an object of the derived category of filtered $\mathbb{Z}/p^n[\mathbb{G}^{\#}_a]$-modules in $D(X_{n-1})$. Equipping $\mathbb{Z}/p^n$ with the trivial filtration $\mathrm{Fil}^{\mathrm{conj}}_0 \mathbb{Z}/p^n = \mathbb{Z}/p^n, \mathrm{Fil}^{\mathrm{conj}}_{-1}\mathbb{Z}/p^n = 0$, we get an induced tensor product filtration $\mathbb{Z}/p^n \otimes_{\mathbb{Z}/p^n[\mathbb{G}^{\#}_a]} \Omega^{\not{D}}_{X_{n-1}}$, and hence on the complex $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$.

We will first check that the filtered complex $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} \otimes_{\mathcal{O}_{X_{n-1}}} \mathcal{O}_{X_0}$ is equivalent to dR_{X_0}, and that there is an equivalence of filtered complexes $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n} \simeq \Omega^{\not{D}}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{n-1}}$ compatible with the Sen operators if X is a flat formal $\mathbb{Z}_p$-scheme lifting X_{n-1}. Since we assume that X_{n-1} is quasisyntomic over $\mathbb{Z}/p^n$, the formal scheme X is quasisyntomic over $\mathbb{Z}_p$, and $\Omega^{\not{D}}_X$ coincides with the derived pushforward of the structure sheaf along the map of stacks $X^{\not{D}} \to X$, by specializing to the Hodge-Tate locus the isomorphism of [BL22b, Theorem 7.20(2)]. We have the following relations between the relevant Cartier-Witt stacks:

$$\mathrm{WCart}^{\mathrm{HT}}_{X_{n-1}} \times_{\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n}} \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{F}_p} \simeq \mathrm{WCart}^{\mathrm{HT}}_{X_0}$$
$$\mathrm{WCart}^{\mathrm{HT}}_{X} \times_{\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}_p}} \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n} \simeq \mathrm{WCart}^{\mathrm{HT}}_{X_{n-1}} \tag{6.22}$$

These follows directly from the definition of $\mathrm{WCart}^{\mathrm{HT}}_{(-)}$, as in [BL22b, Remark 3.5]. The identification $\mathrm{dR}_{X_0} \simeq \Omega^{\not{D}}_{X_{n-1}} \otimes_{\mathbb{Z}/p^n} \mathbb{F}_p$ follows from the fact that $\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{F}_p} \simeq \mathrm{Spec}\,\mathbb{F}_p$ and the map $\mathrm{WCart}^{\mathrm{HT}}_{\mathbb{F}_p} \to \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n}$ factors through the map η. The identification $\Omega^{\not{D}}_X/p^n \simeq \Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$ is implied by the fact that the composition $\mathrm{Spec}\,\mathbb{Z}/p^n \xrightarrow{\eta} \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}/p^n} \to \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}_p}$ is equal to the composition $\mathrm{Spec}\,\mathbb{Z}/p^n \to \mathrm{Spf}\,\mathbb{Z}_p \to \mathrm{WCart}^{\mathrm{HT}}_{\mathbb{Z}_p}$.

Finally, we will compute the quotients of the conjugate filtration on $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$. The associated graded object $\mathrm{gr}^\bullet_{\mathrm{conj}}(\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}) \simeq \mathrm{gr}^\bullet_{\mathrm{conj}}(\mathbb{Z}/p^n \otimes_{\mathbb{Z}/p^n[\mathbb{G}_a^\#]} \Omega^{\not{D}}_{X_{n-1}})$ is equivalent to $\mathbb{Z}/p^n \otimes_{\mathbb{Z}/p^n\langle t\rangle} \bigoplus\limits_{i\geq 0} L\Omega^i_{X_{n-1}/\mathbb{Z}_p}[-i]$ where we identified $\mathrm{gr}^\bullet_{\mathrm{conj}}\,\mathbb{Z}/p^n[\mathbb{G}_a^\#]$ with the divided power algebra $\mathbb{Z}/p^n\langle t\rangle$ in one variable, such that $\Theta(t) = -t$.

We have a natural map $\mathrm{gr}^\bullet_{\mathrm{conj}}(\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}) \to \bigoplus\limits_{i\geq 0}(L\Omega^i_{X_{n-1}/\mathbb{Z}/p^n}[-i], -i)$ of graded objects of $D_{\mathbb{N}}(X_{n-1})$. To check that it is an equivalence we may assume that X_{n-1} is isomorphic to the spectrum of a polynomial ring over $\mathbb{Z}/p^n$, and the result follows because such X_{n-1} lifts over $\mathbb{Z}_p$. □

We will now establish analogs of Corollary 6.6 and Lemmas 6.7, 6.8 over $\mathbb{Z}/p^2$, thus proving Lemma 6.12. Given a flat scheme X_1 over $\mathbb{Z}/p^2$ and two objects $M, N \in D(X_1)$, consider the complex of morphisms $\mathrm{RHom}_{D_{\mathbb{N}}(X_1)}((M,p),(N,0))$, viewed as an object of $D(\mathbb{Z}/p^2)$. The restriction along $i : X_0 \hookrightarrow X_1$ induces the map

$$\begin{aligned}&\mathrm{RHom}_{D_{\mathbb{N}}(X_1)}((M,p),(N,0)) \to \\ &\quad \mathrm{RHom}_{D_{\mathbb{N}}(X_0)}((i^*M,0),(i^*N,0)) \simeq \mathrm{RHom}_{X_0}(i^*M.i^*N) \oplus \mathrm{RHom}_{X_0}(i^*M, i^*N[-1])\end{aligned} \tag{6.23}$$

We denote by $r : \mathrm{RHom}_{D_{\mathbb{N}}(X_1)}((M,p),(N,0)) \to \mathrm{RHom}_{X_0}(i^*M, i^*N[-1])$ the second component of this composition. Explicitly, the data of a 1-morphism $f : (M,p) \to (N,0)$ amounts to a map $f : M \to N$ in $D(X_1)$ and a homotopy between $p\cdot f$ and 0. Restricting to X_0, this gives a homotopy from the zero morphism $i^*M \xrightarrow{0} i^*N$ to itself, which is equivalent to the data of a 1-morphism $i^*M \to i^*N[-1]$. By Lemma 6.1 this map is nothing but $r(f)$.

Lemma 6.15. *Let X_1 be a flat scheme over $\mathbb{Z}/p^2$. Denote by $X_0 = X_1 \times_{\mathbb{Z}/p^2} \mathbb{F}_p \xhookrightarrow{i} X_1$ its special fiber. For any two objects $M, N \in D(X_1)$ the composition*

$$\mathrm{RHom}_{D_{\mathbb{N}}(X_1)}((M,p),(N,0)) \to \mathrm{RHom}_{D(X_1)}(M,N) \to \mathrm{RHom}_{D(X_0)}(i^*M, i^*N) \tag{6.24}$$

where the first map is forgetting the endomorphisms and the second map is induced by i, can be identified with the composition

(6.25)
$$\mathrm{RHom}_{D_{\mathbb{N}}(X_1)}((M,p),(N,0)) \xrightarrow{r} \mathrm{RHom}_{D(X_0)}(i^*M, i^*N[-1]) \xrightarrow{\mathrm{Bock}} \mathrm{RHom}_{D(X_0)}(i^*M, i^*N)$$

where the second map is Bockstein morphism associated to the object $\mathrm{RHom}_{D(X_1)}(M, N[-1])$.

Proof. By Lemma 6.1 we can identify $\mathrm{RHom}_{D_{\mathbb{N}}(X_1)}((M,p),(N,0))$ with $\mathrm{RHom}_{D(X_1)}(M,N^{p=0})$, and the forgetful map to $\mathrm{RHom}_{D(X_1)}(M,N)$ is given by composing with the natural map $N^{p=0}\to N$. Moreover, we can identify $\mathrm{RHom}_{D(X_1)}(M,N^{p=0})$ with $\mathrm{RHom}_{D(X_1)}(M,N)^{p=0}$ so that the desired identification becomes a consequence of the following Lemma 6.16, applied to $A=\mathrm{RHom}_{D(X_1)}(M,N)$. □

Lemma 6.16. *For any object $A\in D(\mathbb{Z}/p^2)$ the composition $A^{p=0}\to A\to i_*i^*A$ is naturally identified with the composition*

$$A^{p=0}\xrightarrow{r_A} i_*i^*A[-1]\xrightarrow{\mathrm{Bock}_{A[-1]}} i_*i^*A \tag{6.26}$$

where the map r_A is induced by the identification $i_*i^*(A^{p=0})=i_*(i^*A)^{p=0}\simeq i_*i^*A\oplus i_*i^*A[-1]$.

Proof. Recall that $i_*i^*A\simeq A\otimes_{\mathbb{Z}/p^2}\mathbb{F}_p$ and there is a natural fiber sequence $i_*i^*A\xrightarrow{a_1}A\xrightarrow{a_2}i_*i^*A$ obtained by taking the tensor product of the sequence $\mathbb{F}_p\to\mathbb{Z}/p^2\to\mathbb{F}_p$ with A. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "a_2"] & A \arrow[d, equal] \\ i_*i^*A \arrow[r, "a_1"] & A \end{tikzcd} \tag{6.27}$$

Taking the fibers of the horizontal arrows induces a commutative diagram

$$\begin{tikzcd} \mathrm{fib}(A\xrightarrow{p}A) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{fib}(i_*i^*A\xrightarrow{a_1}A) \arrow[r] & i_*i^*A \end{tikzcd} \tag{6.28}$$

The induced map $A^{p=0}=\mathrm{fib}(A\xrightarrow{p}A)\to\mathrm{fib}(i_*i^*A\xrightarrow{a_1}A)\simeq i_*i^*A[-1]$ is the map r_A, and the map $i_*i^*A[-1]\simeq\mathrm{fib}(i_*i^*A\xrightarrow{a_1}A)\to i_*i^*A$ is $\mathrm{Bock}_{A[-1]}$, by definition. Hence the statement of the lemma amounts to commutativity of the diagram (6.28). □

The upshot is that, given a quasisyntomic $\mathbb{Z}/p^n$-scheme X_{n-1} for $n\geq 2$, the Sen operator on $\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$ induces an operator on dR_{X_0}, and all of the information about the restriction of the latter operator to $\mathrm{Fil}_p\mathrm{dR}_{X_0}$ is captured by the class $c_{X_{n-1},p}$. If X_{n-1} lifts to a flat formal $\mathbb{Z}_p$-scheme X, then this $c_{X_{n-1},p}$ also remembers the data of the Sen operator on $\mathrm{Fil}_p\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$, by Lemma 6.3. In the absence of a lift to $W(k)$ it is at the moment unclear to me whether the Sen operator on $\mathrm{Fil}_p\Omega^{\not{D}}_{X_{n-1}/\mathbb{Z}/p^n}$ contains more information than its mod p reduction.

7. Sen operator via descent from semiperfectoid rings

In this section we prove Theorem 7.1 via descent from quasiregular semiperfectoid rings, this approach was suggested by Bhargav Bhatt. As a consequence, we extend Theorem 5.10 to the situation where X_0 is only liftable to $\mathbb{Z}/p^2$ rather than $\mathbb{Z}_p$.

Theorem 7.1. *Let X_1 be a quasisyntomic scheme over $\mathbb{Z}/p^2$ with the special fiber $X_0 = X_1\times_{\mathbb{Z}/p^2}\mathbb{F}_p$. The map $c_{X_1,p}: L\Omega^p_{X_0}\to\mathcal{O}_{X_0}[p]$ arising from the Sen operator on the de Rham complex of X_0 is naturally homotopic to the composition*

$$L\Omega^p_{X_0}\xrightarrow{\alpha(L\Omega^1_{X_0})}F^*_{X_0}L\Omega^1_{X_0}[p-1]\xrightarrow{\mathrm{ob}_{F,X_1}}\mathcal{O}_{X_0}[p]. \tag{7.1}$$

The particular interpretation of $c_{X_1,p}$ that will be used in the proof is that the map $\Theta_{X_1}-\Theta^p_{X_1}: L\Omega^p_{X_0}[-p]\to\mathrm{Fil}^{\mathrm{conj}}_{p-1}\mathrm{dR}_{X_0}\simeq\bigoplus\limits_{i=0}^{p-1}L\Omega^i_{X_0}[-p]$ factors as $L\Omega^p_{X_0}[-p]\xrightarrow{c_{X_1,p}}\mathcal{O}_{X_0}\xrightarrow{\oplus}\mathrm{Fil}^{\mathrm{conj}}_{p-1}\mathrm{dR}_{X_0}$, as we established in Lemma 6.12.

We start by recalling the description of diffracted Hodge cohomology of quasiregular semiperfectoid algebras. Let S be a quasiregular semiperfectoid flat $\mathbb{Z}/p^n\mathbb{Z}$-algebra, as defined in [BMS19, Definition 4.20]. The cotangent complex $L\Omega^1_{S/\mathbb{Z}/p^n}$ is concentrated in degree (-1) and $H^{-1}(L\Omega^1_{S/\mathbb{Z}/p^n})$ is a flat S-module. For brevity, we denote $H^{-1}(L\Omega^1_{S/\mathbb{Z}/p^n})$ by M_S. The objects $L\Omega^i_{S/\mathbb{Z}/p^n}[-i]=\Lambda^i(H^{-1}(L\Omega^1_{S/\mathbb{Z}/p^n})[1])[-i]\simeq\Gamma^iM_S$ are also flat S-modules placed in degree zero. Since $\Omega^{\not D}_{S/\mathbb{Z}/p^n}$ admits an exhaustive filtration with graded pieces given by $L\Omega^i_{S/\mathbb{Z}/p^n}[-i]$, the diffracted Hodge cohomology complex $\Omega^{\not D}_{S/\mathbb{Z}/p^n}$ is a commutative flat S-algebra concentrated in degree 0.

We denote by $S_0:=S/p$ the mod p reduction of S, and by $\Omega^{\not D}_{S_0}\simeq\Omega^{\not D}_{S/\mathbb{Z}/p^n}\otimes_S S_0$ the diffracted Hodge cohomology of S_0 which coincides with the derived de Rham cohomology $\mathrm{dR}_{S_0/\mathbb{F}_p}$. Thanks to the fact that $\Omega^{\not D}_{S/\mathbb{Z}/p^n}$ is concentrated in degree 0, the Sen operator $\Theta_S:\Omega^{\not D}_{S/\mathbb{Z}/p^n}\to\Omega^{\not D}_{S/\mathbb{Z}/p^n}$ is a genuine endomorphism of a discrete S-algebra, and we may use tools from ordinary algebra rather than higher algebra to study it. The reader is encouraged to view the next result as an analog of Theorem 4.1.

Proposition 7.2. *Let S be a quasiregular semiperfectoid flat $\mathbb{Z}/p^2$-algebra. Suppose that $f:\Omega^{\not D}_{S/\mathbb{Z}/p^2}\to\Omega^{\not D}_{S/\mathbb{Z}/p^2}$ is a derivation of S-algebras that preserves the conjugate filtration, acting on $\mathrm{gr}^{\mathrm{conj}}_i\,\Omega^{\not D}_{S/\mathbb{Z}/p^2}\simeq\Gamma^iM_S$ by $-i$. Denote by $s:M_S\to\mathrm{Fil}^{\mathrm{conj}}_1\Omega^{\not D}_{S/\mathbb{Z}/p^2}$ the splitting of the conjugate filtration given by $M_S\simeq\ker(f+1:\mathrm{Fil}^{\mathrm{conj}}_1\Omega^{\not D}_{S/\mathbb{Z}/p^2})\subset\mathrm{Fil}^{\mathrm{conj}}_1\Omega^{\not D}_{S/\mathbb{Z}/p^2}$. Then the map $f-f^p:\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_{S_0}\to\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_{S_0}$ on the mod p reduction of $\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_{S/\mathbb{Z}/p^2}$ factors as*

(7.2)

$$\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_{S_0}\twoheadrightarrow\Gamma^p_{S_0}(M_S/p)\twoheadrightarrow F^*_{S_0}(M_S/p)\xrightarrow{F^*_{S_0}s}F^*_{S_0}\mathrm{Fil}^{\mathrm{conj}}_1\Omega^{\not D}_{S_0}\xrightarrow{\varphi_{S_0}}\mathrm{Fil}^{\mathrm{conj}}_1\Omega^{\not D}_{S_0}\hookrightarrow\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_{S_0}$$

where the surjection $\Gamma^p_{S_0}(M_S/p)\twoheadrightarrow F^*_{S_0}(M_S/p)$ is the map $\psi_{M_S/p}$ defined in (2.7).

Proof. To prove the proposition it is enough to check that $f-f^p$ coincides with the composition (7.2) on elements $y\in\mathrm{Fil}^{\mathrm{conj}}_p\Omega^{\not D}_{S_0}$ whose image in $\mathrm{gr}^{\mathrm{conj}}_p\simeq\Gamma^p_{S_0}(M_S/p)$ has the form $x^{[p]}$ for some $x \in M_S/p$. Indeed, the S_0-module $\mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S_0}^{\not{D}}$ is spanned by elements of this form and the submodule $\mathrm{Fil}_{p-1}^{\mathrm{conj}}\Omega_{S_0}^{\not{D}}$, and both $f - f^p$ and the composition (7.2) are identically zero on $\mathrm{Fil}_{p-1}^{\mathrm{conj}}\Omega_{S_0}^{\not{D}}$. The composition (7.2) takes the element y to $s(x)^p$, and we will prove that $f - f^p$ does the same.

Consider the endomorphism of $\mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S/\mathbb{Z}/p^2}^{\not{D}}$ given by $F := (-1)^p \prod_{i=0}^{p-1}(f+i)$. It reduces modulo p to the map $f - f^p : \mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S_0}^{\not{D}} \to \mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S_0}^{\not{D}}$. Note also that F annihilates $\mathrm{Fil}_{p-1}^{\mathrm{conj}}\Omega_{S}^{\not{D}}$. Let $\widetilde{y} \in \mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S/\mathbb{Z}/p^2}^{\not{D}}$ be a lift of y such that the image of $\widetilde{y}$ in $\mathrm{gr}_p^{\mathrm{conj}}\,\Omega_{S/\mathbb{Z}/p^2}^{\not{D}} \simeq \Gamma_S^p(M_S)$ has the form $\widetilde{x}^{[p]}$ for some $\widetilde{x} \in M_S$ lifting $x \in M_S/p$. Then the image of $p! \cdot \widetilde{y}$ in $\mathrm{gr}_p^{\mathrm{conj}}\,\Omega_{S/\mathbb{Z}/p^2}^{\not{D}}$ is equal to $\widetilde{x}^p$, hence $p! \cdot \widetilde{y} - s(\widetilde{x})^p$ lies in the submodule $\mathrm{Fil}_{p-1}^{\mathrm{conj}}\Omega_{S/\mathbb{Z}/p^2}^{\not{D}} \subset \mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S/\mathbb{Z}/p^2}^{\not{D}}$. Therefore $F(p!\cdot\widetilde{y} - s(\widetilde{x})^p) = 0$. By definition of the section s, we have $f(s(\widetilde{x})) = -s(\widetilde{x})$. Since f is a derivation, $f(s(\widetilde{x})^p) = -p \cdot s(\widetilde{x})^p$, and $F(s(\widetilde{x})^p) = (-1)^p \prod_{i=0}^{p-1}(-p+i)\cdot s(\widetilde{x})^p$. Hence we get the following equation on the element $F(\widetilde{y})$:

$$p!F(\widetilde{y}) = \prod_{i=0}^{p-1}(p-i)\cdot s(\widetilde{x})^p \tag{7.3}$$

We now use crucially that $\Omega_{S/\mathbb{Z}/p^2}^{\not{D}}$ is flat over $\mathbb{Z}/p^2$: we can cancel out $p!$ on both sides of (7.3) to get $F(\widetilde{y}) = s(\widetilde{x})^p + p\cdot a$ for some $a \in \mathrm{Fil}_p^{\mathrm{conj}}\Omega_{S/\mathbb{Z}/p^2}^{\not{D}}$. Reducing modulo p gives the desired equality

$$(f - f^p)(y) = s(x)^p. \tag{7.4}$$

$\square$

We will now deduce a computation of the Sen operator for an arbitrary quasisyntomic $\mathbb{Z}/p^2$-scheme using descent.

Proof of Theorem 7.1. We will prove that $c_{X_1,p}$ is equivalent to the composition

(7.5)
$$L\Omega^p_{X_0} \xrightarrow{\alpha(L\Omega^1_{X_0})} F^*_{X_0}L\Omega^1_{X_0}[p-1] \xrightarrow{F^*_{X_0}s[p-1]} F^*_{X_0}\mathrm{dR}_{X_0}[p] \to F^*_{X_0}F_{X_0*}\mathcal{O}_{X_0}[p] \to \mathcal{O}_{X_0}[p]$$

This is equivalent to the desired statement because the composition $F^*_{X_0}L\Omega^1_{X_0} \xrightarrow{F^*_{X_0}s[1]} F^*_{X_0}\mathrm{dR}_{X_0}[1] \to F^*_{X_0}F_{X_0*}\mathcal{O}_{X_0}[1] \to \mathcal{O}_{X_0}[1]$ is homotopic to the obstruction class $\mathrm{ob}_{F,X_1} : F^*_{X_0}L\Omega^1_{X_0} \to \mathcal{O}_{X_0}[1]$ as was established in Proposition 5.2. Denote by $d_{X_1,p}$ the composition of maps in (7.5). We denote by $\mathrm{QSyn}_{\mathbb{Z}/p^2}$ the category of schemes quasisyntomic over $\mathbb{Z}/p^2$, by $\mathrm{AffQSyn}_{\mathbb{Z}/p^2} \subset \mathrm{QSyn}_{\mathbb{Z}/p^2}$ the full subcategory of affine quasisyntomic schemes, and by $\mathrm{QRSPrfd}_{\mathbb{Z}/p^2} \subset \mathrm{AffQSyn}_{\mathbb{Z}/p^2}$ the opposite of the category of quasiregular semiperfectoid $\mathbb{Z}/p^2$-algebras.

Proposition 7.2 established that $c_{X_1,p} = d_{X_1,p}$ when $X_1 = \mathrm{Spec}\, S$ is the spectrum of a quasiregular semiperfectoid $\mathbb{Z}/p^2$-algebra, because the map $\Gamma^p_{S_0}(M_S/p) \to F^*_{S_0}(M_S/p)$

is the shift by $[-p]$ of the map $L\Omega^p_{S_0}\xrightarrow{\alpha(L\Omega^1_{S_0})} F^*_{S_0}L\Omega^1_{S_0}[p-1]$. The case of an arbitrary $X_1\in \mathrm{Sch}_{\mathbb{Z}/p^2}$ will follow by a descent argument in the spirit of [BMS19], cf. [BLM21, Proposition 10.3.1] and [LM21, Proposition 4.4].

Let $\mathcal{QC}_{\mathrm{obj}}$ be the ∞-category of ∞-categories equipped with a distinguished object, as defined in [Lur22, Tag 020S]. Its objects are pairs $(C,\mathcal{C})$ where $\mathcal{C}$ is a ∞-category and C is an object of $\mathcal{C}$, and 1-morphisms from $(C,\mathcal{C})$ to $(D,\mathcal{D})$ are pairs $(F:\mathcal{C}\to\mathcal{D},\alpha: F(C)\to D)$ where F is a functor and α is a 1-morphism in $\mathcal{D}$.

Assigning to a scheme $X_1\in \mathrm{QSyn}_{\mathbb{Z}/p^2}$ the pair $(L\Omega^i_{X_0/\mathbb{F}_p}, D(X_0))$ defines a functor $L\Omega^i:\mathrm{Sch}^{\mathrm{op}}_{\mathbb{Z}/p^2}\to\mathcal{QC}_{\mathrm{obj}}$, where a morphism $f:X_1\to Y_1$ is sent to $f_0^*:D(Y_0)\to D(X_0)$ and $\Lambda^i df_0: f_0^*L\Omega^i_{Y_0}\to L\Omega^i_{X_0}$ (this functor factors through $\mathrm{QSyn}_{\mathbb{F}_p}$). For $i=0$ we denote this functor simply by $\mathcal{O}$. The formation of classes $c_{X_1,p}$ and $d_{X_1,p}$ defines morphisms from $L\Omega^p[-p]$ to $\mathcal{O}$. Denote by $\mathrm{Func}^0(\mathrm{QSyn}^{\mathrm{op}}_{\mathbb{Z}/p^2},\mathcal{QC}_{\mathrm{obj}})$ the category of functors for which the composition with the forgetful functor $\mathcal{QC}_{\mathrm{obj}}\to\mathrm{Cat}_\infty$ is $X_1\mapsto D(X_0)$.

Lemma 7.3. *For any i,j the restriction induces an equivalence*

$$\mathrm{Map}_{\mathrm{Func}^0(\mathrm{QSyn}^{\mathrm{op}}_{\mathbb{Z}/p^2},\mathcal{QC}_{\mathrm{obj}})}(L\Omega^i[-i],L\Omega^j[-j])\to \mathrm{Map}_{\mathrm{Func}^0(\mathrm{QRSPrfd}^{\mathrm{op}}_{\mathbb{Z}/p^2},\mathcal{QC}_{\mathrm{obj}})}(L\Omega^i[-i],L\Omega^j[-j]). \tag{7.6}$$

Proof. Affine schemes corresponding to quasiregular semiperfectoid rings form a basis in the flat topology on the category of all quasisyntomic $\mathbb{Z}/p^2$-schemes by [BMS19, Lemma 4.28] which implies the result by flat descent for the exterior powers of the cotangent complex. □

In general, specifying a natural transformation between functors F_1,F_2 into an ∞-category requires (among further higher homotopies) specifying maps $a_{X_1}:F_1(X_1)\to F_2(X_1)$ for all objects X_1 together with the additional data of homotopies between $F_2(f)\circ a_{X_1}$ and $a_{Y_1}\circ F_1(f)$ for every map $f:X_1\to Y_1$. Crucially, if $X=\mathrm{Spec}\, S$ is the spectrum of a quasiregular semiperfectoid $\mathbb{Z}/p^2$-algebra, then the mapping space $\mathrm{Map}_{D(X_0)}(L\Omega^p_{X_0}[-p],\mathcal{O}_{X_0})$ is discrete because both $L\Omega^p_{X_0}[-p]$ and $\mathcal{O}_{X_0}$ are concentrated in degree zero. Therefore the target of the map (7.6) is a discrete space and an element of it is completely determined by its values on the objects: functoriality with respect to morphisms in $\mathrm{QRSPrfd}_{\mathbb{Z}/p^2}$ amounts to checking a condition rather than specifying additional structure.

Therefore it is enough to compare the values of $c_{X_1,p}$ and $d_{X_1,p}$ on objects of $\mathrm{QRSPrfd}_{\mathbb{Z}/p^2}$, and the corollary is proven. □

We can now generalize the results of Section 5 on extensions in the conjugate filtration on the de Rham complex to schemes over k that only admit a lift to $W_2(k)$ and are not necessarily smooth.

Corollary 7.4. *For a quasisyntomic scheme X_0 over $\mathbb{F}_p$ equipped with a flat lift X_1 over $\mathbb{Z}/p^2$ the class $e_{X_1,p}\in\mathrm{RHom}_{D(X_1)}(L\Omega^p_{X_0},\mathcal{O}_{X_0}[p+1])$ of the extension $\bigoplus\limits_{i=0}^{p-1}L\Omega^i_{X_0}[-i]\to \mathrm{Fil}^{\mathrm{conj}}_p\mathrm{dR}_{X_0}\to L\Omega^p_{X_0}[-p]$ is naturally homotopic to $\mathrm{Bock}_{X_1}(\mathrm{ob}_{F,X_1}\circ\alpha(L\Omega^1_{X_0}))$.*

Proof. This is Lemma 6.12 combined with Theorem 7.1. □

For convenience of applications, let us also explicitly state this result on the level of cohomology classes in the case of smooth varieties:

Corollary 7.5. *For a smooth scheme X_0 over k equipped with a smooth lift X_1 over $W_2(k)$ the class $c_{X_1,p} \in H^p(X_0, \Lambda^p T_{X_0/k})$ is equal to $\mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0/k})$, and $e_{X_1,p} \in H^{p+1}(X_0, \Lambda^p T_{X_0/k})$ is equal to $\mathrm{Bock}_{X_1}(\mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0}))$.*

Hence the first potentially non-trivial differentials in the conjugate spectral sequence of a liftable scheme are described as follows:

Corollary 7.6. *For a smooth scheme X_0 over k equipped with a lift X_1 over $W_2(k)$ the conjugate spectral sequence $E^2_{ij} = H^i(X_0^{(1)}, \Omega^j_{X_0^{(1)}}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X_0/k)$ has no non-zero differentials on pages $E_2, \dots, E_p$. The differentials $d^{i,p}_{p+1} : H^i(X_0^{(1)}, \Omega^p_{X_0^{(1)}}) \to H^{i+p+1}(X_0^{(1)}, \mathcal{O})$ on page E_{p+1} can be described as*

$$\mathrm{Bock}_{\mathcal{O}_{X_1^{(1)}}} \circ (\mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0^{(1)}})) - (\mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0^{(1)}})) \circ \mathrm{Bock}_{\Omega^p_{X_1^{(1)}}} \tag{7.7}$$

where $\mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0^{(1)}})$ denotes the map $H^j(X_0^{(1)}, \Omega^p_{X_0^{(1)}}) \to H^{j+p}(X_0,^{(1)}, \mathcal{O}_{X_0^{(1)}})$ induced by the product with the class $c_{X_1^{(1)},p} = \mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0^{(1)}}) \in H^p(X_0^{(1)}, \Lambda^p T_{X_0^{(1)}})$, for $j = i, i+1$.

Proof. As usual, denote the inclusion of the special fiber by $i : X_0 \hookrightarrow X_1$. The differential $d^{i,p}_{p+1}$ is induced by the map $i^* e_{X_1^{(1)},p} : \Omega^p_{X_0^{(1)}} \to \mathcal{O}_{X_0^{(1)}}[p+1]$ which is the image of $\mathrm{ob}_{F,X_1} \cup \alpha(\Omega^1_{X_0^{(1)}}) \in \mathrm{RHom}_{X_0^{(1)}}(\Omega^p_{X_0^{(1)}}, \mathcal{O}_{X_0^{(1)}}[p])$ under the Bockstein homomorphism associated with the $W_2(k)$-module $\mathrm{RHom}_{X_1^{(1)}}(\Omega^p_{X_1^{(1)}}, \mathcal{O}_{X_1^{(1)}}[p])$. For the purposes of computing the effect of $i^* e_{X_1^{(1)},p}$ on the cohomology groups it is enough to compute $i_* i^* e_{X_1^{(1)},p} : i_* \Omega^p_{X_0^{(1)}} \to i_* \mathcal{O}_{X_0^{(1)}}[p+1]$. This is achieved by Lemma 7.7 below. ☐

Lemma 7.7. *Let X_1 be a flat scheme over $W_2(k)$ with the special fiber $X_0 := X_1 \times_{W_2(k)} k \overset{i}{\hookrightarrow} X_1$. For any two objects $M, N \in D(X_1)$ the composition*

$$\mathrm{RHom}_{D(X_0)}(i^*M, i^*N) \xrightarrow{\mathrm{Bock}_{\mathrm{RHom}_{X_1}(M,N)}} \mathrm{RHom}_{D(X_0)}(i^*M, i^*N)[1] \xrightarrow{i_*} \mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)[1] \tag{7.8}$$

is given by sending a map $f : i^*M \to i^*N$ to $\mathrm{Bock}_N \circ i_* f - i_* f[1] \circ \mathrm{Bock}_M$.

Proof. For any object $K \in D(X_1)$ there is a natural fiber sequence $i_*i^*K \to K \to i_*i^*K$ that we will view as a two-step filtration on K, defining therefore a functor $B : D(X_1) \to D_{\mathrm{Fil}}(X_1)$ to the category of filtered objects of $D(X_1)$. The complex of filtered morphisms between M and N fits into the fiber sequence

$$\mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N) \to \mathrm{RHom}_{D_{\mathrm{Fil}}(X_1)}(M, N) \to \mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)^{\oplus 2} \tag{7.9}$$

Here the first term is identified with the complex of morphisms from M to N that shift the filtration down by 1, and the second map associates to a filtered morphism its effect on the graded pieces of the filtrations. This fiber sequence is the Baer sum of the following fiber sequences

(7.10)
$$\mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N) \to \mathrm{RHom}_{D(X_1)}(M, i_*i^*N) \to \mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)$$
$$\mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N) \to \mathrm{RHom}_{D(X_1)}(i_*i^*M, N) \to \mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)$$

induced by applying the functor $\mathrm{RHom}_{D(X_1)}(-, i_*i^*N)$ to $i_*i^*M \to M \to i_*i^*M$, and the functor $\mathrm{RHom}_{D(X_1)}(i_*i^*M, -)$ to $i_*i^*N \to N \to i_*i^*N$, respectively.

Therefore the connecting map $\mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)^{\oplus 2} \to \mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)[1]$ induced by (7.9) is given by $(f_0, f_1) \mapsto \mathrm{Bock}_N \circ f_0 - f_1[1] \circ \mathrm{Bock}_M$.

The map $\mathrm{RHom}_{D(X_1)}(M, N) \to \mathrm{RHom}^{\mathrm{Fil}}_{D(X_1)}(M, N)$ induced by the functor B then induces a map of fiber sequences

(7.11)
$$\begin{tikzcd}
\mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N) \arrow[r] & \mathrm{RHom}^{\mathrm{Fil}}_{D(X_1)}(M, N) \arrow[r] & \mathrm{RHom}_{D(X_1)}(i_*i^*M, i_*i^*N)^{\oplus 2} \\
\mathrm{RHom}_{D(X_0)}(i^*M, i^*N) \arrow[r] \arrow[u, "i_*"] & \mathrm{RHom}_{D(X_1)}(M, N) \arrow[r] \arrow[u] & \mathrm{RHom}_{D(X_0)}(i^*M, i^*N) \arrow[u, "(i_*, i_*)"']
\end{tikzcd}$$

and this proves the lemma because $\mathrm{Bock}_{\mathrm{RHom}_{X_1}(M,N)}$ is precisely the connecting morphism induced by the bottom row of (7.11). □

8. Sen operator of a fibration in terms of Kodaira-Spencer class

In this section we specialize the formula for the class $c_{X_1,p}$ from Corollary 7.5 to p-dimensional $W_2(k)$-schemes that are fibered over a curve. The fact that the cotangent bundle of such a scheme admits a line subbundle allows us (Theorem 8.1) to relate the class $\alpha(\Omega^1_{X_0})$ to the Kodaira-Spencer class of the fibration. We then use this relation in Proposition 8.4 to give examples showing that the Sen operator Θ_X on dR_{X_0} might be non-semisimple. By Corollary 6.9, this is the case in a situation when the conjugate spectral sequence is non-degenerate, so Corollary 10.7 does also provides such examples. But we would like to demonstrate that non-semi-simplicity of Θ_X is a much more frequent phenomenon than non-degeneration of the Hodge-to-de Rham spectral sequence, appearing even for familiar classes of varieties.

We start by introducing the notation needed to state Theorem 8.1. Let $f: X_1 \to Y_1$ be a smooth morphism of smooth $W_2(k)$-schemes with $\dim Y_1 = 1$, $\mathrm{rel.dim}(f) = p-1$. We will describe the class $c_{X_1,p} \in H^p(X_0, \Lambda^p T_{X_0})$ in terms of the Kodaira-Spencer class of f and the obstruction to lifting the Frobenius morphism of Y_0 to Y_1. Recall that the Kodaira-Spencer class $\mathrm{ks}_{f_0}: T_{Y_0} \to R^1 f_{0*} T_{X_0/Y_0}$ is obtained from the fundamental triangle

$$T_{X_0/Y_0} \to T_{X_0} \to f_0^* T_{Y_0} \tag{8.1}$$

by applying the functor of 0th cohomology to the morphism $T_{Y_0} \to R f_{0*} T_{X_0/Y_0}[1]$ corresponding to the class of the extension (8.1) by adjunction. Denote by $\mathrm{ks}^{p-1}_{f_0}$ the composition $T^{\otimes p-1}_{Y_0} \xrightarrow{\mathrm{ks}^{\otimes p-1}_{f_0}} (R^1 f_{0*} T_{X_0/Y_0})^{\otimes p-1} \to R^{p-1} f_{0*} \Lambda^{p-1} T_{X_0/Y_0}$ where the second map is the cup product on cohomology.

By our assumption on the dimensions, the Leray spectral sequence for f_0 identifies $H^p(X_0, \Lambda^p T_{X_0})$ with $H^1(Y_0, R^{p-1}f_{0*}\Lambda^p T_{X_0}) = H^1(Y_0, T_{Y_0} \otimes R^{p-1}f_{0*}\Lambda^{p-1}T_{X_0/Y_0})$.

Theorem 8.1. *The class $c_{X,p} \in H^p(X_0, \Lambda^p T_{X_0})$ is equal, up to multiplying by an element of $\mathbb{F}_p^\times$, to the product of the class $\mathrm{ob}_{F,Y} \in H^1(Y_0, F_{Y_0}^* T_{Y_0}) = H^1(Y_0, T_{Y_0}^{\otimes p})$ with $\mathrm{ks}_{f_0}^{p-1} \in H^0(Y_0, T_{Y_0}^{\otimes 1-p} \otimes R^{p-1}f_{0*}\Lambda^{p-1}T_{X_0/Y_0})$.*

We have the following description of $\alpha(E)$ for a rank p vector bundle admitting a line subbundle. It will be proven, for $p > 2$, as a consequence of our computations with group cohomology in Section 13, under Lemma 13.12. For the proof in the case $p = 2$ see Remark 8.3 below.

Lemma 8.2. *Let E be a vector bundle on X_0 of rank p that fits into an extension*

$$0 \to L \to E \to E' \to 0 \tag{8.2}$$

where L is a line bundle, and E' is a vector bundle of rank $p-1$. The class of this extension defines an element $v(E) \in \mathrm{Ext}^1_{X_0}(E', L) = H^1(X_0, L \otimes (E')^\vee)$. Denote by $v(E)^{p-1} \in H^{p-1}(X_0, L^{\otimes p-1} \otimes (\det E')^\vee)$ the image of $v(E)^{\otimes p-1} \in H^{p-1}(X_0, (L \otimes (E')^\vee)^{\otimes p-1})$ under the map induced by $(L \otimes (E')^\vee)^{\otimes p-1} \to \Lambda^{p-1}(L \otimes (E')^\vee) = L^{\otimes p-1} \otimes (\det E')^\vee$.

The class $\alpha(E) \in \mathrm{Ext}^{p-1}_{X_0}(\Lambda^p E, F_{X_0}^* E) = H^{p-1}(X_0, F_{X_0}^* E \otimes L^\vee \otimes (\det E')^\vee)$ is equal, up to multiplying by a scalar from $\mathbb{F}_p^\times$, to the image of $v(E)^{p-1}$ under the map induced by $L^{\otimes p-1} \otimes (\det E')^\vee = F_{X_0}^* L \otimes L^\vee \otimes (\det E')^\vee \hookrightarrow F_{X_0}^* E \otimes L^\vee \otimes (\det E')^\vee$.

Remark 8.3. The lemma is straightforward when $p = 2$. It follows from the existence of the following map of extensions:

$$\begin{tikzcd}
F_{X_0}^* E \arrow[r] & S^2 E \arrow[r] & \Lambda^2 E \\
L^{\otimes 2} \arrow[r] \arrow[u, hook] & L \otimes E \arrow[r] \arrow[u, hook] & L \otimes E' \arrow[u, equal]
\end{tikzcd} \tag{8.3}$$

Here the left vertical map is the pullback along F_{X_0} of the inclusion $L \hookrightarrow E$, the top row is the extension representing $\alpha(E)$, and the bottom row is the tensor product of (8.2) with L.

Our proof of this lemma for $p > 2$ is rather ad hoc: it relies on the fact that the class $\alpha(V)$ for the tautological representation V of GL_p is non-zero, as proven in Proposition 12.1. Given that both $\alpha(E)$ and $v(E)^{p-1}$ admit explicit representatives as Yoneda extensions ((12.10) for the former and a Koszul complex for the latter), it would be nicer to have a direct proof of the identity claimed in Lemma 8.2, in the spirit of the proof for $p = 2$.

Proof of Theorem 8.1. By Corollary 7.5 the class $c_{X_1,p}$ is the product of $\alpha(\Omega^1_{X_0}) \in H^{p-1}(X_0, \Lambda^p T_{X_0} \otimes (F_{X_0}^* T_{X_0})^\vee)$ with $\mathrm{ob}_{F,X_1} \in H^1(X_0, F_{X_0}^* T_{X_0})$. The vector bundle $\Omega^1_{X_0}$ fits into the exact sequence dual to (8.1):

$$0 \to f^*\Omega^1_{Y_0} \to \Omega^1_{X_0} \to \Omega^1_{X_0/Y_0} \to 0 \tag{8.4}$$

and we can apply Lemma 8.2 to $E = \Omega^1_{X_0}$. It gives that the class $\alpha(\Omega^1_{X_0}) \in H^{p-1}(X_0, F_{X_0}^*\Omega^1_{X_0} \otimes \Lambda^p T_{X_0})$ is the image of the class $v(\Omega^1_{X_0})^{p-1} \in H^{p-1}(X_0, F_{X_0}^* f_0^* \Omega^1_{Y_0} \otimes \Lambda^p T_{X_0})$. Therefore the product $\alpha(\Omega^1_{X_0}) \cdot \mathrm{ob}_{F,X_1} \in H^p(X_0, \Lambda^p T_{X_0})$ is equal to the product of $v(\Omega^1_{X_0})^{p-1}$ with the image of ob_{F,X_1} under the map $H^1(X_0, F^*_{X_0}T_{X_0}) \to H^1(X_0, F^*_{X_0}f_0^*T_{Y_0})$.

By functoriality of obstructions, this image is equal to the image of ob_{F,Y_1} under the pullback map $H^1(Y_0, F^*_{Y_0}T_{Y_0}) \to H^1(X_0, F^*_{X_0}f_0^*T_{Y_0})$. By the Leray spectral sequence the receptacle of the class $v(\Omega^1_{X_0})^{p-1}$ fits into the exact sequence

$$(8.5)\quad 0 \to H^1(Y_0, R^{p-2}f_{0*}(F^*_{X_0}f_0^*\Omega^1_{Y_0} \otimes \Lambda^p T_{X_0})) \to H^{p-1}(X_0, F^*_{X_0}f_0^*\Omega^1_{Y_0} \otimes \Lambda^p T_{X_0}) \xrightarrow{\rho} H^0(Y_0, R^{p-1}f_{0*}(F^*_{X_0}f_0^*\Omega^1_{Y_0} \otimes \Lambda^p T_{X_0})) \to 0$$

Since $H^2(Y_0, \mathcal{F}) = 0$ for any quasicoherent sheaf $\mathcal{F}$ on the curve Y_0, our product only depends on the image of $v(\Omega^1_{X_0})^{p-1}$ in $H^0(Y_0, R^{p-1}f_{0*}(F^*_{X_0}f_0^*\Omega^1_{Y_0} \otimes \Lambda^p T_{X_0}))$ and is equal to the product of this image $\rho(v(\Omega^1_{X_0})^{p-1})$ with ob_{F,Y_1} under the identification $H^p(X_0, \Lambda^p T_{X_0}) \simeq H^1(Y_0, T_{Y_0} \otimes R^{p-1}f_{0*}\Lambda^{p-1}T_{X_0/Y_0})$.

It remains to observe that the image of the class $v(\Omega^1_{X_0}) \in H^1(X_0, f^*\Omega^1_{Y_0} \otimes T_{X_0/Y_0})$ in $H^0(Y_0, R^1(f_0^*\Omega^1_{Y_0} \otimes T_{X_0/Y_0})) = H^0(Y_0, \Omega^1_{Y_0} \otimes R^1 f_{0*}T_{X_0/Y_0})$ is the Kodaira-Spencer map ks_{f_0}. Therefore $\rho(\alpha(\Omega^1_{X_0})^{p-1}) \in H^0(Y_0, R^{p-1}f_{0*}(F^*_{X_0}f_0^*\Omega^1_{Y_0} \otimes \Lambda^p T_{X_0})) = H^0(Y_0, F^*_{Y_0}\Omega^1_{Y_0} \otimes T_{Y_0} \otimes R^{p-1}f_{0*}\Lambda^{p-1}T_{X_0/Y_0})$ is equal to $\mathrm{ks}_{f_0}^{p-1}$ and the desired formula for $c_{X_1,p}$ is proven. □

We will now demonstrate that there does exist a fibration as in Theorem 8.1 for which $c_{X_1,p}$ is non-zero. From now until the end of this section assume that $k = \overline{\mathbb{F}}_p$.

Proposition 8.4. *For every p there exists a smooth projective scheme X over $W(k)$ with $\dim_{W(k)}(X) = p$ such that the class $c_{X,p} \in H^p(X_0, \Lambda^p T_{X_0})$ is non-zero, and therefore the Sen operator on dR_{X_0} is not semi-simple.*

Remark 8.5. Since X has relative dimension p, the de Rham complex dR_{X_0} of X_0 is necessarily decomposable, by [DI87, Corollaire 2.3].

Proof. We will construct X as the $(p-1)$th relative Cartesian power $S^{\times_Y (p-1)}$ of an appropriate smooth projective morphism $h : S \to Y$ of relative dimension 1, where Y is a geometrically connected smooth projective relative curve over $W(k)$. Assume that

1. The Kodaira-Spencer map $\mathrm{ks}_{h_0} : T_{Y_0} \to R^1h_{0*}T_{S_0/Y_0}$ is an injection of vector bundles
2. $\mathrm{coker}\,\mathrm{ks}_{h_0}$ is a direct sum $\bigoplus L_j$ of line bundles such that $\deg L_j < \frac{1}{p-1}\deg \Omega^1_{Y_0}$ for all j.

Injectivity of the Kodaira-Spencer map implies that the curve fibration h is not isotrivial, which forces the genus of Y and all of the fibers of h to be larger than or equal to 2 by [Szp81, Théoremè 4]. In particular, the Frobenius endomorphism of Y_0 does not lift to $Y \times_{W(k)} W_2(k)$ so the class $\mathrm{ob}_{Y_1,F} \in H^1(Y_0, F^*_{Y_0}T_{Y_0})$ is non-zero, cf. [Ray83, Lemma I.5.4]. We will now prove that under these conditions on S the class $c_{X,p} \in H^p(X_0, \Lambda^p T_{X_0})$ for the p-dimensional $W(k)$-scheme X is non-zero, and then will check that a family of curves h satisfying (1) and (2) does indeed exist.

Denote by $\pi_1, \ldots, \pi_{p-1} : X = S^{\times_Y (p-1)} \to S$ the projection maps, and by $f : X \to Y$ the map down to Y (so that $f = h \circ \pi_i$, for all i). We have $T_{X/Y} \simeq \bigoplus\limits_{i=1}^{p-1} \pi_i^* T_{S/Y}$. For each i the pushforward $R^1 f_*(\pi_i^* T_{S/Y}) = H^1(Rh_* \circ R\pi_{i*}(\pi_i^* T_{S/Y})) = H^1(Rh_*(T_{S/Y} \otimes R\pi_{i*}\mathcal{O}_X))$

is equal to $R^1h_*T_{S/Y}$, because $h_*T_{S/Y}=0$ by our assumption on the genus of the fibers of h. The Kodaira-Spencer class $\mathrm{ks}_f:T_Y\to R^1f_*T_{X/Y}=\bigoplus\limits_{i=1}^{p-1}R^1h_*T_{S/Y}$ is then equal to the diagonal map $\bigoplus\limits_{i=1}^{p-1}\mathrm{ks}_h$, because the extension $T_{X/Y}\to T_X\to f^*T_Y$ is the Baer sum of the pullbacks along π_i of the extension $T_{S/Y}\to T_S\to h^*T_Y$.

The pushforward $R^{p-1}f_*\Lambda^{p-1}T_{X/Y}=R^{p-1}f_*(\bigotimes\limits_{i=1}^{p-1}\pi_i^*T_{S/Y})$ is likewise identified with $(R^1h_*T_{S/Y})^{\otimes p-1}$, and the $(p-1)$th power ks_f^{p-1} of the Kodaira-Spencer map is equal to $\mathrm{ks}_h^{\otimes p-1}:T_Y^{\otimes p-1}\to(R^1h_*T_{S/Y})^{\otimes p-1}$.

By Theorem 8.1, the class $c_{X,p}\in H^p(X_0,\Lambda^pT_{X_0})=H^1(Y_0,T_{Y_0}\otimes R^{p-1}\Lambda^{p-1}T_{X_0/Y_0})$ is (up to a scalar from $\mathbb{F}_p^\times$) the image of the obstruction to lifting Frobenius $\mathrm{ob}_{F,Y_1}\in H^1(Y_0,F^*T_{Y_0})=H^1(Y_0,T_{Y_0}^{\otimes p})$ under the map $\mathrm{id}_{T_{Y_0}}\otimes\mathrm{ks}_{h_0}^{\otimes p-1}:T_{Y_0}^{\otimes p}\to T_{Y_0}\otimes(R^1h_{0*}T_{S_0/Y_0})^{\otimes p-1}$. The proof of non-vanishing of $c_{X,p}$ will be completed if we can show that the induced map $H^1(Y_0,T_{Y_0}^{\otimes p})\to H^1(Y_0,T_{Y_0}\otimes(R^1h_{0*}T_{S_0/Y_0})^{\otimes p-1})$ is injective.

For brevity, denote the vector bundle $\operatorname{coker}\mathrm{ks}_{h_0}$ by Q. The map $\mathrm{id}_{T_{Y_0}}\otimes\mathrm{ks}_{h_0}^{\otimes p-1}$ is an injection of vector bundles whose cokernel admits a filtration with graded pieces isomorphic to $T_{Y_0}^{\otimes i}\otimes Q^{\otimes p-i}$ for $i=1,\dots,p-1$. By our assumption (2), each $T_{Y_0}^{\otimes i}\otimes Q^{\otimes p-i}$ is a direct sum of line bundles of degree $<\frac{1}{p-1}\deg\Omega^1_{Y_0}\cdot(p-i)-\deg\Omega^1_{Y_0}\cdot i<0$ because $\deg\Omega^1_{Y_0}=2g(Y_0)-2$ is positive. Therefore $H^0(Y_0,\operatorname{coker}(\mathrm{id}_{T_{Y_0}}\otimes\mathrm{ks}_{h_0}^{\otimes p-1}))=0$ which implies that the map induced by $\mathrm{id}_{T_{Y_0}}\otimes\mathrm{ks}_{h_0}^{\otimes p-1}$ on cohomology in degree 1 is injective.

We will now prove that there exists a family of curves $h:S\to Y$ satisfying properties (1), (2). We will construct Y as a complete intersection of ample divisors in an appropriate compactification $\mathcal{M}_g^*$ of the moduli space of curves of genus g, following the idea of Mumford ([Oor74, §1]) for constructing proper subvarieties of moduli spaces of curves. We only need to make sure that this construction goes through over $W(k)$, and check that the condition (2) is fulfilled. Denote by $M_{g,W(k)}$ the coarse moduli scheme of the stack $\mathcal{M}_{g,W(k)}$, and by $\overline{M}_{g,W(k)}$ the coarse moduli scheme of its Deligne-Mumford compactification.

By [FC90, Theorem V.2.5] there exists a projective scheme $A^*_{g,W(k)}$ over $W(k)$ that contains the coarse moduli scheme $A_{g,W(k)}$ of principally polarized abelian varieties of dimension g as a dense open subscheme such that $\dim_{W(k)}(A^*_{g,W(k)}\setminus A_{g,W(k)})=g$. Moreover, the Torelli map extends to a morphism $j:\overline{M}_{g,W(k)}\to A^*_{g,W(k)}$ that induces a locally closed immersion of the locus $U\subset\overline{M}_{g,W(k)}$ of smooth curves without nontrivial automorphisms into $A^*_{g,W(k)}$. Denote also by $\pi:\mathcal{C}\to U$ the universal curve over U.

Assume from now on that $g\geq\max(4,\frac{p}{3}+1)$. The inequality $g\geq 4$ ensures that the locus of curves without nontrivial automorphisms has complement of codimension ≥ 2 in $\mathcal{M}_g$, and the condition $g\geq\frac{p}{3}+1$ will be used to ensure property (2). Denote by $M^*_{g,W(k)}$ the closure of $j(M_{g,W(k)})$ inside $A^*_{g,W(k)}$. This is a flat projective scheme over $W(k)$ that contains U as a dense open subscheme, such that fibers of $M^*_{g,W(k)}\setminus U$ over both points of $\operatorname{Spec}W(k)$ have codimension ≥ 2 (though we do not claim that this complement is flat over $W(k)$). Denote by ω the line bundle $\det(\pi_*\Omega^1_{\mathcal{C}/U})$ on U. By the property [FC90, Theorem V.2.5(1)] of the Satake compactification, some positive power $\omega^{\otimes m}$ extends to a very ample L line bundle on $M^*_{g,W(k)}$.

Take $Y\subset U\subset M^*_{g,W(k)}$ to be a smooth proper complete intersection of zero loci of sections of L that has $\dim(Y/W(k))=1$. It is possible to find such a complete intersection entirely contained in U because the codimension of the complement of U is at least 2. Let $h:S\to Y$ be the restriction of the universal curve $\mathcal{C}$ to Y. We have the exact sequence corresponding to the closed embedding $\iota:Y\hookrightarrow U$:

$$0\to T_Y\xrightarrow{d\iota}T_{M_{g,W(k)}}|_Y\to L|_Y^{\oplus 3g-4}\to 0 \tag{8.6}$$

where we identified the normal bundle to Y with $L|_Y^{\oplus 3g-4}$. Since U is the locus where the map $\mathcal{M}_{g,W(k)}\to M_{g,W(k)}$ from the moduli stack of curves to the coarse moduli space is an isomorphism, the sheaf $T_{M_{g,W(k)}}|_Y$ is identified with $R^1h_*T_{S/Y}$ in a way that identifies $d\iota$ with the Kodaira-Spencer map ks_f.

By [HM82, Theorem 2], the determinant line bundle $\det(T_{M_g,W(k)}|_Y)$ is isomorphic to $\omega^{\otimes -13}|_Y$ and, in particular, has negative degree. Therefore $\deg(T_Y)+(3g-4)\cdot\deg L|_Y\leq 0$ giving that $\deg L|_Y\leq -\deg(T_Y)\cdot\frac{1}{3g-4}<-\deg(T_Y)\cdot\frac{1}{p-1}$ by our assumption on g. Therefore the family $h:S\to Y$ satisfies all of the desired assumptions. □

Remark 8.6. At the moment it is unclear to me whether the class $c_{X,p}$ is non-zero for other natural classes of varieties. For instance, when $p=2$ is $c_{X,2}\in H^2(X_0,\Lambda^2T_{X_0})\simeq k$ non-zero for a general K3 surface X over $W_2(k)$?

9. Cohomology of abelian varieties

In this section, we apply Theorem 4.1 to the de Rham, coherent, and étale cohomology of abelian varieties equipped with a group action. These results will play the key role in our example of a liftable variety with a non-degenerate conjugate spectral sequence, which ultimately relies on the existence of a supersingular abelian variety with an action of a group, whose Hodge cohomology is equivariantly decomposable, but de Rham cohomology is not.

In this section, for a commutative ring R and a discrete group G we denote by $D_G(R)$ the derived ∞-category of complexes of R-modules equipped with a G-action.

9.1. **Coherent cohomology.** Recall that for a group G acting on a finite-dimensional k-vector space V we have a class $\alpha(V)\in\mathrm{Ext}^{p-1}_G(\Lambda^pV,V^{(1)})=H^{p-1}(G,V^{(1)}\otimes(\Lambda^pV)^\vee)$ defined in Definition 3.4, corresponding to the extension $V^{(1)}[-2]\to\tau^{\geq 2}S^p(V[-1])\to\Lambda^pV[-p]$ in $D_G(k)$. If there exists a representation of G on a free $W_2(k)$-module $\widetilde{V}$ such that $\widetilde{V}/p\simeq V$ then the module $\widetilde{V}^{(1)}\otimes(\Lambda^p\widetilde{V})^\vee$ defines the Bockstein homomorphisms $\mathrm{Bock}^i:H^i(G,V^{(1)}\otimes(\Lambda^pV)^\vee)\to H^{i+1}(G,V^{(1)}\otimes(\Lambda^pV)^\vee)$.

Proposition 9.1. *Let A be an abelian scheme over $W(k)$ equipped with an action of a discrete group G.*

(0) There exists a G-equivariant equivalence $\tau^{\leq p-1}\mathrm{R}\Gamma(A,\mathcal{O})\simeq\bigoplus\limits_{i=0}^{p-1}H^i(A,\mathcal{O})[-i]$.

(1) If $F^*_{A_0}:H^1(A_0,\mathcal{O})\to H^1(A_0,\mathcal{O})$ is zero, then there exists a G-equivariant equivalence

$$\tau^{\leq p}\mathrm{R}\Gamma(A,\mathcal{O})\simeq\bigoplus_{i=0}^{p}H^i(A,\mathcal{O})[-i] \tag{9.1}$$

(2) *Suppose that A admits an endomorphism $\widetilde{F}_A : A \to A$ that lifts the absolute Frobenius endomorphism F_{A_0} (in particular, A_0 is ordinary) and commutes with the action of G. Then the extension class $H^p(A,\mathcal{O}) \to \bigoplus_{i=0}^{p-1} H^i(A,\mathcal{O})[p-i+1]$ in $D_G(W(k))$ corresponding to $\tau^{\leq p}\mathrm{R}\Gamma(A,\mathcal{O})$ lands in the direct summand $H^1(A,\mathcal{O})[p]$ and the resulting class in $\mathrm{Ext}^p_{G,W(k)}(H^p(A,\mathcal{O}),H^1(A,\mathcal{O})) = H^p(G, H^1(A,\mathcal{O})\otimes\Lambda^p H^1(A,\mathcal{O})^\vee)$ is equal to*

$$\widetilde{F}_A^*(\mathrm{Bock}^{p-1}(\alpha(H^1(A_0,\mathcal{O})))) \tag{9.2}$$

where $\alpha(H^1(A_0,\mathcal{O})) \in H^{p-1}(G, H^1(A_0,\mathcal{O})^{(1)} \otimes \Lambda^p H^1(A_0,\mathcal{O})^\vee)$ is the class corresponding to the representation of G on the k-vector space $H^1(A_0,\mathcal{O})$, and Bock^{p-1} is the Bockstein homomorphism induced by the $W(k)$-module $H^1(A,\mathcal{O})^{(1)}\otimes \Lambda^p H^1(A,\mathcal{O})^\vee$.

Proof. The identity section $e : \operatorname{Spec} W(k) \to A$ induces the augmentation $e^* : \mathrm{R}\Gamma(A,\mathcal{O}) \to W(k)$, and the cohomology algebra $H^*(A,\mathcal{O})$ is the exterior algebra on $H^1(A,\mathcal{O})$ so we can apply Theorem 4.4 to the G-equivariant derived commutative algebra $\mathrm{R}\Gamma(A,\mathcal{O})$. Part (0) then follows. To prove statements (1) and (2), we will apply (4.11). The formula for the extension $H^p(A,\mathcal{O}) \to \tau^{\leq p-1}\mathrm{R}\Gamma(A,\mathcal{O})[p+1]$ in $D_G(W(k))$ reads:

$$\Lambda^p H^1(A,\mathcal{O}) \xrightarrow{\alpha(H^1(A_0,\mathcal{O}))} H^1(A_0,\mathcal{O})^{(1)}[p-1] \xrightarrow{F^*_{A_0}} H^1(A_0,\mathcal{O})[p-1] \xrightarrow{\mathrm{Bock}_{H^1(A,\mathcal{O})}} H^1(A,\mathcal{O})[p] \tag{9.3}$$

Here we identified the Frobenius endomorphism of the cosimplicial algebra $\mathrm{R}\Gamma(A_0,\mathcal{O})$ with $F^*_{A_0}$ by Lemma 5.8. In particular, if $F^*_{A_0}$ on $H^1(A_0,\mathcal{O})$ is zero, then the composition (9.3) vanishes, which proves part (1).

Regarding part (2), note that the lift of Frobenius $\widetilde{F}_A^*$ induces the following commutative diagram in $D_G(W(k))$:

$$\begin{tikzcd} H^1(A_0,\mathcal{O})^{(1)} \arrow[r, "F^*_{A_0}"] \arrow[d, "\mathrm{Bock}^{(1)}_A"] & H^1(A_0,\mathcal{O}) \arrow[d, "\mathrm{Bock}_A"] \\ H^1(A,\mathcal{O})^{(1)}[1] \arrow[r, "\widetilde{F}^*_A"] & H^1(A,\mathcal{O})[1] \end{tikzcd} \tag{9.4}$$

This allows us to rewrite (9.3) as

$$\Lambda^p H^1(A,\mathcal{O}) \xrightarrow{\alpha(H^1(A_0,\mathcal{O}))} H^1(A_0,\mathcal{O})^{(1)}[p-1] \xrightarrow{\mathrm{Bock}_{H^1(A,\mathcal{O})^{(1)}}[p]} H^1(A,\mathcal{O})^{(1)}[p] \xrightarrow{\widetilde{F}^*_A} H^1(A,\mathcal{O})[p] \tag{9.5}$$

The desired formula (9.2) now follows from Lemma 5.11. □

Corollary 9.2. *Let $n \geq 2p$ be an integer and E_0 be an elliptic curve over k. The group $GL_n(\mathbb{Z})$ acts on the abelian variety E_0^n and therefore acts on the complex $\mathrm{R}\Gamma(E_0^n,\mathcal{O})$. The truncation $\tau^{\leq p}\mathrm{R}\Gamma(E_0^n,\mathcal{O})$ decomposes in $D_{GL_n(\mathbb{Z})}(k)$ as a direct sum of its cohomology modules if and only if E_0 is supersingular.*

Proof. We can choose a lift E of E_0 to an elliptic curve over $W(k)$ and apply Proposition 9.1 to the abelian scheme E^n that is being acted on by the group $G = GL_n(\mathbb{Z})$. If E_0 is supersingular then Proposition 9.1(1) gives that $\tau^{\leq p}\mathrm{R}\Gamma(E_0^n,\mathcal{O})$ is decomposable, for all n.

If E_0 is ordinary, assume moreover that our chosen lift E is the canonical one, so that F_{E_0} lifts to a map $\widetilde{F}_E : E \to E$. Following Proposition 9.1(2), we need to check that the mod p reduction of the class $\widetilde{F}^*_{E^n}(\mathrm{Bock}^{p-1}(\alpha(H^1(E_0,\mathcal{O})))) \in H^p(GL_n(\mathbb{Z}), H^1(E^n,\mathcal{O}) \otimes \Lambda^p H^1(E^n,\mathcal{O})^\vee)$ is non-zero. Note that the map $\widetilde{F}^*_{E^n}$ is a bijection, so we only need to check that its argument is non-zero modulo p.

Let $F/\mathbb{Q}$ be the quadratic extension provided by Proposition 14.1. Identifying the abelian group $\mathcal{O}_F$ with $\mathbb{Z}^{\oplus 2}$ we get an embedding $SL_p(\mathcal{O}_F) \subset GL_{2p}(\mathbb{Z}) \subset GL_n(\mathbb{Z})$. The induced action of $SL_p(\mathcal{O}_F)$ on $H^1(E^n,\mathcal{O}) = H^1(E,\mathcal{O}) \otimes_{W(k)} W(k)^{\oplus n} \simeq W(k)^{\oplus n}$ contains the tautological representation $W(k)^{\oplus p}$ as a direct summand, hence the mod p reduction of the class $\mathrm{Bock}^{p-1}(\alpha(H^1(E^n,\mathcal{O})))$ does not vanish in $H^p(SL_p(\mathcal{O}_F), H^1(E_0^n,\mathcal{O})^{(1)} \otimes \Lambda^p H^1(E_0,\mathcal{O})^\vee)$. Therefore the analogous class in $H^p(GL_n(\mathbb{Z}), H^1(E_0^n,\mathcal{O})^{(1)} \otimes \Lambda^p H^1(E_0,\mathcal{O})^\vee)$ does not vanish either, and the non-decomposability of $\tau^{\leq p}\mathrm{R}\Gamma(E_0^n,\mathcal{O})$ in $D_{GL_n(\mathbb{Z})}(k)$ follows. $\square$

9.2. **De Rham cohomology.** We can also apply Theorem 4.1 to the de Rham cohomology of an abelian variety equipped with a group action.

Proposition 9.3. *Let A_0 be an abelian variety over k equipped with an action of a discrete group G.*

(1) $\tau^{\leq p-1}\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)$ is G-equivariantly equivalent to $\bigoplus_{i=0}^{p-1} H^i_{\mathrm{dR}}(A_0/k)[-i]$.

(2) The extension class $H^p_{\mathrm{dR}}(A_0/k) \to \bigoplus_{i=0}^{p-1} H^i_{\mathrm{dR}}(A_0/k)[p+1-i]$ corresponding to $\tau^{\leq p}\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)$ lands in the direct summand $H^1(A_0/k)[p]$ and the resulting class in $\mathrm{Ext}^p_{G,k}(H^p_{\mathrm{dR}}(A_0/k), H^1_{\mathrm{dR}}(A_0/k)) = H^p(G, H^1_{\mathrm{dR}}(A_0/k) \otimes \Lambda^p H^1_{\mathrm{dR}}(A_0/k)^\vee)$ equals to

$$F^*_{A_0} \circ \mathrm{Bock}^{p-1}(\alpha(H^1_{\mathrm{dR}}(A_0/k))) \tag{9.6}$$

where we use the crystalline cohomology $H^1_{\mathrm{cris}}(A_0/W_2(k))$ as a lift of the G-representation $H^1_{\mathrm{dR}}(A_0/k)$ to define the Bockstein homomorphism.

Proof. The derived commutative algebra $\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)$ in the category of G-representations on k-vector spaces admits a lift to $W(k)$ given by the crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cris}}(A_0/W(k))$. This algebra has an augmentation $e^* : \mathrm{R}\Gamma_{\mathrm{cris}}(A_0/W(k)) \to \mathrm{R}\Gamma_{\mathrm{cris}}(\mathrm{Spec}\, k/W(k)) = W(k)$ so we can apply Theorem 4.4 to it. This gives decomposability of $\tau^{\leq p-1}\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)$ and describes the extension class $H^p_{\mathrm{cris}}(A_0/W(k)) \to H^1_{\mathrm{cris}}(A_0/W(k))[p]$ as the composition

$$H^p_{\mathrm{cris}}(A_0/W(k)) = \Lambda^p H^1_{\mathrm{cris}}(A_0/W(k)) \to \Lambda^p H^1_{\mathrm{dR}}(A_0/k) \xrightarrow{\alpha(H^1_{\mathrm{dR}}(A_0/k))} \tag{9.7}$$

$$H^1_{\mathrm{dR}}(A_0/k)^{(1)}[p-1] \xrightarrow{\varphi_{\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)}} H^1_{\mathrm{dR}}(A_0/k)[p-1] \xrightarrow{\mathrm{Bock}_{H^1_{\mathrm{cris}}(A_0/W(k))}} H^1_{\mathrm{cris}}(A_0/W(k))[p]$$

Since $\varphi_{\mathrm{R}\Gamma_{\mathrm{dR}}(A_0/k)}$ lifts to an endomorphism $\varphi^*_{A_0,\mathrm{cris}}$ of $\mathrm{R}\Gamma_{\mathrm{cris}}(A_0/W(k))$, the composition of the last two arrows of (9.7) is equivalent to

(9.8)
$$H^1_{\mathrm{dR}}(A_0/k)^{(1)}[p-1] \xrightarrow{\mathrm{Bock}_{H^1_{\mathrm{cris}}(A_0/W(k))^{(1)}}} H^1_{\mathrm{cris}}(A_0/W(k))^{(1)}[p] \xrightarrow{\varphi^*_{A_0,\mathrm{cris}}} H^1_{\mathrm{cris}}(A_0/W(k))[p]$$

As in the deduction of Theorem 5.10 from Theorem 5.9, we now apply Lemma 5.11 to conclude that the class defined by the composition (9.7) equals to $\varphi^*_{A_0}\circ \mathrm{Bock}_{p-1}(\alpha(H^1_{\mathrm{dR}}(A_0/k)))\in \mathrm{Ext}^p_G(H^p_{\mathrm{cris}}(A_0/W(k)),H^1_{\mathrm{cris}}(A_0/W(k)))$. Reducing this class modulo p gives the desired result. $\square$

9.3. **Galois action on étale cohomology of abelian varieties.** This application was suggested by Alexei Skorobogatov who had independently conjectured the validity of Proposition 9.4 for $p=2$.

Let A be an abelian variety over an arbitrary field F of characteristic not equal to p. Denote by $\overline{F}$ a separable closure of F and by $G_F:=\mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. We view $\mathrm{R}\Gamma_{\mathrm{et}}(A_{\overline{F}},\mathbb{Z}_p)$ as an object of the derived category $\widehat{D}_{G_F}(\mathbb{Z}_p)$ of p-complete $\mathbb{Z}_p$-modules equipped with a continuous action of G_F. More precisely, it is the inverse limit of categories $D_{G_F}(\mathbb{Z}/p^n)$ of discrete $\mathbb{Z}/p^n$-modules over the profinite group G_F.

Proposition 9.4. *(1) The truncation $\tau^{\leq p-1}\mathrm{R}\Gamma_{et}(A_{\overline{F}},\mathbb{Z}_p)$ is equivalent to $\bigoplus\limits_{i=0}^{p-1}H^i_{et}(A_{\overline{F}},\mathbb{Z}_p)[-i]$ in $\widehat{D}_{G_F}(\mathbb{Z}_p)$.*

(2) The extension class $H^p_{et}(A_{\overline{F}},\mathbb{Z}_p)\to\bigoplus\limits_{i=0}^{p-1}H^i_{et}(A_{\overline{F}},\mathbb{Z}_p)[p+1-i]$ corresponding to the fiber sequence $\tau^{\leq p-1}\mathrm{R}\Gamma_{et}(A_{\overline{F}},\mathbb{Z}_p)\to\tau^{\leq p}\mathrm{R}\Gamma_{et}(A_{\overline{F}},\mathbb{Z}_p)\to H^p_{et}(A_{\overline{F}},\mathbb{Z}_p)[-p]$ can be described as

$$H^p_{et}(A_{\overline{F}},\mathbb{Z}_p)\xrightarrow{\alpha(H^1_{et}(A_{\overline{F}},\mathbb{F}_p))}H^1_{et}(A_{\overline{F}},\mathbb{F}_p)[p-1]\xrightarrow{\mathrm{Bock}_{H^1_{et}(A_{\overline{F}},\mathbb{Z}_p)}}H^1_{et}(A_{\overline{F}},\mathbb{Z}_p)[p]\xrightarrow{\oplus}\bigoplus_{i=0}^{p-1}H^i_{et}(A_{\overline{F}},\mathbb{Z}_p)[p+1-i]\tag{9.9}$$

Proof. The étale cohomology $\mathrm{R}\Gamma_{\mathrm{et}}(A_{\overline{F}},\mathbb{Z}_p)$ has a structure of a derived commutative algebra in $\widehat{D}_{G_F}(\mathbb{Z}_p)$ by Lemma 2.16. The pullback along the identity section $e:\mathrm{Spec}\,F\to A$ induces an augmentation, and multiplication on cohomology induces isomorphisms $H^i_{\mathrm{et}}(A_{\overline{F}},\mathbb{Z}_p)\simeq\Lambda^iH^1_{\mathrm{et}}(A_{\overline{F}},\mathbb{Z}_p)$. Hence we may apply Theorem 4.4 to $\mathrm{R}\Gamma_{\mathrm{et}}(A_{\overline{F}},\mathbb{Z}_p)$ which readily implies part (1).

Part (2) follows from the formula (4.11) by the fact that the Frobenius map $\varphi_{\mathrm{R}\Gamma_{\mathrm{et}}(A_{\overline{F}},\mathbb{F}_p)}:H^1_{\mathrm{et}}(A_{\overline{F}},\mathbb{F}_p)\to H^1_{\mathrm{et}}(A_{\overline{F}},\mathbb{F}_p)$ is the identity map, by Lemma 2.17. $\square$

We can apply this to describe some of the differentials in the Hochschild-Serre spectral sequence of A with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$. Here $H^i(G_F,-)$ denotes continuous cohomology of the Galois group G_F.

Corollary 9.5. *(1) In the spectral sequence $E_2^{i,j}=H^i(G_F,H^j_{et}(A_{\overline{F}},\mathbb{Z}_p))\Rightarrow H^{i+j}_{et}(A,\mathbb{Z}_p)$ there are no non-zero differentials on pages $E_2,\dots,E_{p-1}$, and the differentials $d_p^{i,p}:H^i(G_F,H^p_{et}(A_{\overline{F}},\mathbb{Z}_p))\to H^{i+p}(G_F,H^1_{et}(A_{\overline{F}},\mathbb{Z}_p))$ can be described as*

$$H^i(G_F,\Lambda^pH^1_{et}(A_{\overline{F}},\mathbb{Z}_p))\xrightarrow{\alpha(H^1_{et}(A_{\overline{F}},\mathbb{F}_p))}H^{i+p-1}(G_F,H^1_{et}(A_{\overline{F}},\mathbb{F}_p))\xrightarrow{\mathrm{Bock}^{i+p-1}_{H^1_{et}(A_{\overline{F}},\mathbb{Z}_p)}}H^{i+p}(G_F,H^1_{et}(A_{\overline{F}},\mathbb{Z}_p))\tag{9.10}$$

(2) Likewise, the the spectral sequence $E_2^{i,j}=H^i(G_F,H^j_{et}(A_{\overline{F}},\mathbb{F}_p))\Rightarrow H^{i+j}_{et}(A,\mathbb{F}_p)$ has no non-zero differentials on pages $E_2,\dots,E_{p-1}$, and the differentials $d_p^{i,p}:H^i(G_F,H^p_{et}(A_{\overline{F}},\mathbb{F}_p))\to H^{i+p}(G_F,H^1_{et}(A_{\overline{F}},\mathbb{F}_p))$ are equal to

$$\mathrm{Bock}^{i+p-1}_{H^1_{et}(A_{\overline{F}},\mathbb{Z}/p^2)}\circ\alpha(H^1_{et}(A_{\overline{F}},\mathbb{F}_p))-\alpha(H^1_{et}(A_{\overline{F}},\mathbb{F}_p))\circ\mathrm{Bock}^i_{H^p_{et}(A_{\overline{F}},\mathbb{Z}/p^2)} \tag{9.11}$$

Proof. Part (1) follows immediately from the formula (9.9) because $d_p^{i,p}$ is the result of applying the functor $H^i(G_F,-)$ to that map. Part (2) follows by applying Lemma 7.7 to describe the effect of the mod p reduction of (9.9) on cohomology. $\square$

10. Liftable variety with non-degenerate conjugate spectral sequence

In this section we construct an example of a smooth projective variety over k that lifts to a smooth projective scheme over $W(k)$ but whose conjugate spectral sequence has a non-zero differential. Let us start by describing our example.

Choose an elliptic curve E over $W(k)$ such that the special fiber $E_0=E\times_{W(k)}k$ is supersingular. Denote p^2 by q, and consider the finite flat group scheme $H:=E[p]\otimes_{\mathbb{F}_p}\mathbb{F}_q^{\oplus p}$. It is isomorphic to a product of $2p$ copies of $E[p]$, and we write it this way to define an action of the group $GL_p(\mathbb{F}_q)$ on it via the tautological representation on $\mathbb{F}_q^{\oplus p}$. Define the non-commutative finite flat group scheme

$$G:=SL_p(\mathbb{F}_q)\ltimes(E[p]\otimes_{\mathbb{F}_p}\mathbb{F}_q^{\oplus p}) \tag{10.1}$$

The main result of this section is that the classifying stack BG_0 of the special fiber of this finite group scheme has a non-degenerate conjugate spectral sequence. In Corollary 10.7 we then approximate the stack BG by a smooth projective scheme whose special fiber also has a non-zero differential in its conjugate spectral sequence.

Theorem 10.1. *The differential $d_{p+1}^{0,p}:H^0(BG_0,L\Omega^p_{BG_0/k})\to H^{p+1}(BG_0,\mathcal{O})$ on the $(p+1)$st page of the conjugate spectral sequence of the classifying stack BG_0 is non-zero.*

In general, if X_0 is an algebraic stack over k, the spectral sequence associated with the conjugate filtration on its de Rham complex starts with the second page of the form $E_2^{i,j}=H^i(X_0,L\Omega^j_{X_0/k})$. If X_0 admits a lift over $W_2(k)$ then there are no non-zero differentials on pages $E_2,\dots,E_p$, by the main result of [KP22]. In this case we denote by $d_{p+1}^{i,j}:H^i(X_0,L\Omega^j_{X_0/k})\to H^{i+p+1}(X_0,L\Omega^{j-p}_{X_0/k})$ the differentials on page E_{p+1}.

Although we already have a formula for the differential $d_{p+1}^{0,p}$, provided by Theorem 5.10, we take a somewhat roundabout approach to proving Theorem 10.1 by proving as an intermediary that for a certain n the quotient stack of the abelian variety $E_0^{\times n}$ by a well-chosen infinite group has a non-degenerate conjugate spectral sequence. This non-degeneracy will arise from the results of Section 9 by contrasting the extensions in the canonical filtrations on de Rham and Hodge cohomology of $E_0^{\times n}$, viewed as complexes with a group action.

Let F be a number field provided by Proposition 14.1. It is a quadratic extension of $\mathbb{Q}$ such that $\mathcal{O}_F/p\simeq\mathbb{F}_{p^2}=\mathbb{F}_q$. Consider the abelian scheme $A:=E\otimes_{\mathbb{Z}}\mathcal{O}_F^{\oplus p}$ equipped with the natural action of the group $GL_p(\mathcal{O}_F)$. The '$\otimes$' symbol refers here to Serre's tensor product [CCO14, 1.7.4]. Explicitly, choosing a $\mathbb{Z}$-basis in $\mathcal{O}_F$ we get an identification $A\simeq E^{\times 2p}$ and the group $GL_p(\mathcal{O}_F)$ acts through the embedding $GL_p(\mathcal{O}_F)\hookrightarrow GL_{2p}(\mathbb{Z})$.

We can view the multiplication-by-p map $A \xrightarrow{[p]} A$ as an $A[p]$-torsor on A and therefore get a $GL_p(\mathcal{O}_F)$-equivariant classifying map $A \to BA[p]$. The action of $GL_p(\mathcal{O}_F)$ on the p-torsion group scheme $A[p]$ factors through $GL_p(\mathcal{O}_F/p) = GL_p(\mathbb{F}_q)$ and $A[p]$ is $GL_p(\mathbb{F}_q)$-equivariantly isomorphic to H. Hence the classifying map can be viewed as a $GL_p(\mathcal{O}_F)$-equivariant morphism $A \to BH$ and therefore induces a morphism

$$f : [A/SL_p(\mathcal{O}_F)] \to [(BH)/SL_p(\mathbb{F}_q)] \simeq B(SL_p(\mathbb{F}_q) \ltimes H) = BG \tag{10.2}$$

of quotient stacks. Pullback along f induces a morphism between conjugate spectral sequences of the special fibers of these stacks. In particular, there is a commutative square

$$\begin{array}{ccc} H^0(BG_0^{(1)}, L\Omega^p_{BG_0^{(1)}}) & \xrightarrow{f_0^*} & H^0([A_0^{(1)}/SL_p(\mathcal{O}_F)], L\Omega^p_{[A_0^{(1)}/SL_p(\mathcal{O}_F)]}) \\ \Big\downarrow{\scriptstyle d^{0,p}_{p+1}} & & \Big\downarrow{\scriptstyle d^{0,p}_{p+1}} \\ H^{p+1}(BG_0^{(1)}, \mathcal{O}) & \xrightarrow{f_0^*} & H^{p+1}([A_0^{(1)}/SL_p(\mathcal{O}_F)], \mathcal{O}) \end{array} \tag{10.3}$$

Therefore the following Lemma 10.2 and Proposition 10.3 imply Theorem 10.1.

Lemma 10.2. *The map*

$$f_0^* : H^0(BG_0, L\Omega^p_{BG_0/k}) \to H^0([A_0/SL_p(\mathcal{O}_F)], L\Omega^p_{[A_0/SL_p(\mathcal{O}_F)]}) \tag{10.4}$$

is an isomorphism.

Proposition 10.3. *The differential $d^{0,0}_{p+1} : H^0(Y_0^{(1)}, L\Omega^p_{Y_0/k}) \to H^{p+1}(Y_0^{(1)}, \mathcal{O})$ in the conjugate spectral sequence for $Y_0 = [A_0/SL_p(\mathcal{O}_F)]$ is non-zero.*

Remark 10.4. A trivialization of the cotangent bundle to the abelian variety A_0 provides a decomposition of the de Rham complex of A_0 [DI87, Remarque 2.6 (iv)]. Proposition 10.3 demonstrates that this decomposition in general cannot be chosen to be compatible with the action of the group of automorphisms of the lift A.

Proof of Lemma 10.2. For a finite flat group scheme Γ over k denote by $\pi : \operatorname{Spec} k \to B\Gamma$ the natural quotient morphism, and by $e : \operatorname{Spec} k \to \Gamma$ the identity section. Recall that the cotangent complex $L\Omega^1_{B\Gamma/k}$ is described as the shift $e^* L\Omega^1_{\Gamma/k}[-1]$ of the co-Lie complex of Γ, where we view $L\Omega^1_{B\Gamma/k} \in D(B\Gamma)$ as an object of the category of Γ-equivariant objects in $D(k)$.

In particular, $L\Omega^1_{BG_0/k}$ is concentrated in degrees ≥ 0, its pth exterior power $L\Omega^p_{BG_0/k}$ is concentrated in non-negative degrees as well, so $H^0(BG_0, L\Omega^p_{BG_0/k})$ is simply the invariant subspace $(H^0(\pi^* L\Omega^p_{BG_0/k}))^{G_0} = (H^0(\pi^* L\Omega^p_{BG_0/k}))^{SL_p(\mathbb{F}_q)}$. In the last equality we used that H_0 is commutative, hence the adjoint action of G_0 on the cotangent complex $\pi^* L\Omega^p_{BG_0} \simeq \pi^* L\Omega^p_{BH_0}$ factors through $SL_p(\mathbb{F}_q) = G_0/H_0$. Similarly, $H^0([A_0/SL_p(\mathcal{O}_F)], L\Omega^p_{[A_0/SL_p(\mathcal{O}_F)]}) = H^0(A_0, \Omega^p_{A_0})^{SL_p(\mathcal{O}_F)} = H^0(A_0, \Omega^p_{A_0})^{SL_p(\mathbb{F}_q)}$.

By Lemma 10.5 below, f_0 induces an isomorphism $H^0(f_0^* L\Omega^p_{BG_0}) \simeq \Omega^p_{A_0}$. Therefore it induces an isomorphism on the subspaces of $SL_p(\mathbb{F}_q)$-invariants, which finishes the proof of Lemma 10.2. $\square$

Lemma 10.5. *Let T be an abelian scheme over a base S such that $p\mathcal{O}_S = 0$. We denote the dual Lie algebra $e^*\Omega^1_{T/S}$ by ω_T, where $e : S \to T$ is the identity section. Denote by*

$q : BT[p] \to S$ the structure morphism. The cotangent complex of $BT[p]$ can be described as $L\Omega^1_{BT[p]/S} \simeq q^*\omega_T \oplus q^*\omega_T[-1]$.

The classifying map $f : T \to BT[p]$ corresponding to the torsor $T \xrightarrow{[p]} T$ induces the map $df : f^*L\Omega^1_{BT[p]/S} \to \Omega^1_{T/S}$ that gives an isomorphism $H^0(f^*L\Omega^1_{BT[p]/S}) \simeq \Omega^1_{T/S}$.

Proof. By definition, the classifying map fits into the Cartesian square

$$\begin{tikzcd} T \arrow[r, "q_T"] \arrow[d, "{[p]}"] & S \arrow[d, "\pi"] \\ T \arrow[r, "f"] & BT[p] \end{tikzcd} \tag{10.5}$$

that induces an equivalence $L\Omega^1_{T \xrightarrow{[p]} T} \simeq q_T^* L\Omega^1_{S/BT[p]} \simeq q_T^* e^* L\Omega^1_{T[p]/S}$. Transitivity triangles for the two vertical morphisms in (10.5) fit into the following commutative diagram of sheaves on T

$$\begin{tikzcd} q_T^*\pi^* L\Omega^1_{BT[p]/S} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & q_T^* e^* L\Omega^1_{T[p]/S} \arrow[d, "\sim"] \\ {[p]}^*\,\Omega^1_{T/S} \arrow[r, "{d[p]}"] & \Omega^1_{T/S} \arrow[r] & q_T^* e^* L\Omega^1_{T[p]/S} \end{tikzcd} \tag{10.6}$$

We have $d[p] = 0$, and the bottom triangle gives that $L\Omega^1_{T[p]/S} \simeq \omega_T \oplus \omega_T[1]$ which implies that $L\Omega^1_{BT[p]/S} \simeq q^*\omega_T \oplus q^*\omega_T[-1]$. The left vertical map in (10.6) is the pullback of the map $df : f^*L\Omega^1_{BT[p]/S} \to \Omega^1_{T/S}$ along $[p] : T \to T$. Since $[p]$ is a faithfully flat map, to prove that df induces an isomorphism on H^0 it is enough to prove that the left vertical map in (10.6) induces an isomorphism on H^0. But this follows from the fact that the right vertical map in this diagram is an equivalence. □

To prove Proposition 10.3 we observe that the conjugate spectral sequence of a quotient stack of a scheme X_0 by an action of a discrete group Γ has a non-zero differential in a certain situation where the Hodge cohomology of X_0 is Γ-equivariantly decomposable while the de Rham cohomology is not. The precise statement is:

Lemma 10.6. *Let X_0 be a smooth scheme over k equipped with an action of a discrete group Γ such that X_0 admits a lift X_1 over $W_2(k)$ to which the action of Γ lifts. Suppose that for all $i \leq p$ the complex $\tau^{\leq p-i}\mathrm{R}\Gamma(X_0, \Omega^i_{X_0/k})$ is Γ-equivariantly equivalent to $\bigoplus_{j=0}^{p-i} H^j(X, \Omega^i_{X/k})[-j]$ but the map $H^p_{\mathrm{dR}}([X_0/\Gamma]/k) \to H^p_{\mathrm{dR}}(X_0/k)^{\Gamma}$ is not surjective. Then the conjugate spectral sequence for the stack $Y_0 = [X_0/\Gamma]$ has a non-zero differential $d^{0,p}_{p+1} : H^0(Y_0^{(1)}, L\Omega^p_{Y_0^{(1)}/k}) \to H^{p+1}(Y_0^{(1)}, \mathcal{O}_{Y_0^{(1)}})$ on the $(p+1)$th page.*

Proof. For an algebraic stack Z over k that satisfies $L\Omega^i_{Z/k} \in D^{\geq 0}(Z)$ for all i, we have $H^p_{\mathrm{dR}}(Z/k) = \mathrm{Fil}^{\mathrm{conj}}_p H^p_{\mathrm{dR}}(Z/k)$, and we denote by $a_{Z,p} : H^p_{\mathrm{dR}}(Z/k) \to H^0(Z^{(1)}, L\Omega^p_{Z^{(1)}/k})$ the map on cohomology induced by the morphism $\mathrm{Fil}^{\mathrm{conj}}_p \mathrm{dR}_Z \to \mathrm{gr}^{\mathrm{conj}}_p \mathrm{dR}_Z \simeq L\Omega^p_{Z^{(1)}/k}[-p]$. Note that both X_0 and $[X_0/\Gamma]$ satisfy this condition.

The conclusion of the lemma is equivalent to saying that the map $a_{[X_0/\Gamma],p}$: $H^p_{\mathrm{dR}}([X_0/\Gamma]/k) \to H^0([X_0^{(1)}/\Gamma], L\Omega^p_{[X_0^{(1)}/\Gamma]/k}) = H^0(X_0^{(1)}, \Omega^p_{X_0^{(1)}/k})^{\Gamma}$ is not surjective.

Consider the commutative diagram induced by the pullback along the map $\pi : X_0 \to [X_0/\Gamma]$:

$$\begin{tikzcd}
0 \to \mathrm{Fil}^{\mathrm{conj}}_{p-1} H^p_{\mathrm{dR}}([X_0/\Gamma]) \arrow[r] \arrow[d] & H^p_{\mathrm{dR}}([X_0/\Gamma]) \arrow[r, "a_{[X_0/\Gamma],p}"] \arrow[d] & H^0([X_0^{(1)}/\Gamma], L\Omega^p_{[X_0^{(1)}/\Gamma]}) \arrow[d, equal] \\
0 \to (\mathrm{Fil}^{\mathrm{conj}}_{p-1} H^p_{\mathrm{dR}}(X_0))^\Gamma \arrow[r] & H^p_{\mathrm{dR}}(X_0)^\Gamma \arrow[r, "a_{X_0,p}"] & H^0(X_0^{(1)}, \Omega^p_{X_0^{(1)}})^\Gamma
\end{tikzcd} \tag{10.7}$$

Both rows are exact on the left and in the middle, and our goal is to show that the top row is not exact on the right. For the sake of a contradiction, assume that $a_{[X_0/\Gamma],p}$ is surjective. Then the second map in the bottom row is of course surjective as well, so both rows are exact sequences. However, we will now check that the left vertical map is surjective which will yield the contradiction with our assumption that the middle vertical map is not surjective.

By the assumption that X_0 lifts to $W_2(k)$ together with the group action, we have splittings $\mathrm{Fil}^{\mathrm{conj}}_{p-1} H^p_{\mathrm{dR}}([X_0/\Gamma]) \simeq \bigoplus\limits_{i=0}^{p-1} H^{p-i}([X_0^{(1)}/\Gamma], L\Omega^i_{[X_0^{(1)}/\Gamma]})$ and $\mathrm{Fil}^{\mathrm{conj}}_{p-1} H^p_{\mathrm{dR}}(X_0) \simeq \bigoplus\limits_{i=0}^{p-1} H^{p-i}(X_0^{(1)}, \Omega^i_{X_0^{(1)}})$ (the second splitting is moreover Γ-equivariant), and the map π^* is compatible with these decompositions. Thus to prove that the map $\mathrm{Fil}^{\mathrm{conj}}_{p-1} H^p_{\mathrm{dR}}([X_0/\Gamma]) \to (\mathrm{Fil}^{\mathrm{conj}}_{p-1} H^p_{\mathrm{dR}}(X_0))^\Gamma$ is surjective it is enough to show that $H^{p-i}([X_0^{(1)}/\Gamma], L\Omega^i_{[X_0^{(1)}/\Gamma]}) \to H^{p-i}(X_0, \Omega^i_{X_0^{(1)}})^\Gamma$ is surjective for all $0 \leq i \leq p-1$. The group $H^{p-i}([X_0^{(1)}/\Gamma], L\Omega^i_{[X_0^{(1)}/\Gamma]})$ is identified with the group cohomology $H^{p-i}(\Gamma, \mathrm{R}\Gamma(X_0^{(1)}, \Omega^i_{X_0^{(1)}}))$ so this surjectivity is implied by the fact that each of the complexes $\tau^{\leq p-i}\mathrm{R}\Gamma(X_0, \Omega^i_{X_0^{(1)}})$ is Γ-equivariantly decomposable. $\square$

Proof of Proposition 10.3. The results of Section 9 put us in a position to apply Lemma 10.6 to the action of $SL_p(\mathcal{O}_F)$ on $A_0 \simeq E_0 \otimes_{\mathbb{Z}} \mathcal{O}_F^{\oplus p}$. On the one hand, since A_0 is a product of supersingular elliptic curves, Proposition 9.1 asserts that $\tau^{\leq p}\mathrm{R}\Gamma(A, \mathcal{O})$ is $SL_p(\mathcal{O}_F)$-equivariantly decomposable, and hence so is the mod p reduction $\tau^{\leq p}\mathrm{R}\Gamma(A_0, \mathcal{O})$ of this complex. This implies that the cohomology complexes $\tau^{\leq p}\mathrm{R}\Gamma(A_0, \Omega^i_{A_0})$ are decomposable as well because $\Omega^i_{A_0}$ is trivial as a G-equivariant bundle, so $\mathrm{R}\Gamma(A_0, \Omega^i_{A_0})$ is G-equivariantly quasi-isomorphic to $\mathrm{R}\Gamma(A_0, \mathcal{O}) \otimes_k \Lambda^i(\mathrm{Lie}\, A_0)^\vee$. Hence the assumptions on decomposability of the Hodge cohomology in Lemma 10.6 are satisfied.

On the other hand, using Proposition 9.3, we will see that the map $H^p_{\mathrm{dR}}([A_0/SL_p(\mathcal{O}_F)]) \to H^p_{\mathrm{dR}}(A_0)^{SL_p(\mathcal{O}_F)}$ is not surjective. This map fits into a long exact sequence

(10.8)
$$\ldots \to H^p_{\mathrm{dR}}([A_0/SL_p(\mathcal{O}_F)]) \to H^p_{\mathrm{dR}}(A_0)^{SL_p(\mathcal{O}_F)} \xrightarrow{\delta} H^{p+1}(SL_p(\mathcal{O}_F), \bigoplus_{i=0}^{p-1} H^i_{\mathrm{dR}}(A_0)[-i]) \to \ldots$$

Here δ lands in the direct summand $H^p(SL_p(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0))$ and it is given by the composition
(10.9)

$$(\Lambda^p H^1_{\mathrm{dR}}(A_0))^{SL_p(\mathcal{O}_F)} \xrightarrow{\mathrm{Bock}^{p-1}(\alpha(H^1_{\mathrm{dR}}(A_0/k)))} H^p(SL_p(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0)^{(1)}) \xrightarrow{F^*_{A_0}} H^p(SL_p(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0))$$

in the notation of Proposition 9.3. We will prove that δ is non-zero, thus checking that $H^p_{\mathrm{dR}}([A_0/SL_p(\mathcal{O}_F)]) \to H^p_{\mathrm{dR}}(A_0)^{SL_p(\mathcal{O}_F)}$ is not surjective.

Non-vanishing of δ can be checked after replacing the base field k by a finite extension, hence we may assume that k contains $\mathbb{F}_q = \mathbb{F}_{p^2}$. The k-vector space $H^1_{\mathrm{dR}}(A_0)$ can be $SL_p(\mathcal{O}_F)$-equivariantly identified with

$$H^1_{\mathrm{dR}}(E_0) \otimes_{\mathbb{Z}} \mathcal{O}_F^{\oplus p} = H^1_{\mathrm{dR}}(E_0) \otimes_k (k \otimes_{\mathbb{F}_p} \mathcal{O}_F^{\oplus p}/p) \tag{10.10}$$

The group $SL_p(\mathcal{O}_F)$ acts on the RHS via its tautological action on $\mathcal{O}_F^{\oplus p}/p = \mathbb{F}_q^{\oplus p}$. The k-vector space $k \otimes_{\mathbb{F}_p} \mathcal{O}_F^{\oplus p}/p$ is $SL_p(\mathcal{O}_F)$-equivariantly isomorphic to $\bigoplus\limits_{\tau \in \mathrm{Gal}(F/\mathbb{Q})} V^\tau$ where V^τ is isomorphic to $V := \mathbb{F}_q^{\oplus p} \otimes_{\mathbb{F}_q} k$ as a k-vector space but the $SL_p(\mathcal{O}_F)$-action is modified by precomposing with the automorphism $\tau \in \mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbb{Z}/2$. We get the following $SL_p(\mathcal{O}_F)$-equivariant description of 1st de Rham cohomology of A_0:

$$H^1_{\mathrm{dR}}(A_0) \simeq H^1_{\mathrm{dR}}(E_0) \otimes_k \bigoplus_{\tau \in \mathrm{Gal}(F/\mathbb{Q})} V^\tau \tag{10.11}$$

Choose any element $\xi \in H^1_{\mathrm{dR}}(E_0/k)$ such that $F^*_{E_0}(\xi) \neq 0$, and choose an arbitrary lift $\widetilde{\xi} \in H^1_{\mathrm{cris}}(E_0/W_2(k))$ of ξ. This defines a split $SL_p(\mathcal{O}_F)$-equivariant embedding ι_ξ : $V \xrightarrow{\xi \otimes \mathrm{id}_V} H^1_{\mathrm{dR}}(E_0) \otimes_k V \subset H^1_{\mathrm{dR}}(A_0)$ that lifts to an embedding $\widetilde{V} \to H^1_{\mathrm{cris}}(A_0/W_2(k))$. We have a commutative diagram

(10.12)

$$\begin{tikzcd} k = (\Lambda^p V)^{SL_p(\mathcal{O}_F)} \arrow[r, "\mathrm{Bock}^{p-1}_{\widetilde{V}}(\alpha(V))"] \arrow[d, "\Lambda^p \iota_\xi"] & H^p(SL_p(\mathcal{O}_F), V^{(1)}) \arrow[r, "\iota_\xi^{(1)}"] & H^p(SL_p(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0)^{(1)}) \arrow[d, "\varphi^*_{A_0}"] \\ H^p_{\mathrm{dR}}(A_0)^{SL_p(\mathcal{O}_F)} \arrow[rr, "\delta"] & & H^p(SL_p(\mathcal{O}_F), H^1_{\mathrm{dR}}(A_0)) \end{tikzcd}$$

By Proposition 14.1 the class $\mathrm{Bock}^{p-1}_{\widetilde{V}}(\alpha(V))$ induces a non-zero map $k = (\Lambda^p V)^{SL_p(\mathcal{O}_F)} \to H^p(SL_p(\mathcal{O}_F), V^{(1)})$. By our choice of ξ the composition $\varphi^*_{A_0} \circ \iota_\xi^{(1)}$: $V^{(1)} \to H^1_{\mathrm{dR}}(A_0)$ is a split injection. Therefore the composition of the top row and right vertical arrow in (10.12) is injective, so δ is non-zero.

We have thus checked that all the conditions of Lemma 10.6 are satisfied, hence the differential $d^{0,p}_{p+1} : H^0(Y_0, L\Omega^p_{Y_0/k}) \to H^{p+1}(Y_0, \mathcal{O})$ in the conjugate spectral sequence for the stack $Y_0 = [A_0/SL_p(\mathcal{O}_F)]$ is non-zero, as desired. ☐

Theorem 10.1 is therefore proven, by combining Proposition 10.3 with Lemma 10.2. We can now produce an example of a liftable smooth projective variety with a non-degenerate conjugate spectral sequence:

Corollary 10.7. *Let k be any perfect field of characteristic p. There exists a smooth projective scheme X over $W(k)$ of relative dimension $p+1$, such that the differential*

$H^0(X_0, \Omega^p_{X_0}) \to H^{p+1}(X_0, \mathcal{O})$ on the $(p+1)$st page of the conjugate spectral sequence of X_0 is non-zero.

Proof. We will use the classical technique of approximating the classifying stack of a finite flat group scheme by a projective variety, originated by Serre [Ser58], cf. [ABM21, Theorem 1.2].

We choose, using Lemma 10.8 below, a complete intersection $Z \subset \mathbb{P}^N_{W(k)}$ of relative dimension $p+1$ over $W(k)$, equipped with a free action of the group scheme G, such that Z/G is smooth over $W(k)$. Our example is $X := Z/G$, it is a smooth projective scheme over $W(k)$ equipped with the classifying map $f : X \to BG$. The induced map $f_0^* : H^{p+1}(BG_0, \mathcal{O}) \to H^{p+1}(X_0, \mathcal{O})$ is seen to be injective by considering the Hochschild-Serre spectral sequence for the morphism $Z_0 \to Z_0/G_0$ using the fact that $H^i(Z_0, \mathcal{O}) = 0$ for $i \leq p$. Therefore the differential $H^0(X_0, \Omega^p_{X_0}) \to H^{p+1}(X_0, \mathcal{O})$ is non-zero. □

Lemma 10.8. *Let k be any perfect field of characteristic p. For any finite flat group scheme Γ over $W(k)$ and an integer $d \geq 0$ there exists a complete intersection $Z \subset \mathbb{P}^N_{W(k)}$ of relative dimension d equipped with a free action of Γ such that the quotient Z/Γ is a smooth scheme.*

Proof. If k is infinite, this is proven in [BMS18, 2.7-2.9]. For finite k we will check that the same construction goes through if we appeal to a Bertini theorem over finite fields due to Gabber and Poonen. [BMS18, Lemma 2.7] provides us with an action of Γ on a projective space $\mathbb{P}^N_{W(k)}$ such that there is an open subscheme $U \subset \mathbb{P}^N_{W(k)}$ preserved by Γ on which Γ acts freely, the quotient U/Γ is smooth over $W(k)$, and the dimension of every component of both fibers of $\mathbb{P}^N_{W(k)} \setminus U$ is at most $N-d-1$. Consider the quotient scheme $Q := \mathbb{P}^N_{W(k)}/\Gamma$ and denote by $M \subset Q$ the image of $\mathbb{P}^N_{W(k)} \setminus U$ under the quotient map. By construction, $Q \setminus M$ is smooth over $W(k)$, and M has fiber-wise codimension $\geq d+1$ inside Q.

Following the argument below [BMS18, Lemma 2.9], it is enough for us to find very ample line bundles $\mathcal{L}_1, \dots, \mathcal{L}_{N-d}$ on Q and sections $s_i \in H^0(Q, \mathcal{L}_i)$ such that their common vanishing locus is smooth and does not intersect M. By induction on the number $N-d$, it is enough to produce a very ample line bundle $\mathcal{L}$ on Q and a section $s \in H^0(Q, \mathcal{L})$ such that $V(s) \cap M \subset V(s)$ is of fiber-wise codimension $\geq d+1$, and $V(s) \setminus V(s) \cap M$ is smooth. Note that both conditions can be checked just on the special fiber.

Let $\mathcal{L}$ be some very ample line bundle on Q. We choose a closed point on every irreducible component of M_0, and denote B the set comprised of these points. Let n_0 be an integer such that $H^1(Q, \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$. Applying [Gab01, Corollary 1.6], for some $n > n_0$ we can find a section $s \in H^0(Q_0, \mathcal{L}^{\otimes n})$ of the line bundle $\mathcal{L}^{\otimes n}$ on the special fiber $Q_0 := Q \times_{W(k)} k$, such that $V(s) \setminus M_0 \cap V(s)$ is smooth of dimension $N-1$, and $V(s)$ does not contain any of the points from B. The latter property implies that the codimension of $M_0 \cap V(s)$ inside $V(s)$ is at least $d+1$. By our assumption on the vanishing of $H^1(Q, \mathcal{L}^{\otimes n})$, we can lift s to a section of $\mathcal{L}^{\otimes n}$ on Q whose vanishing locus satisfies the desired properties.

Having constructed the sections $s_1, \dots, s_{N-d}$, we define $Z \subset \mathbb{P}^N_{W(k)}$ to be the preimage of $V(s_1, \dots, s_{N-d}) \subset Q \setminus M \subset Q = \mathbb{P}^N_{W(k)}/\Gamma$. The action of Γ on Z is free and the quotient $Z/\Gamma = V(s_1, \dots, s_{N-d})$ is smooth, by the choice of sections $s_1, \dots, s_{N-d}$. □

Remark 10.9. (1) If in the construction of the abelian scheme A we used an elliptic curve E with *ordinary* reduction, the statement of Proposition 10.3 would be false. Indeed, if E_0 is ordinary then the stack $[A_0/SL_p(\mathcal{O}_F)]$ admits a lift to $W_2(k)$ together with its Frobenius endomorphism, hence the conjugate filtration is split in all degrees by [DI87, Remarque 2.2(ii)]. The part of the proof showing that the map $H^p_{\mathrm{dR}}([A_0/SL_p(\mathcal{O}_F)]) \to H^p_{\mathrm{dR}}(A_0)^{SL_p(\mathcal{O}_F)}$ is not surjective goes through just as well because it only used that $\varphi^*_{E_0}$ is non-zero on $H^1_{\mathrm{dR}}(E_0)$, but it is no longer true that $\tau^{\leq p}\mathrm{R}\Gamma(A_0, \mathcal{O})$ is $SL_p(\mathcal{O}_F)$-equivariantly decomposable, by Proposition 9.1. So both Hodge and de Rham cohomology of A_0 are not $SL_p(\mathcal{O}_F)$-equivariantly decomposable, but the conjugate filtration on $\mathrm{R}\Gamma_{\mathrm{dR}}(A_0)$ is $SL_p(\mathcal{O}_F)$-equivariantly split.

(2) As in the proof of [DI87, Corollarie 2.4], the variety X_0 in Corollary 10.7 necessarily has a non-zero differential in its Hodge-to-de Rham spectral sequence.

(3) There are several examples in the literature of smooth projective varieties over k with a non-degenerate Hodge-to-de Rham spectral sequence that lift to some ramified extension of $W(k)$, but to the best of my understanding none of them lift to $W_2(k)$.

11. Extensions in higher degrees

It would be interesting to generalize the methods of Section 4 to treat extensions in the canonical filtration on the de Rham complex in degrees $> p$. In this section, which is independent of the rest of the paper, we make some preliminary remarks, roughly amounting to extending part (1) of Theorem 4.1 to higher degrees.

Suppose that X is an arbitrary (not necessarily flat) scheme over $\mathbb{Z}_{(p)}$ and let $A \in \mathrm{DAlg}(X)$ be a derived commutative algebra such that $H^0(A) = \mathcal{O}_X$, $H^1(A)$ is a locally free sheaf, and multiplication induces isomorphisms $\Lambda^i H^1(A) \simeq H^i(A)$ for all i.

Lemma 11.1. *Given maps $s_i : \Lambda^i H^1(A)[-i] \to A, s_j : \Lambda^j H^1(A)[-j] \to A$ in $D(X)$ that induce isomorphisms on ith and jth cohomology respectively, there exists a map $s_{i+j} : \Lambda^{i+j} H^1(A)[-i-j] \to A$ that induces an isomorphism on H^{i+j} if the integer $\binom{i+j}{i}$ is not divisible by p.*

Proof. For any object $M \in D(X)$, there is a natural map $m_{i,j} : S^i M \otimes S^j M \to S^{i+j} M$ that, in case M is a projective module over a ring, is given by the surjection $(M^{\otimes(i+j)})_{S_i \times S_j} \to (M^{\otimes(i+j)})_{S_{i+j}}$ where we view $S_i \times S_j$ as a subgroup of S_{i+j}. Choosing a set of representatives $\Sigma \subset S_{i+j}$ for left cosets of this subgroup, we can define a map (that is independent of the choice of Σ) $N_{S_{i+j}/S_i \times S_j} : S^{i+j} M \to S^i M \otimes S^j M$ given by $\sum\limits_{\sigma \in \Sigma} \sigma$. The composition $S^{i+j} M \xrightarrow{N_{S_{i+j}/S_i \times S_j}} S^i M \otimes S^j M \xrightarrow{m_{i,j}} S^{i+j} M$ is equal to multiplication by $\binom{i+j}{i} = [S_i \times S_j : S_{i+j}]$.

Applying this to $M = H^1(A)[1]$ and using the decalage equivalences $S^i(H^1(A)[1]) \simeq (\Lambda^i H^1(A))[i]$, we obtain a map $N_{S_{i+j}/S_i \times S_j} : \Lambda^{i+j} H^1(A) \to \Lambda^i H^1(A) \otimes \Lambda^j H^1(A)$ whose composition with the wedge product map is the multiplication by $\binom{i+j}{i}$ map on $\Lambda^{i+j} H^1(A)$.

Consider the product of sections s_i and s_j given by

$$s'_{i+j} : \Lambda^i H^1(A)[-i] \otimes \Lambda^j H^1(A)[-j] \xrightarrow{s_i \otimes s_j} A \otimes A \xrightarrow{m} A. \tag{11.1}$$

It induces exterior multiplication $\Lambda^i H^1(A) \otimes \Lambda^j H^1(A) \to \Lambda^{i+j} H^1(A)$ when evaluated on H^{i+j}, so precomposing s'_{i+j} with the map $N_{S_{i+j}/S_i \times S_j}$ gives rise to a map

$s_{i+j} : \Lambda^{i+j} H^1(A)[-i-j] \to A$ that induces multiplication by $\binom{i+j}{i}$ on H^{i+j}. Under the assumption that $p \nmid \binom{i+j}{i}$ the map s_{i+j} thus gives the desired splitting in degree $i+j$. □

Corollary 11.2. *Given an integer $n \geq 0$, suppose that there exist morphisms $s_{p^i} : H^{p^i}(A)[-p^i] \to A$ inducing an isomorphism on H^{p^i}, for all $i = 0, 1, \dots n$. Then there exists an equivalence $\tau^{\leq p^{n+1}-1} A \simeq \bigoplus_{i=0}^{p^{n+1}-1} H^i(A)[-i]$.*

Proof. We need to construct sections $s_m : H^m(A)[-m] \to A$ for all $0 \leq m \leq p^{n+1}-1$. We will construct these by induction on m, the base case being $m = 0$, where $s_0 : H^0(A) \to A$ is simply the natural map that exists for every complex concentrated in non-negative degrees. Suppose that the sections $s_{m'}$ have been constructed for all $m' < m$. Let $m = a_r p^r + \dots + a_1 p + a_0$ be the base p expansion of m. The splitting s_{p^r} is given to us and s_{m-p^r} has already been constructed, so s_m is provided by Lemma 11.1 applied to $(i, j) = (p^r, m - p^r)$. □

Example 11.3. If X_0 is a smooth scheme over k that lifts to a scheme X_1 over $W_2(k)$, such that the class $e_{X_1,p}$ vanishes, the conditions of Corollary 11.2 are satisfied for $A = \mathrm{dR}_{X_0/k}$ on X_0 with $n = 1$, hence the truncation $\tau^{\leq p^2-1}\mathrm{dR}_{X_0/k}$ decomposes as a direct sum of its cohomology sheaves.

12. Non-vanishing in rational group cohomology

In this section we prove non-vanishing results related to the class α of Definition 3.4. Let V be a finite-dimensional k-vector space equipped with the tautological action of the algebraic group $GL(V)$, viewed as a group scheme over k. We denote by $V^{(1)}$ the vector space $V \otimes_{k,\mathrm{Fr}_p} k$ on which $GL(V)$ acts through the relative Frobenius $F_{GL(V)/k} : GL(V) \to GL(V)^{(1)} = GL(V^{(1)})$. Under the equivalence between representations of $GL(V)$ and quasicoherent sheaves on the classifying stack $BGL(V)$ the representation $V^{(1)}$ corresponds to the pullback $F^*_{BGL(V)}V$ under absolute Frobenius, because V and $GL(V)$ can be descended to $\mathbb{F}_p \subset k$.

The construction of Definition 3.4 applied to the stack $BGL(V)$ with its tautological vector bundle associates to V a class $\alpha(V) \in \mathrm{Ext}^{p-1}_{GL(V)}(\Lambda^p V, V^{(1)})$ where $V^{(1)}$ is the Frobenius twist of the representation V. Recall that $\alpha(V)$ is the extension class of

$$V^{(1)}[-2] \to \tau^{\geq 2} S^p(V[-1]) \to \Lambda^p V[-p] \tag{12.1}$$

We will start by proving that this extension is generally non-split when viewed as an extension of algebraic representations

Proposition 12.1. *If $\dim V = p$ then the class $\alpha(V)|_{SL(V)} \in \mathrm{Ext}^{p-1}_{SL(V)}(\Lambda^p V, V^{(1)}) = H^{p-1}(SL(V), V^{(1)})$ is non-zero.*

Remark 12.2. In this section, we will work in the derived category $D(\mathrm{Rep}_{SL(V)})$ of the abelian category of $k[SL(V)]$-comodules. Its full subcategory $D^+(\mathrm{Rep}_{SL(V)})$ of bounded below complexes is equivalent to $D^+(BSL(V))$, so we may view $\alpha(V)$ as a morphism in $D^+(\mathrm{Rep}_{SL(V)})$. The unbounded derived categories $D(BSL(V))$ and $D(\mathrm{Rep}_{SL(V)})$ are not equivalent though, and we will need to consider complexes unbounded from below in the intermediate steps of the proof.

We will witness the non-vanishing of $\alpha(V)$ by representing $S^p(V[-1])$ as a direct summand of an explicit complex of $SL(V)$-modules and then comparing the result of applying $\mathrm{R}\Gamma(SL(V),-)$ to the terms of this complex and the cohomology groups of this complex.

Remark 12.3. When $p=2$, an explicit model for $\tau^{\geq 2}S^p(V[-1])$ is at our disposal by Lemma 3.6. In this case Proposition 12.1 is asserting that the extension

$$0\to V^{(1)}\to S^2V\to \Lambda^2 V\to 0$$

does not admit an $SL(V)$-equivariant section, for a 2-dimensional space V, and this is elementary to show. Indeed, if e_1,e_2 is a basis in V then a matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ satisfying $ad-bc=1$ sends an element $\lambda_{11}e_1^2+\lambda_{12}e_1e_2+\lambda_{22}e_2^2\in S^2V$ to $(\lambda_{11}a^2+\lambda_{12}\cdot ab+\lambda_{22}b^2)e_1^2+\lambda_{12}e_1e_2+(\lambda_{11}c^2+\lambda_{12}\cdot cd+\lambda_{22}d^2)e_2^2$, so the invariants $(S^2V)^{SL_2}$ vanish. But the action of SL_2 on Λ^2V is trivial, so the surjection $S^2V\to\Lambda^2V$ does not admit a section.

This remark proves Proposition 12.1 for $p=2$. For the remainder of this section we assume that $p>2$. We start by relating $S^p(V[-1])$ to the homology of the symmetric group S_p acting on $V[-1]^{\otimes p}$ via permutation of the factors. For an object M of the derived category of modules over a discrete group G we denote by M_{hG} the derived coinvariants of M.

Lemma 12.4. *There is a $GL(V)$-equivariant equivalence $\tau^{\geq 1}((V[-1]^{\otimes p})_{hS_p})\simeq S^pV[-1]$.*

Proof. This is Lemma 3.8 applied to the classifying stack $X_0=BGL(V)$ and E being the tautological vector bundle corresponding to the representation V. □

The advantage of working with $(V[-1]^{\otimes p})_{hS_p}$ to understand the extension in the canonical filtration on $S^p(V[-1])$ is that we can represent $(V[-1]^{\otimes p})_{hS_p}$ as a direct summand of an explicit complex. Denote by $C_p\subset S_p$ a cyclic subgroup of order p generated by the long cycle $\sigma=(12\dots p)$. We have a natural map $(V[-1]^{\otimes p})_{hC_p}\to(V[-1]^{\otimes p})_{hS_p}$ that admits a section $N_{S_p/C_p}:(V[-1]^{\otimes p})_{hS_p}\to(V[-1]^{\otimes p})_{hC_p}$ because the index of $C_p\subset S_p$ is coprime to p, so $(V[-1]^{\otimes p})_{hS_p}$ is a direct summand of $(V[-1]^{\otimes p})_{hC_p}$.

Denote by $N:=\sum\limits_{i=0}^{p-1}\sigma^i\in k[C_p]$ the norm element in the group algebra of C_p. Using the 2-periodic resolution

$$\dots\xrightarrow{1-\sigma}k[C_p]\xrightarrow{N}k[C_p]\xrightarrow{1-\sigma}k[C_p] \tag{12.2}$$

for the trivial C_p-module we have the following complex representing $(V[-1]^{\otimes p})_{hC_p}$:

$$\dots\xrightarrow{1-\sigma}V^{\otimes p}\xrightarrow{N}V^{\otimes p}\xrightarrow{1-\sigma}\overset{p}{V^{\otimes p}} \tag{12.3}$$

With this explicit complex in hand, we will deduce Proposition 12.1 from the following collection of facts about the cohomology of $SL(V)$:

Lemma 12.5. *(1) We have $H^i(SL(V),V^{\otimes n})=0$ for all $i>0$ and $n\geq 0$.*

(2) *The embedding* $\varepsilon : \Lambda^p V \to V^{\otimes p}$ *given by* $v_1 \wedge \ldots \wedge v_p \mapsto \sum\limits_{g\in S_p} \mathrm{sgn}(g) v_{g(1)} \otimes \ldots \otimes v_{g(p)}$ *induces an isomorphism* $\Lambda^p V \simeq H^0(SL(V), V^{\otimes p})$.

(3) $H^i(SL(V), V^{(1)}) = 0$ *for all* $i \neq p-1$ *and* $H^{p-1}(SL(V), V^{(1)}) = k$.

Remark 12.6. First assertion of (3) for $p = 2$ is elementary to prove analogously to Remark 12.3, and for $p = 3$ it is a result of Stewart [Ste12, Theorem 1]. It appears to be a new result for $p > 3$.

We will now explain how Proposition 12.1 follows from Lemma 12.5 and the rest of the subsection is devoted to its proof. Proving that the extension $V^{(1)}[-2] \to \tau^{\geq 2} S^p(V[-1]) \to \Lambda^p V[-p]$ is non-split is equivalent to showing that the map

$$H^p(SL(V), \tau^{\geq 2} S^p(V[-1])) \to H^p(SL(V), \Lambda^p V[-p]) = H^0(SL(V), k) = k$$

is zero. By Lemma 12.4 the object $S^p(V[-1])$ is identified with $\tau^{\geq 2}(V[-1]^{\otimes p})_{hS_p}$. First, by Lemma 12.5 (3) and the fact that all non-zero cohomology modules of the complex $\tau^{\leq 1}(V[-1]^{\otimes p})_{hS_p}$ are isomorphic to $V^{(1)}$ (by Lemma 3.9), the complex of derived invariants $\mathrm{R}\Gamma(SL(V), \tau^{\leq 1}(V[-1]^{\otimes p})_{hS_p})$ is concentrated in degrees $\leq p$. Therefore the map $H^p(SL(V), (V[-1])^{\otimes p}_{hS_p}) \to H^p(SL(V), \tau^{\geq 2}(V[-1])^{\otimes p}_{hS_p})$ is surjective, and it is enough to prove that the map $H^p(SL(V), (V[-1])^{\otimes p}_{hS_p}) \to H^p(SL(V), \Lambda^p V[-p])$ is zero. Moreover, since $(V[-1]^{\otimes p})_{hS_p}$ is a direct summand of $(V[-1]^{\otimes p})_{hC_p}$, it is enough to prove that the composition $H^p(SL(V), (V[-1])^{\otimes p}_{hC_p}) \to H^p(SL(V), (V[-1])^{\otimes p}_{hS_p}) \to H^p(SL(V), \Lambda^p V[-p])$ is zero.

By Lemma 12.5 (1) and (2), the map $\varepsilon : k = \Lambda^p V \to V^{\otimes p}$ induces an isomorphism $k = \mathrm{R}\Gamma(SL(V), \Lambda^p V) \to \mathrm{R}\Gamma(SL(V), V^{\otimes p})$. Lemma 12.7 below implies that this map induces an equivalence $k[-p]_{hC_p} = \mathrm{R}\Gamma(SL(V), k[-p]_{hC_p}) \simeq \mathrm{R}\Gamma(SL(V), (V[-1]^{\otimes p})_{hC_p})$.

This allows us to compute the map

$$H^p(SL(V), (V[-1]^{\otimes p})_{hC_p}) \to H^p(SL(V), (\Lambda^p V)[-p]) \tag{12.4}$$

as the result of passing to derived $SL(V)$-invariants in the composition $\Lambda^p V \xrightarrow{\varepsilon} V^{\otimes p} \to \Lambda^p V$. But this composition is zero, which finishes the proof of Proposition 12.1 modulo Lemma 12.5.

Lemma 12.7. *Let* M *be a finite-dimensional representation of a reductive group* G *over* k*, equipped with an action of* C_p *that commutes with* G*. Then there is an equivalence* $\mathrm{R}\Gamma(G, M_{hC_p}) \simeq \mathrm{R}\Gamma(G, M)_{hC_p}$

Proof. This is not a priori obvious as it entails commuting a limit with a colimit. We will first show that the natural map $M_{hC_p} \to \mathrm{Rlim}\, \tau^{\geq -n}(M_{hC_p})$ induces an equivalence $\mathrm{R}\Gamma(G, M_{hC_p}) \simeq R\Gamma(G, \mathrm{Rlim}\, \tau^{\geq -n}(M_{hC_p}))$ The object $M_{hC_p} \in D_G(k)$ is represented by a 2-periodic complex

$$\ldots \xrightarrow{1-\sigma} M \xrightarrow{N} M \xrightarrow{1-\sigma} M \tag{12.5}$$

In particular, as i varies $H^i(M)$ runs through at most 3 different finite-dimensional representations of G. Since for a reductive G and any finite-dimensional representation W the complex $\mathrm{R}\Gamma(G, W)$ is bounded [CPSvdK77, Theorem 2.4(b)], there exists a constant $c \in \mathbb{N}$ such that $\mathrm{R}\Gamma(G, H^i(M)) \in D^{\leq c}(k)$ for all i, hence $\mathrm{R}\Gamma(G, \tau^{<-n}(M_{hC_p})) \in D^{<c-n}(k)$. The fiber of the natural map $\mathrm{R}\Gamma(G, M_{hC_p}) \to \mathrm{R}\Gamma(G, \mathrm{Rlim}\, \tau^{\geq -n}(M_{hC_p}))$ is given by $\mathrm{Rlim}\, \mathrm{R}\Gamma(G, \tau^{<-n}(M_{hC_p}))$, and this limit is forced to vanish.

Next, let $c' \in \mathbb{N}$ be such that $\mathrm{R\Gamma}(G, M/\ker(1-\sigma))$, $\mathrm{R\Gamma}(G, M/\ker N)$, and $\mathrm{R\Gamma}(G, M)$ belong to $D^{\leq c}(k)$. Then for every n the natural map $\mathrm{R\Gamma}(G, M)_{hC_p} \to \mathrm{R\Gamma}(G, \tau^{\geq -n}(M_{hC_p}))$ induces an isomorphism on cohomology in degrees $\geq c' - n$ because $\tau^{\geq -n}(M_{hC_p})$ can be represented by the complex

$$0 \to \overset{-n-1}{M/\ker f} \to \ldots \xrightarrow{1-\sigma} M \xrightarrow{N} M \xrightarrow{1-\sigma} M \tag{12.6}$$

where f is N or $1-\sigma$, depending on the parity of n. This implies that the map $\mathrm{R\Gamma}(G, M)_{hC_p} \to \mathrm{Rlim}\, \mathrm{R\Gamma}(G, \tau^{\leq -n}(M_{hC_p}))$ is an equivalence which finishes the proof of the lemma. $\square$

Proof of Lemma 12.5. Part (1) follows from classical results on good filtrations, see [Jan03, II.4]. We recall here relevant facts from this theory, from which the desired vanishing follows immediately. In general, consider a split reductive group G over k with a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Recall that for a weight $\lambda \in X^*(T)$ there is the associated G-equivariant line bundle $\mathcal{O}(\lambda)$ on the flag variety G/B, obtained by applying the equivalence $\{G\text{-equivariant vector bundles on } G/B\} \simeq \{\text{representations of } B\}$ to the character $B \to T \xrightarrow{\lambda} \mathbb{G}_m$. The global sections $H^0(\lambda) := H^0(G/B, \mathcal{O}(\lambda))$ is a finite-dimensional representation of G which is non-zero if and only if λ is a dominant weight. A filtration $\ldots \subset W_i \subset W_{i-1} \subset \ldots$ on a finite-dimensional representation W is called *good* if every quotient W_{i-1}/W_i is isomorphic to the representation $H^0(G/B, \mathcal{O}(\lambda_i))$ for some dominant weight λ. If a representation W admits a good filtration then $H^i(G, W) = 0$ for all $i > 0$, by Kempf vanishing [Jan03, Proposition II.4.13].

For $n = 0$, part (1) is asserting that $H^{>0}(SL(V), k) = 0$ which is the case because $k = H^0(\lambda)$ for the trivial character λ. The tautological representation V of $SL(V)$ has the form $H^0(\lambda)$ for λ the highest weight of V, so it tautologically has a good filtration. Tensor product of two representations with good filtrations admits a good filtration as well [Jan03, Proposition II.4.21], hence $V^{\otimes n}$ admits a good filtration which implies that $H^{>0}(SL(V), V^{\otimes n}) = 0$ for $n > 0$.

Part (2) is proven by a direct computation. Note that ε is an $SL(V)$-invariant map so all we need is to prove that $(V^{\otimes p})^{SL(V)}$ has dimension at most 1. Let $e_1, \ldots, e_p$ be a basis for V and consider first the action of the maximal torus $T = \{\mathrm{diag}(t_1, \ldots, t_{p-1}, (t_1 \cdot \ldots \cdot t_{p-1})^{-1}) | t_i \in \mathbb{G}_m\}$ on $V^{\otimes p}$. An element $e_{i_1} \otimes \ldots \otimes e_{i_p}$ is being acted on by T via the character $t_1^{d_1 - d_p} \ldots t_{p-1}^{d_{p-1} - d_p}$ where d_j is the number of indices $r = 1, \ldots, p$ such that $i_r = j$. Therefore the subset of T-invariants $(V^{\otimes p})^T \subset V^{\otimes p}$ is spanned by tensors $e_{i_1} \otimes \ldots \otimes e_{i_p}$ for which $(i_1, \ldots, i_p)$ is a permutation of the sequence $(1, 2 \ldots, p)$. We can embed the group S_p into $SL(V)$ by sending $\tau \in S_p$ to the operator $e_i \mapsto \mathrm{sgn}(\tau) e_{\tau(i)}$. The subspace $(V^{\otimes p})^T$ is preserved by S_p and is isomorphic to the regular representation $k[S_p]$ of S_p. Hence $S_p \ltimes T$-invariants in $V^{\otimes p}$ are 1-dimensional and part (2) is proven.

To prove part (3) we will use an auxiliary complex of $SL(V)$-modules obtained from the de Rham complex. The idea of using the de Rham complex of an affine space to study cohomology of representations of $GL(V)$ appears already in [FLS94] and [FS97], and the idea of our computation is specifically quite similar to [FS97, Theorem 4.5].

Consider the affine space $\mathbb{A}(V) := \mathrm{Spec}\, S^\bullet(V)$ corresponding to the vector space $V^\vee$. It is equipped with an action of $GL(V)$ and the de Rham complex $\Omega^\bullet_{\mathbb{A}(V)/k}$ can be viewed as a complex of representations of $GL(V)$ on k-vector spaces. Explicitly, $\Omega^\bullet_{\mathbb{A}(V)/k}$ has the form

$$S^*(V) \xrightarrow{d} S^*(V) \otimes_k V \xrightarrow{d} S^*(V) \otimes_k \Lambda^2 V \xrightarrow{d} \dots \tag{12.7}$$

where the de Rham differential $d : S^*(V) \otimes_k \Lambda^i V \to S^*(V) \otimes_k \Lambda^{i+1} V$ is given by

$$d(v_1 \cdot \dots \cdot v_n \otimes \omega) = \sum_{i=1}^{d} v_1 \cdot \dots \cdot \widehat{v_i} \cdot \dots \cdot v_d \otimes v_i \wedge \omega \tag{12.8}$$

The action of the center $\mathbb{G}_m \subset GL(V)$ introduces a grading on the de Rham complex with the n-th graded piece given by

$$\Omega_n^\bullet := S^n V \xrightarrow{d} S^{n-1} V \otimes V \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n V \tag{12.9}$$

The Cartier isomorphism describes the cohomology modules of this complex as follows:

Lemma 12.8 (cf. [FS97, Theorem 4.1]). *When $p \nmid n$ the complex $\Omega_n^\bullet$ is acyclic. If $n = pk$ for an integer k then there are $GL(V)$-equivariant isomorphisms $H^i(\Omega_{pk}^\bullet) \simeq \Lambda^i(V^{(1)}) \otimes S^{k-i} V^{(1)}$ for all i.*

In particular, for $n = p$ the only non-zero cohomology groups of the complex $\Omega_p^\bullet$ are in degrees 0 and 1, both isomorphic to $V^{(1)}$. For completeness, we note the remarkable coincidence that will however not be used in our proof:

Lemma 12.9 ([FS97, Lemma 4.12]). *There is an equivalence $T_p(V[-1]) \simeq \tau^{\leq 1}\Omega_p^\bullet$ of representations of $GL(V)$.*

Suppose that $\dim V = p$. Consider the complex $\Omega_p^{\leq p-1}$ obtained from $\Omega_p^\bullet$ by removing the last term:

$$\Omega_p^{\leq p-1} := S^p V \to S^{p-1} V \otimes V \to \dots \to V \otimes \Lambda^{p-1} V \tag{12.10}$$

This is a complex with $H^0 \simeq H^1 \simeq V^{(1)}$, $H^{p-1} = \Lambda^p V$ and all other cohomology groups are equal to zero. We will use $\Omega_p^{\leq p-1}$ to compute cohomology of the module $V^{(1)}$. We will compute $\mathrm{R}\Gamma(SL(V), -)$ applied to the complex (12.10) in two ways. First, Lemma 12.10 below implies that the terms of the complex have no cohomology in positive degrees and the only term having non-zero H^0 is $V \otimes \Lambda^{p-1} V$. Hence $\mathrm{R}\Gamma(SL(V), \Omega_p^{\leq p-1}) = k[1-p]$, and the map $\Lambda^{p-1} V \otimes V[1-p] \to \Omega_p^{\leq p-1}$ given by the embedding of the top term of $\Omega_p^{\leq p-1}$ induces a quasi-isomorphism on $\mathrm{R}\Gamma(SL(V), -)$. Therefore the map $\Omega_p^{\leq p-1} \to H^{p-1}(\Omega_p^{\leq p-1})[1-p] = (\Lambda^p V)[1-p]$ induces the zero map on $\mathrm{R}\Gamma(SL(V), -)$ because the de Rham differential $V \otimes \Lambda^{p-1} V \to \Lambda^p V$ induces the zero map on $SL(V)$-invariants.

On the other hand, the E_2 page spectral sequence associated with the canonical filtration looks as follows:

$$
\begin{tikzcd}
p-1 & H^0(SL(V),\Lambda^p V) \arrow[ddrrr] & 0 & 0 & \ldots \\
\vdots & 0 & 0 & 0 & \\
1 & H^0(SL(V),V^{(1)}) & H^1(SL(V),V^{(1)}) & \ldots & H^{p-1}(SL(V),V^{(1)}) \\
0 & H^0(SL(V),V^{(1)}) & H^1(SL(V),V^{(1)}) & \ldots & H^{p-1}(SL(V),V^{(1)}) \\
 & 0 & 1 & \ldots & p-1
\end{tikzcd}
$$

We used here that $H^{>0}(SL(V),\Lambda^p V)=H^{>0}(SL(V),k)=0$, as we proved in part (1). Since we computed that $H^{p-1}(SL(V),\Omega_p^{\leq p-1})=k$, and $H^i(SL(V),\Omega_p^{\leq p-1})=0$ for $i\neq p-1$, the E_∞ page of the spectral sequence has no non-zero terms away from the diagonal $i+j=p-1$. Moreover, the induced map $H^{p-1}(SL(V),\Omega_p^{\leq p-1})\to H^0(SL(V),\Lambda^p V)$ is zero, so there has to be at least one non-zero differential coming out of the entry $E_2^{0,p-1}=H^0(SL(V),\Lambda^p V)$. Therefore $H^i(G,V^{(1)})\neq 0$ for at least one i.

Let i_- be the minimal such i and i_+ be the maximal one. The latter is well-defined because cohomology of a finite-dimensional module over a reductive group is always concentrated in finitely many degrees by [CPSvdK77, Theorem 2.4(b)]. If $i_+>p-1$ then the entry $E_2^{i^+,1}=H^{i^+}(SL(V),V^{(1)})$ will necessarily survive to the infinite page giving that $H^{i_++1}(SL(V),\Omega_p^{\leq p-1})\neq 0$ which is not the case. Similarly, if $i_-<p-1$ then the entry $E_2^{i_-,0}$ will survive to the infinite page contradicting that $H^{i_-}(SL(V),\Omega_p^{\leq p-1})=0$. Therefore $H^i(G,V^{(1)})\neq 0$ for all $i\neq p-1$ and $d_{p-1}:H^0(SL(V),\Lambda^p V)\to H^{p-1}(SL(V),V^{(1)})$ is an isomorphism. $\square$

Lemma 12.10. *(1) For all a,b the cohomology $H^i(SL(V),S^aV\otimes\Lambda^bV)$ vanishes in all positive degrees i.*
(2) $H^0(SL(V),S^aV\otimes\Lambda^bV)=0$ for $a\geq 2$ and any b.
(3) $(V\otimes\Lambda^{p-1}V)^{SL(V)}$ is a 1-dimensional space, but the multiplication map $V\otimes\Lambda^{p-1}V\to\Lambda^pV$ induces the zero map on subspaces of invariants.

Proof. (1), (2), and the first assertion of (3) are special cases of [Jan03, Proposition II.4.13]. For the second assertion of (3), the multiplication map $V\otimes\Lambda^{p-1}V\to\Lambda^pV$ is equal, up to a unit, to the composition $V\otimes\Lambda^{p-1}V\hookrightarrow V^{\otimes p}\to\Lambda^pV$ where the fist map is induced by the antisymmetrization map $\Lambda^{p-1}V\to V^{\otimes p-1}$. In Lemma 12.5(2) we established that the map $V^{\otimes p}\to\Lambda^pV$ induces the zero map on $SL(V)$-invariants, so part (3) is proven. $\square$

Having done this computation, we can conclude that $\Omega_p^{\leq p-1}$ in fact coincides with $S^p(V[-1])$, up to a shift:

Corollary 12.11. *The complex $\Omega_p^{\leq p-1}[-1]$ is equivalent to $S^p(V[-1])$ in $D_G(k)$.*

Remark 12.12. It would be much nicer to directly identify $\Omega_p^{\leq p-1}[-1]$ with $\tau^{\geq 1}(V[-1]^{\otimes p})_{hS_p}$ thus making the proof of Proposition 12.1 more straightforward.

Proof. By Proposition 12.9 the object $\Omega_p^{\leq p-1}[-1]$ fits into a fiber sequence

$$T_p(V[-2]) \to \Omega_p^{\leq p-1}[-1] \to \Lambda^p V[-p] \tag{12.11}$$

By the proof of Proposition 12.5(3) this fiber sequence is not split. Our object $S^p(V[-1])$ also fits into a non-split fiber sequence with the same first and third terms. Equivalence classes of such fiber sequences are parametrized by $\mathrm{Ext}^{p-1}_{SL(V)}(\Lambda^p V, T_p(V)) = H^{p-1}(SL(V), T_p(V))$. Since $T_p(V)$ fits into a fiber sequence $V^{(1)}[1] \to T_p(V) \to V^{(1)}$, Lemma 12.5(3) implies that this Ext space is 1-dimensional. Therefore any two non-split extensions are isomorphic, as desired. $\square$

13. From algebraic cohomology to cohomology of the group of $\mathbb{F}_q$-points

Let now $k = \mathbb{F}_q$ be a finite field of characteristic p. In this section we show that $\alpha(V)$ remains non-zero when restricted to the discrete group $SL_p(\mathbb{F}_q)$, provided that $q > p$:

Proposition 13.1. *If $\dim V = p$ and $q > p$ then the restriction map $H^{p-1}_{\mathrm{alg}}(SL(V), V^{(1)}) \to H^{p-1}(SL_p(\mathbb{F}_q), V^{(1)})$ is injective.*

In this and next sections we use the notation H^*_{alg} for the cohomology of representations of algebraic groups, and H^* is reserved for cohomology of discrete groups. In general, it follows from results of Cline, Parshall, Scott, and van der Kallen that the restriction from cohomology of a split reductive group G to that of its $\mathbb{F}_q$-points is injective for a large enough q:

Theorem 13.2 ([CPSvdK77, Theorem 6.6]+[CPS83, Theorem 2.1])**.** *Let G be a split reductive group over $\mathbb{F}_p$, and W be a finite-dimensional representation of it. For large enough $q = p^r$ the restriction map*

$$H^n_{\mathrm{alg}}(G, W) \to H^n(G(\mathbb{F}_q), W) \tag{13.1}$$

is injective for all n.

Our Proposition 13.1 is only marginally stronger than this result in that we show that for any $q > p$ injectivity holds in the particular case $G = SL_p, W = V^{(1)}$. We give here an argument that follows closely the proof of [CPSvdK77, Theorem 6.6] in order to introduce the techniques that will be used in Section 14.

Let us briefly describe the method of [CPSvdK77]. Let $B \subset G$ be a Borel subgroup of a split reductive group G over k. We have a commutative diagram

$$\begin{tikzcd} H^n_{\mathrm{alg}}(G, W) \arrow[r] \arrow[d, "\sim"] & H^n(G(\mathbb{F}_q), W) \arrow[d] \\ H^n_{\mathrm{alg}}(B, W) \arrow[r] & H^n(B(\mathbb{F}_q), W) \end{tikzcd} \tag{13.2}$$

The fact that the left vertical map is an isomorphism is a consequence of the vanishing $H^{>0}(G/B,\mathcal{O})=0$ ([Kem76, §6 Theorem 1(a)]) of the cohomology of the structure sheaf on the flag variety:

Theorem 13.3 ([CPSvdK77, Theorem 2.1])**.** *For every algebraic G-module W restriction induces an isomorphism $H^i_{\mathrm{alg}}(G,W)\simeq H^i_{\mathrm{alg}}(B,W)$ for all i.*

Therefore it is enough to prove that the bottom horizontal arrow is injective. This is now a more tangible question as the Borel subgroup is isomorphic to the semi-direct product $T\ltimes U$ of the maximal torus T and the unipotent radical U of B. Since the algebraic cohomology of a torus, as well as the cohomology of the finite group $T(\mathbb{F}_q)$ with p-torsion coefficients vanish in positive degrees, we have $H^i_{\mathrm{alg}}(B,W)=H^i_{\mathrm{alg}}(U,W)^T$ and $H^i(B(\mathbb{F}_q),W)=H^i(U(\mathbb{F}_q),W)^{T(\mathbb{F}_q)}$. This reduces the problem to the study of the action of T on $H^i_{\mathrm{alg}}(U,W)$ which, for the purposes of proving the desired eventual injectivity, can be reduced to the study of the action of $\mathbb{G}_m$ on $H^\bullet_{\mathrm{alg}}(\mathbb{G}_a,k)$ and $\mathbb{F}_q^\times$ on $H^\bullet(\mathbb{F}_q,k)$ which is performed in Proposition 13.4.

Choose a basis $e_1,\dots,e_p$ of V, and let $B_p\subset SL(V)$ be the subgroup of matrices preserving each of the subspaces $\langle e_1,\dots,e_i\rangle\subset V$. Denote also by $T_p\subset B_p$ the maximal torus of the diagonal matrices, and by $U_p\subset B_p$ the subgroup of strictly upper-triangular matrices.

13.1. **Proof of Proposition 13.1 when** $p=2$**.** We will now prove that when $p=2$ and $q>p$ the restriction map $H^1_{\mathrm{alg}}(B_2,V^{(1)})\to H^1(B_2(\mathbb{F}_q),V^{(1)})$ is injective. We treat the case $p=2$ separately both because there are slight notational differences from the case $p>2$, arising from the fact that $H^\bullet_{\mathrm{alg}}(\mathbb{G}_a,k)$ has a different-looking ring structure, and because it explains the idea of the general proof in a setting less burdened by the combinatorial difficulties.

Denote by $\chi_1:T_2\to\mathbb{G}_m$ the character $\mathrm{diag}(a,a^{-1})\mapsto a$ identifying the maximal torus T_2 with $\mathbb{G}_m$. The conjugation action of T_2 on $U_2\simeq\mathbb{G}_a$ is then given by χ_1^2.

When restricted to B_2, the representation $V^{(1)}$ fits into an exact sequence $0\to\chi_1^2\to V^{(1)}\to\chi_1^{-2}\to 0$. It induces the long exact sequences in cohomology of B_2 and $B_2(\mathbb{F}_q)$:

(13.3)
$$\begin{tikzcd}
 & H^1_{\mathrm{alg}}(U_2,\chi_1^2)^{T_2} \arrow[r]\arrow[d] & H^1_{\mathrm{alg}}(U_2,V^{(1)})^{T_2} \arrow[r]\arrow[d] & H^1_{\mathrm{alg}}(U_2,\chi_1^{-2})^{T_2} \arrow[d] \\
H^0(U(\mathbb{F}_q),\chi_1^{-2})^{T_2(\mathbb{F}_q)} \arrow[r] & H^1(U_2(\mathbb{F}_q),\chi_1^2)^{T_2(\mathbb{F}_q)} \arrow[r] & H^1(U_2(\mathbb{F}_q),V^{(1)})^{T_2(\mathbb{F}_q)} \arrow[r] & H^1(U_2(\mathbb{F}_q),\chi_1^{-2})^{T_2(\mathbb{F}_q)}
\end{tikzcd}$$

We have $H^1_{\mathrm{alg}}(U_2,\chi_1^2)\simeq\mathrm{Hom}_{\mathrm{grp}}(\mathbb{G}_a,\mathbb{G}_a)$ with the action of T_2 described as follows: any k-algebra R, an element $a\in T_2(R)=R^\times$ acts by sending a homomorphism $f:\mathbb{G}_a\to\mathbb{G}_a$ to $a^2\cdot f(a^{-2}\cdot -)$. Similarly, $H^1_{\mathrm{alg}}(U_2,\chi_1^{-2})\simeq\mathrm{Hom}_{\mathrm{grp}}(\mathbb{G}_a,\mathbb{G}_a)$ but the torus action sends f to $a^{-2}\cdot f(a^{-2}\cdot -)$. All homomorphisms $f:\mathbb{G}_a\to\mathbb{G}_a$ have the form $f(x)=\sum\limits_{i=0}^N a_ix^{p^i}$ for some $N\geq 0$ and $a_i\in k$, where x is a coordinate on $\mathbb{G}_a$. It follows that $H^1_{\mathrm{alg}}(U_2,\chi_1^{-2})^{T_2}=0$ and $H^1_{\mathrm{alg}}(U_2,\chi_1^2)^{T_2}\subset\mathrm{Hom}_{\mathrm{grp}}(\mathbb{G}_a,\mathbb{G}_a)$ is the one-dimensional space of homomorphisms of the form $x\mapsto a_0\cdot x$. Therefore the map $H^1_{\mathrm{alg}}(U_2,\chi_1^2)^{T_2}\to H^1_{\mathrm{alg}}(U_2,V^{(1)})^{T_2}$ is surjective, and the restriction map $H^1_{\mathrm{alg}}(U_2,\chi_1^2)^{T_2}\to H^1(U_2(\mathbb{F}_q),\chi_1^2)^{T(\mathbb{F}_q)}$ is injective for any q.

It remains to observe that $H^0(U_2(\mathbb{F}_q),\chi_1^{-2})$ is a one-dimensional vector space on which $T_2(\mathbb{F}_q)=\mathbb{F}_q^\times$ acts via the character χ_1^{-2}. Therefore the invariant subspace of this 0th cohomology group is trivial as soon as $q>2$. This implies that the map $H^1(U_2(\mathbb{F}_q),\chi_1^2)^{T_2(\mathbb{F}_q)}\to H^1(U_2(\mathbb{F}_q),V^{(1)})^{T_2(\mathbb{F}_q)}$ is injective, and hence the restriction $H^1_{\mathrm{alg}}(U_2,V^{(1)})^{T_2}\to H^1(U_2(\mathbb{F}_q),V^{(1)})^{T_2(\mathbb{F}_q)}$ is injective.

13.2. **Proof of Proposition 13.1 when $p>2$.** As in the special case $p=2$ that we dealt with above, the key to the proof is to analyze the action of the maximal torus on the cohomology of the unipotent radical of a Borel subgroup of SL_p. For a split torus T over k we denote by $X^*(T)=\mathrm{Hom}_{\mathrm{grp}}(T,\mathbb{G}_m)$ its lattice of characters. We use additive notation for the group operation in $X^*(T)$. If U is a unipotent algebraic group equipped with an action of T, we denote by $\Delta_U\subset X^*(T)$ the set of characters appearing in the T-representation $\mathrm{Lie}\,U$.

Here is the main computation in the case $U\simeq\mathbb{G}_a^n$ using which we will study the case of an arbitrary unipotent U by a dévissage argument.

Proposition 13.4. *Assume that $p>2$. Let A be a group scheme over $\mathbb{Z}$ isomorphic to $\mathbb{G}^n_{a,\mathbb{Z}}$, equipped with an action of a split torus T. Denote by $\mathfrak{a}:=(\mathrm{Lie}\,A)^\vee$ the dual Lie algebra of A. We denote by $\mathfrak{a}_k$ and $\mathfrak{a}_{W_2(k)}$ the modules $\mathfrak{a}\otimes_{\mathbb{Z}}k$ and $\mathfrak{a}\otimes_{\mathbb{Z}}W_2(k)$.*

(1) There is a T-equivariant identification

$$H^\bullet_{\mathrm{alg}}(A,k)=\Lambda^*(\bigoplus_{i=0}^{\infty}\mathfrak{a}_k^{(i)}\cdot x_i)\otimes S^*(\bigoplus_{i=1}^{\infty}\mathfrak{a}_k^{(i)}\cdot\beta(x_i))\tag{13.4}$$

where x_i and $\beta(x_i)$ are formal symbols, invariant under T. The cohomological degrees of x_i and $\beta(x_i)$ are 1 and 2, respectively.

(2) Let $\mathbb{F}_q\subset k$ be a finite subfield of k. There is a $T(\mathbb{F}_q)$-equivariant identification

$$H^\bullet(A(\mathbb{F}_{p^r}),k)=\Lambda^*(\bigoplus_{i=0}^{r-1}\mathfrak{a}_k^{(i)}\cdot x_i)\otimes S^*(\bigoplus_{i=1}^{r}\mathfrak{a}_k^{(i)}\cdot\beta(x_i))\tag{13.5}$$

and the map $H^\bullet_{\mathrm{alg}}(A,k)\to H^\bullet(A(\mathbb{F}_{p^r}),k)$ sends x_i and $\beta(x_i)$ to $x_{i \bmod r}$ and $\beta(x_{((i-1)\bmod r)+1})$, respectively. In particular, this map is surjective in every degree.

Suppose that $F\supset\mathbb{Q}$ is a finite Galois extension, and $\mathfrak{p}\subset\mathcal{O}_F$ is an unramified prime ideal such that the residue field $\mathcal{O}_F/\mathfrak{p}$ is identified with $\mathbb{F}_q$. We have the following $T(\mathcal{O}_F)$-equivariant identifications:

(3) $H^\bullet(A(\mathcal{O}_F),k)=\Lambda^\bullet(\bigoplus_{\tau\in\mathrm{Gal}(F/\mathbb{Q})}\mathfrak{a}_k^\tau\cdot x_\tau)$. Here $T(\mathcal{O}_F)$ acts on $\mathfrak{a}_k$ through the chosen map $T(\mathcal{O}_F)\to T(\mathcal{O}_F/\mathfrak{p})=T(\mathbb{F}_q)$, and $\mathfrak{a}_k^\tau$ denotes the composition of this action with the automorphism $T(\mathcal{O}_F)\xrightarrow{\tau}T(\mathcal{O}_F)$. The map on cohomology induced by $A(\mathcal{O}_F)\to A(\mathbb{F}_q)$ annihilates $\beta(x_i)$ and sends x_i to x_{τ_i} where $\tau_i\in\mathrm{Gal}(F/\mathbb{Q})$ is the element of the decomposition group of $\mathfrak{p}$ that induces the automorphism Fr_p^i on $\mathbb{F}_q$.

(4) $H^\bullet(A(\mathcal{O}_F),W_2(k))=\Lambda^\bullet(\bigoplus_{\tau\in\mathrm{Gal}(F/\mathbb{Q})}\mathfrak{a}^\tau_{W_2(k)}\cdot x_\tau)$ such that the mod p reduction of this isomorphism is the isomorphism in (3).

Example 13.5. For cohomology in degree 1 and $A\simeq\mathbb{G}_{a,\mathbb{Z}}$ statement (1) amounts to the fact that $H^1(\mathbb{G}_{a,k},k)=\mathrm{Hom}_{\mathrm{grp}}(\mathbb{G}_{a,k},\mathbb{G}_{a,k})$ is spanned by homomorphisms of the forms $t \mapsto t^{p^i}$ where t is a coordinate on $\mathbb{G}_{a,k}$. Statement (2) in this case is saying that $H^1(\mathbb{G}_a(\mathbb{F}_{p^r}), k) = \mathrm{Hom}(\mathbb{F}_{p^r}, k)$, and every homomorphism $\mathbb{F}_{p^r} \to k$ of additive groups can be represented as $t \mapsto \sum\limits_{i=0}^{r-1} a_i t^{p^i}$ for some $a_i \in k$, in a unique way.

Proof. By Künneth formula, it is enough to consider the case $A \simeq \mathbb{G}_a$. Parts (1) and (2) are [CPSvdK77, Theorem 4.1], see also [Jan03, Proposition I.4.27]. To fix ideas, let us explicitly say that in part (1) the map $\mathfrak{a}_k^{(i)} \cdot x_i \to H^1(A, k) = \mathrm{Hom}_{\mathrm{grp}}(A_k, \mathbb{G}_{a,k})$ sends an element $\alpha \cdot x_i$ with $0 \neq \alpha \in \mathfrak{a}_k$ to the group scheme homomorphism $A_k \xrightarrow{L_\alpha} \mathbb{G}_{a,k} \xrightarrow{t \mapsto t^{p^i}} \mathbb{G}_{a,k}$ where L_α is the unique group scheme homomorphism that induces the functional α on Lie algebras.

We now turn to proving (4). Part (3) can either be proven by the same argument or deduced formally using that cohomology groups $H^\bullet(A(\mathcal{O}_F), W_2(k))$ are flat $W_2(k)$-modules. Additionally to assuming that $A = \mathbb{G}_a$ we may and will assume that $T = \mathbb{G}_m$, acting on A through some power of the standard character. Since $A(\mathcal{O}_F)$ is isomorphic to $\mathbb{Z}^{[F:\mathbb{Q}]}$, the cohomology ring $H^\bullet(A(\mathcal{O}_F), W_2(k))$ is $T(\mathcal{O}_F)$-equivariantly isomorphic to $\Lambda^\bullet(H^1(A(\mathcal{O}_F), W_2(k)))$ via the multiplication on cohomology.

The 1st cohomology module $H^1(A(\mathcal{O}_F), W_2(k))$ is naturally identified with $\mathrm{Hom}(A(\mathcal{O}_F), W_2(k)) = \mathrm{Hom}_{W_2(k)}(\mathcal{O}_F \otimes_{\mathbb{Z}} W_2(k), W_2(k))$ where $T(\mathcal{O}_F) = \mathcal{O}_F^\times$ acts by multiplication on the source of the maps. The algebra $\mathcal{O}_F \otimes_{\mathbb{Z}} W_2(k)$ is isomorphic to $\bigoplus\limits_{\tau \in \mathrm{Gal}(F/\mathbb{Q})} W_2(k)$ via the isomorphism sending $a \otimes b \in \mathcal{O}_F \otimes_{\mathbb{Z}} W_2(k)$ to $\oplus \tau(\kappa(a)b)$ where $\kappa : \mathcal{O}_F \to \mathcal{O}_F/\mathfrak{p}^2 \simeq W_2(\mathbb{F}_q)$ is the unique lift of the chosen identification $\mathcal{O}_F/\mathfrak{p} \simeq \mathbb{F}_q$. Therefore, $A(\mathcal{O}_F) \otimes_{\mathbb{Z}} W_2(k)$ is isomorphic to $\bigoplus\limits_{\tau \in \mathrm{Gal}(F/\mathbb{Q})} (\mathrm{Lie}\, A_{W_2(k)})^\tau$ as a $\mathcal{O}_F^\times$-module which implies the claim by dualizing. □

Since we assume that k contains $\mathbb{F}_q$, every representation of the finite abelian group $T(\mathbb{F}_q)$ on a finite-dimensional k-vector space decomposes as a direct sum of characters. We will sometimes refer to the characters of $T(\mathbb{F}_q)$ appearing as direct summands of a representation as $T(\mathbb{F}_q)$-*weights* of this representation. The image of the restriction map $X^*(T) = \mathrm{Hom}_{\mathrm{grp}}(T, \mathbb{G}_m) \to \mathrm{Hom}(T(\mathbb{F}_q), k^\times)$ is identified with $X^*(T)/(q-1)$.

It follows from Proposition 13.4 that

Corollary 13.6. *(1) If a character $\chi \in X^*(T)$ is a weight of $H^n_{\mathrm{alg}}(A, k)$ then χ can be expressed as a sum of $\leq n$ elements of $-p^{\mathbb{N}} \cdot \Delta_A \subset X^*(T)$.*

(2) If a character χ of $T(\mathbb{F}_q)$ is a weight of $H^n(A(\mathbb{F}_q), k)$ then χ extends to an algebraic character $\widetilde{\chi}$ of T that is congruent modulo $q-1$ to an element of $X^*(T)$ expressible as a sum of $\leq n$ elements of $-p^{\mathbb{N}} \cdot \Delta_A$.

These observations imply the following injectivity criterion

Corollary 13.7 ([CPSvdK77, 5.4])**.** *Let χ be a character of T.*

(1) If every equality of the form $\chi = p^{r_1}\lambda_1 + \ldots + p^{r_l}\lambda_l$ with $\lambda_1, \ldots, \lambda_l \in -\Delta_A$, $l \leq n$, and some $r_1, \ldots, r_l \geq 0$ satisfies $r_1, \ldots, r_l \leq r-1$, then the restriction map $\mathrm{Hom}_T(\chi, H^n_{\mathrm{alg}}(A, k)) \to \mathrm{Hom}_{T(\mathbb{F}_q)}(\chi, H^n(A(\mathbb{F}_q), k))$ is injective.

(2) If χ is not congruent modulo $(q-1) \cdot X^*(T)$ to a sum of the form $r_1 + \ldots + r_l$ with $r_1, \ldots, r_l \in -p^{\mathbb{N}} \cdot \Delta_A$ and $l \leq n$ then $\mathrm{Hom}_{T(\mathbb{F}_q)}(\chi, H^n(A(\mathbb{F}_q), k)) = 0$.

Proof. 1) Proposition 13.4 (1), (2) implies that there exists a $T(\mathbb{F}_q)$-equivariant map $s_n : H^n(A(\mathbb{F}_q), k) \to H^n_{\mathrm{alg}}(A, k)$ such that its composition with the restriction $H^n_{\mathrm{alg}}(A, k) \to H^n(A(\mathbb{F}_q), k)$ is the identity map: we define s_n by sending x_i to x_i, and $\beta(x_i)$ to $\beta(x_i)$. The assumption on character χ implies that every appearance of χ as a T-equivariant direct summand of $H^n_{\mathrm{alg}}(A, k)$ is in the image of s_n, which implies the injectivity.

2) Immediate from Corollary 13.6(2). □

We can deduce from Proposition 13.4 the following results about the action of T on the cohomology of an arbitrary unipotent group.

Lemma 13.8. *Let U be a unipotent algebraic group over k equipped with an action of a split torus T. As before, $\Delta_U \subset X^*(T)$ is the set of characters of T that appear as weights of the action of T on the Lie algebra* $\mathrm{Lie}\, U$.

(1) *Every weight of T on $H^i_{\mathrm{alg}}(U, k)$ can be expressed as a sum of $\leq i$ elements of $-p^{\mathbb{N}} \cdot \Delta_U$*

(2) *Let $U' \subset U$ be a normal T-stable subgroup, and assume the subsets $p^{\mathbb{N}}\Delta_{U'}, p^{\mathbb{N}}\Delta_{U/U'} \subset X^*(T)$ are disjoint. Fix an integer i. Suppose that for every expression $\chi = r_1 + \cdots + r_l$ with $l \leq i$ and $r_1, \ldots, r_l \in -p^{\mathbb{N}} \cdot \Delta_U$, all $r_1, \ldots, r_l$ are contained in $-p^{\mathbb{N}} \cdot \Delta_{U'}$. Then the map $\mathrm{Hom}_T(\chi, H^i_{\mathrm{alg}}(U, k)) \to \mathrm{Hom}_T(\chi, H^i_{\mathrm{alg}}(U', k))$ is injective.*

Proof. Since U is unipotent, there exists a T-equivariant filtration $U_0 = U \supset U_1 \supset \cdots \supset U_n = 1$ by normal subgroups, such that all graded quotients U_m/U_{m+1} are isomorphic to $\mathbb{G}_a$. We have a Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r_{\mathrm{alg}}(U/U_{n-1}, H^s_{\mathrm{alg}}(U_{n-1}, k)) \Rightarrow H^{r+s}_{\mathrm{alg}}(U, k) \tag{13.6}$$

Since U_{n-1} is a central subgroup of U, the term $E_2^{r,s}$ is T-equivariantly isomorphic to $H^r_{\mathrm{alg}}(U/U_{n-1}, k) \otimes H^s_{\mathrm{alg}}(U_{n-1}, k)$. We can use this spectral sequence to prove (1) by induction on $n = \dim U$, with the base case given by Proposition 13.4(1). By the inductive assumption, the T-weights of $H^r_{\mathrm{alg}}(U/U_{n-1}, k)$ are sums of $\leq r$ elements of $-p^{\mathbb{N}} \cdot \Delta_{U/U_{n-1}}$, and the weights of $H^s_{\mathrm{alg}}(U_{n-1}, k)$ are sums of $\leq s$ elements of $-p^{\mathbb{N}} \cdot \Delta_{U_{n-1}}$ by Proposition 13.4(1). From the spectral sequence (13.6) we have that $H^i_{\mathrm{alg}}(U, k)$ is a T-equivariant subquotient of $\bigoplus\limits_{r+s=i} H^r_{\mathrm{alg}}(U/U_{n-1}, k) \otimes H^s_{\mathrm{alg}}(U_{n-1}, k)$ which proves the inductive step, completing the proof of (1).

To prove (2), consider the spectral sequence

$$\widetilde{E}_2^{r,s} = H^r_{\mathrm{alg}}(U/U', H^s_{\mathrm{alg}}(U', k)) \Rightarrow H^{r+s}_{\mathrm{alg}}(U, k) \tag{13.7}$$

To show injectivity of the restriction map

$$\mathrm{Hom}_T(\chi, H^i_{\mathrm{alg}}(U, k)) \to \mathrm{Hom}_T(\chi, E_2^{0,i}) \subset \mathrm{Hom}_T(\chi, H^i_{\mathrm{alg}}(U', k)) \tag{13.8}$$

it is enough to prove that the spaces $\mathrm{Hom}_T(\chi, H^{i-s}_{\mathrm{alg}}(U/U', H^s_{\mathrm{alg}}(U', k)))$ vanish for $s < i$. Since every representation of the solvable group scheme $T \ltimes (U/U')$ is a successive extension of characters, $H^s_{\mathrm{alg}}(U', k)$ has a filtration by U/U'-submodules with 1-dimensional graded pieces which is also respected by T. Therefore T-weights of $H^{i-s}_{\mathrm{alg}}(U/U', H^s_{\mathrm{alg}}(U', k))$ form a subset of the set of T-weights of $H^{i-s}_{\mathrm{alg}}(U/U', k) \otimes H^s_{\mathrm{alg}}(U', k)$.

For $s < i$ the character χ does not appear in the latter set because our assumption implies that χ cannot be written as $\chi' + \chi''$ where χ' is a sum of $\leq s$ elements of $-p^{\mathbb{N}} \cdot \Delta_{U'}$, and χ'' is a sum of $\leq i - s$ elements of $-p^{\mathbb{N}} \cdot \Delta_{U/U'}$. $\square$

We will now compute Δ_{U_p} for the unipotent radical $U_p \subset B_p$ of a Borel subgroup of SL_p. Let $e_1, \dots, e_p$ be a basis of V so that B_p is the subgroup preserving each of the subspaces $\langle e_1, \dots, e_i\rangle$, and $U_p \subset B_p$ is the subgroup of matrices that moreover act trivially on the quotients $\langle e_1, \dots, e_i\rangle / \langle e_1, \dots, e_{i-1}\rangle$. Let $\chi_i \in X^*(T_p)$ be the character of the torus $T_p = B_p/U_p$ through which it acts on e_i. Note that $\chi_p = -\chi_1 - \dots - \chi_{p-1}$, and $\chi_1, \dots, \chi_{p-1}$ form a basis of $X^*(T_p)$. The set of positive roots Δ_{U_p} is equal to

$$\{\chi_i - \chi_j | 1 \leq i < j \leq p-1\} \cup \{\chi_i + (\chi_1 + \dots + \chi_{p-1}) | 1 \leq i \leq p-1\} \tag{13.9}$$

On the other hand, the weights of T_p on the representation $V^{(1)}$ are given by $p\chi_1, \dots, p\chi_{p-1}, p\chi_p = -p(\chi_1 + \dots + \chi_{p-1})$.

Our goal is to prove injectivity of the map $H^{p-1}_{\mathrm{alg}}(U_p, V^{(1)})^T \to H^{p-1}(U_p(\mathbb{F}_q), V^{(1)})$ for $q > p$. To do so we will analyze the T-action on $H^i_{\mathrm{alg}}(U_p, k)$ for $i \leq p-1$ and determine for which i, j the cohomology group $H^i_{\mathrm{alg}}(U_p, \chi_j^p)$ might have non-zero T-invariants. This will be achieved through the following combinatorial computation:

Lemma 13.9. *(1) For $2 \leq j \leq p$ the character $p\chi_j$ does not belong to the submonoid of $X^*(T_p)$ spanned by Δ_{U_p}.*

(2) The only (up to permutation) way to express $p\chi_1$ as a sum of $\leq p-1$ elements of $p^{\mathbb{N}} \cdot \Delta_{U_p}$ is $(\chi_1 - \chi_2) + \dots + (\chi_1 - \chi_{p-1}) + (\chi_1 + (\chi_1 + \chi_2 + \dots + \chi_{p-1}))$.

(3) For $q > p$, and any $2 \leq j \leq p$, the character $p\chi_j$ is not congruent modulo $q-1$ to a sum of $\leq p-1$ elements of $p^{\mathbb{N}} \cdot \Delta_{U_p}$.

(4) For $q > p$, any congruence modulo $q-1$ between $p\chi_1$ and a sum of $\leq p-1$ elements of $p^{\leq r-1} \cdot \Delta_{U_p}$ is an equality. In particular, by (2) there are no such congruences with strictly less than $p-1$ summands.

Proof. The first statement for $j = p$ is clear because the image of every element of Δ_{U_p} under the map $\sigma : X^*(T_p) \xrightarrow{b_1\chi_1 + \dots + b_{p-1}\chi_{p-1} \mapsto b_1 + \dots + b_{p-1}} \mathbb{Z}$ is non-negative. Similarly, a linear combination of elements of Δ_{U_p} that belongs to $\langle \chi_2, \dots, \chi_{p-1}\rangle$ must be a combination of elements $\chi_i - \chi_j, 2 \leq i < j \leq p-1$ and is therefore killed by σ. This shows that $p\chi_j$ for $j = 2, \dots, p-1$ are not in the monoid generated by Δ_{U_p} either.

For the second statement consider an arbitrary expression $p\chi_1 = p^{r_1}\lambda_1 + \dots + p^{r_l}\lambda_l$ with $l \leq p-1$, $\lambda_1, \dots, \lambda_l \in \Delta_{U_p}$. Since $\sigma(p\chi_1) = p$, there is exactly one m such that λ_m is from the set $\{\chi_1 + (\chi_1 + \dots + \chi_{p-1}), \dots, \chi_{p-1} + (\chi_1 + \dots + \chi_{p-1})\}$ and $r_m = 0$ for this m. There are exactly $p-2$ other elements of Δ_{U_p} in which χ_1 appears with a non-zero coefficient: $\chi_1 - \chi_2, \dots, \chi_1 - \chi_{p-1}$. Therefore $l = p-1$, all $r_1, \dots, r_{p-1}$ are equal to 0, and all $\lambda_1, \dots, \lambda_{p-1}$ are elements of the set $\{\chi_1 - \chi_2, \dots, \chi_1 - \chi_{p-1}, 2\chi_1 + \chi_2 + \dots + \chi_{p-1}\}$. Hence there must be no repetitions among $\lambda_1, \dots, \lambda_{p-1}$ for them to sum up to $p\chi_1$, which proves assertion (2).

Suppose that, contrary to the assertion (3), there is a congruence

$$p\chi_j \equiv p^{r_1}\lambda_1 + \dots + p^{r_l}\lambda_l \bmod q-1 \tag{13.10}$$

with $l \leq p-1$ and all $\lambda_m \in \Delta_{U_p}$. The coefficient of χ_1 in $p^{r_1}\lambda_1 + \dots + p^{r_l}\lambda_l$ is a sum of $\leq 2l$ powers of p. Hence it is a number whose sum of digits in base p is at most $2l \leq 2(p-1)$. Moreover, its sum of digits is equal to $2(p-1)$ only if $l = p-1$, and all $\lambda_1, \dots, \lambda_{p-1}$

are equal to $2\chi_1+\chi_2\ldots+\chi_{p-1}$. This would violate (13.10), because the right hand side would have the shape $(p^{\lambda_1}+\ldots+p^{\lambda_{p-1}})\cdot(2\chi_1+\chi_2\ldots+\chi_{p-1})$. Therefore the sum of digits in base p of the coefficient of χ_1 in $p^{r_1}\lambda_1+\ldots+p^{r_l}\lambda_l$ is at most $2(p-1)-1$.

Note that a number with base p expansion $\overline{a_n\ldots a_r a_{r-1}\ldots a_0}$ is congruent to $\overline{a_n\ldots a_r}+\overline{a_{r-1}\ldots a_0}$ modulo p^r-1. Applying this observation repeatedly, we see that for any non-zero number a there exists a number $0<a'<p^r$ congruent to a modulo p^r-1 and with sum of digits less or equal to that of a. For $2\leq j\leq p-1$ the coefficient of χ_1 in $p\chi_j$ is zero, so by this discussion there would have to be an integer $0<a'<p^r$ divisible by p^r-1 with the sum of digits $\leq 2(p-1)-1$, but there is no such number.

For $j=p$ the coefficient of χ_1 in $p\chi_j$ is $-p$, but the only number $0<a'<p^r-1$ congruent to $-p$ modulo p^r-1 is $p^r-p-1=\overline{(p-1)(p-1)\ldots(p-1)(p-2)(p-1)}$ and its sum of digits is $r(p-1)-1$. This finishes the proof of (3) if $r>2$, but for $r=2$ we still need to rule out the possibility that the sum of digits in base p of the coefficient of χ_1 in $p^{r_1}\lambda_1+\ldots+p^{r_l}\lambda_l$ is $2(p-1)-1$. If this was the case, up to reordering the summands, this sum would have the form

$$(p^{r_1}+\ldots+p^{r_{p-2}})(2\chi_1+\chi_2+\ldots+\chi_{p-1})+p^{r_{p-1}}(\chi_i+(\chi_1+\ldots+\chi_{p-1})) \tag{13.11}$$

or

$$(p^{r_1}+\ldots+p^{r_{p-2}})(2\chi_1+\chi_2+\ldots+\chi_{p-1})+p^{r_{p-1}}(\chi_1-\chi_i), \tag{13.12}$$

for some $i=2,\ldots,p-1$. Neither of these expressions can be congruent to $p\chi_p=-p(\chi_1+\ldots+\chi_{p-1})$ modulo p^2-1: if $p>3$ this is clear because for all $j\neq 1,i$ (which exists since $p>3$), the difference between the coefficients of χ_1 and χ_j in these sums is a positive integer with the sum of digits $\leq p-1$, and in particular it cannot be zero modulo p^2-1. If $p=3$ then we have a congruence of the form $-3(\chi_1+\chi_2)\equiv 3^{r_1}(2\chi_1+\chi_2)+3^{r_2}(\chi_1+2\chi_2)$ or $-3(\chi_1+\chi_2)\equiv 3^{r_1}(2\chi_1+\chi_2)+3^{r_2}(\chi_1-\chi_2)$ modulo 8. In the first case we would have $3^{r_1}\equiv 3^{r_2}$ and $3^{r_1+1}\equiv -3$ which is impossible because (-1) is not a power of 3 modulo 8. In the second case comparing the coefficients of χ_2 we arrive at the contradiction as well, finishing the proof of (3).

For the assertion (4), suppose that $p\chi_1\equiv p^{r_1}\lambda_1+\ldots+p^{r_l}\lambda_l$ is such a congruence. The value of σ on its right hand side is a number less than or equal to $p^r(p-1)$ whose sum of digits is at most $p-1$, and which is congruent to p modulo p^r-1. The only such number is p itself (we use here that $r>1$), which implies that for exactly one value of m we have $r_m=0$ and $\lambda_m=\chi_i+(\chi_1+\ldots+\chi_{p-1})$, while for all $m'\neq m$ the character $\lambda_{m'}$ is of the form $\chi_i-\chi_j$.

Next, we consider the coefficient c of χ_1 in the right hand side of our congruence. Summands $p^{r_{m'}}\lambda_{m'}$ for $m'\neq m$ contribute at most p^{r-1} to this coefficients, and $p^{r_m}\lambda_m=\lambda_m$ contributes 1 or 2. Hence c is less than or equal to $p^{r-1}(p-2)+2$. As c also has to be congruent to p modulo p^r-1, it is forced to be equal to p. Given what we already know about the right hand side, this can only happen if $l=p-1$, and all $r_1,\ldots,r_l$ are equal to zero, hence the congruence is forced to be an equality. □

Let $\mathbb{G}_a^{p-1}\simeq A_p\subset U_p$ be the subgroup of matrices that act trivially on the quotient $V/\langle e_1\rangle$. Note that A_p is preserved by the action of T_p and

$$\Delta_{A_p}=\{\chi_1-\chi_i|2\leq i\leq p-1\}\cup\{2\chi_1+\chi_2+\ldots+\chi_{p-1}\}\subset\Delta_{U_p} \tag{13.13}$$

Lemma 13.9 indicates that restriction to the subgroup $A_p\subset U_p$ should detect all cohomology classes of $V^{(1)}$ in degrees $\leq p-1$. We make this precise in Lemma 13.10 below. The deduction of Lemma 13.10 from Lemma 13.9 is analogous to the discussion of injectivity conditions in [CPSvdK77, §5].

Lemma 13.10. *The following restriction maps are injective:*

(1) $H^{p-1}_{\mathrm{alg}}(B_p, V^{(1)}) = H^{p-1}_{\mathrm{alg}}(U_p, V^{(1)})^{T_p} \to H^{p-1}_{\mathrm{alg}}(A_p, V^{(1)})^{T_p}$.

(2) $H^{p-1}_{\mathrm{alg}}(A_p, V^{(1)})^{T_p} \to H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)}$*, if q is strictly larger than p.*

Proof. As a representation of B_p, $V^{(1)}$ admits a filtration with graded pieces given by the characters χ^p_i, for $i = 1, \dots, p$. Given Lemma 13.9(1) and (2), Lemma 13.8(1) implies that $H^i_{\mathrm{alg}}(B_p, \chi^p_j) = H^i_{\mathrm{alg}}(U_p, \chi^p_j)^{T_p} = 0$ for $2 \le j \le p$ and all i, and for $j = 1$ with $i < p-1$. Lemma 13.8(2) shows that the restriction $H^j_{\mathrm{alg}}(U_p, \chi^p_1)^{T_p} \to H^j_{\mathrm{alg}}(A_p, \chi^p_1)^{T_p}$ is an isomorphism (both groups are in fact zero) for $j < p-1$ and is an injection for $j = p-1$. Therefore the restriction $H^{p-1}_{\mathrm{alg}}(B_p, V^{(1)}) \to H^{p-1}_{\mathrm{alg}}(A_p, V^{(1)})^{T_p}$ is injective.

For the second statement, $H^i(A_p(\mathbb{F}_q), \chi^p_j)^{T(\mathbb{F}_q)} = 0$ for $i < p-1$ and all j, by the combination of Corollary 13.6(2) and Lemma 13.9(3), (4). The restriction maps $H^{p-1}_{\mathrm{alg}}(A_p, \chi^p_j)^{T_p} \to H^{p-1}(A_p(\mathbb{F}_q), \chi^p_j)^{T_p(\mathbb{F}_q)}$ are injective by Lemma 13.9(3),(4) and Corollary 13.7 (the source group is in fact zero for $j \neq 1$). This implies that the restriction $H^{p-1}_{\mathrm{alg}}(A_p, V^{(1)})^{T_p} \to H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)}$ is injective. □

Proof of Proposition 13.1. Lemma 13.10 completes the proof of Proposition 13.1, because combined with Theorem 13.3 it even shows that the composition $H^{p-1}_{\mathrm{alg}}(SL_p, V^{(1)}) \simeq H^{p-1}_{\mathrm{alg}}(B_p, V^{(1)}) \to H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})$ is injective. □

Moreover, the 1-dimensional subrepresentation $\chi^p_1 \subset V^{(1)}$ is responsible for all of cohomology of $V^{(1)}$ in degree $p-1$. Precisely, we have the following results that will be used in the next section, and in the proof of Lemma 13.12.

Lemma 13.11. *(1) The map $H^{p-1}_{\mathrm{alg}}(A_p, \chi^p_1)^{T_p} \to H^{p-1}_{\mathrm{alg}}(A_p, V^{(1)})^{T_p}$ is an isomorphism of* 1*-dimensional vector spaces.*

(2) The map $H^{p-1}(A_p(\mathbb{F}_q), \chi^p_1)^{T_p(\mathbb{F}_q)} \to H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)}$ is an isomorphism when $q > p$.

Proof. 1) The kernel and cokernel of this map are, respectively, a quotient and a subgroup of the groups $H^{p-2}_{\mathrm{alg}}(A_p, \chi^p_2 \oplus \dots \oplus \chi^p_p)^{T_p}$ and $H^{p-1}_{\mathrm{alg}}(A_p, \chi^p_2 \oplus \dots \oplus \chi^p_p)^{T_p}$ and we saw in the proof of Lemma 13.10 that they both vanish. The fact that the T_p-invariant subspace of $H^{p-1}_{\mathrm{alg}}(A_p, \chi^p_1)$ is 1-dimensional follows from Lemma 13.9(2).

2) Just like in the case of algebraic cohomology, the kernel and cokernel of this map are subquotients of the groups $H^{p-2}(A_p(\mathbb{F}_q), \chi^p_2 \oplus \dots \oplus \chi^p_p)^{T_p(\mathbb{F}_q)}$ and $H^{p-1}(A_p(\mathbb{F}_q), \chi^p_2 \oplus \dots \oplus \chi^p_p)^{T_p(\mathbb{F}_q)}$ and they vanish by the proof of Lemma 13.10. □

We can deduce from the results obtained so far the following expression for the class $\alpha(E)$ when E is a vector bundle of rank p that admits a line sub-bundle, which was used in Section 8 (as Lemma 8.2) to relate the class $\alpha(\Omega^1_{X_0})$ to the Kodaira-Spencer map of a fibration.

Lemma 13.12. *Let X_0 be arbitrary algebraic stack over $\mathbb{F}_p$. Suppose that a vector bundle E of rank p on X_0 fits into an extension*

$$0 \to L \to E \to E' \to 0 \tag{13.14}$$

where L is a line bundle, and E' is a vector bundle of rank $p-1$. The class of this extension defines an element $v(E) \in \mathrm{Ext}^1_{X_0}(E', L) = H^1(X_0, L \otimes (E')^\vee)$. Denote by $v(E)^{p-1} \in H^{p-1}(X_0, L^{\otimes p-1} \otimes (\det E')^\vee)$ the image of $v(E)^{\otimes p-1} \in H^{p-1}(X_0, (L \otimes (E')^\vee)^{\otimes p-1})$ under the map induced by $(L \otimes (E')^\vee)^{\otimes p-1} \to \Lambda^{p-1}(L \otimes (E')^\vee) = L^{\otimes p-1} \otimes (\det E')^\vee$.

The class $\alpha(E) \in \mathrm{Ext}^{p-1}_{X_0}(\Lambda^p E, F^*E) = H^{p-1}(X_0, F^*E \otimes L^\vee \otimes (\det E')^\vee)$ is equal, up to multiplying by a scalar from $\mathbb{F}_p^\times$, to the image of $v(E)^{p-1}$ under the map induced by $L^{\otimes p-1} \otimes (\det E')^\vee = F^*L \otimes L^\vee \otimes (\det E')^\vee \hookrightarrow F^*E \otimes L^\vee \otimes (\det E')^\vee$.

Proof. The GL_p-torsor $\underline{\mathrm{Isom}}(E, \mathcal{O}^{\oplus p})$ associated to the bundle E naturally reduces to a maximal parabolic subgroup $P := \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ & & \ddots & \vdots \\ 0 & * & & \\ 0 & * & \dots & * \end{pmatrix} \subset GL_p$. Therefore E arises as the pullback of the tautological rank p vector bundle V along the classifying map $X_0 \to BP$ to the classifying stack of the group P over $\mathbb{F}_p$. Therefore it is enough to prove the corresponding expression for the class $\alpha(V) \in \mathrm{Ext}^{p-1}_{P,\mathrm{alg}}(\Lambda^p V, V^{(1)}) = H^{p-1}_{\mathrm{alg}}(P, V^{(1)} \otimes (\Lambda^p V)^*)$.

Denote by $B_p \subset P \cap SL_p$ the subgroup of upper triangular matrices. By [Jan03, Corollary II.4.7(c)] the restriction map $H^i(P \cap SL_p, W) \to H^i(B_p, W|_{B_p})$ is an isomorphism for any $P \cap SL_p$-module W. Since restriction along the inclusion $P \cap SL_p \subset P$ is an injection on cohomology, it is enough to prove the desired expression for $\alpha(V)$ in $\mathrm{Ext}^{p-1}_{B_p,\mathrm{alg}}(\Lambda^p V, V^{(1)}) = H^{p-1}_{\mathrm{alg}}(B_p, V^{(1)})$.

Recall that we denote by $T_p \subset B_p$ the maximal torus of diagonal matrices and by $\chi_i : T_p \to \mathbb{G}_m$, for $i = 1, \dots, p$, the character sending $\mathrm{diag}(a_1, \dots, a_p)$ to a_i. We denote by the same symbol the composite character $B_p \to T_p \xrightarrow{\chi_i} \mathbb{G}_m$. Since we are working inside SL_p the character χ_p can be expressed as $(\chi_1 \cdot \dots \cdot \chi_{p-1})^{-1}$. Next, there is a subgroup $\mathbb{G}_a^{p-1} \simeq A_p = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & 0 \\ & & \ddots & \vdots \\ 0 & 0 & & \\ 0 & 0 & \dots & 1 \end{pmatrix} \subset B_p$. The Frobenius twist $V^{(1)}$ viewed as a representation of B_p admits a filtration with quotients $\chi_1^p, \dots, \chi_p^p$.

By Lemma 13.10(1), we may further restrict to the subgroup $T_p \ltimes A_p \subset B_p$ to prove the desired equality of cohomology classes. The class $v(V|_{T_p \ltimes A_p}) \in H^1_{\mathrm{alg}}(T_p \ltimes A_p, \chi_1 \otimes (\chi_2 \oplus \ldots \oplus \chi_p)^\vee) = \mathrm{Hom}_{\mathrm{grp}}(A_p, \chi_1 \otimes (\chi_2 \oplus \ldots \oplus \chi_p)^\vee)^{T_p}$ is an isomorphism between A_p and the underlying vector space of the representation $\chi_1 \otimes (\chi_2 \oplus \ldots \oplus \chi_p)^\vee$, and its power $v(V|_{T_p \ltimes A_p})^{p-1} \in H^{p-1}_{\mathrm{alg}}(A_p, \chi_1^{p-1} \otimes \chi_2^{-1} \otimes \ldots \otimes \chi_p^{-1})^{T_p} = H^{p-1}(A_p, \chi_1^p)^{T_p}$ is therefore non-zero.

By Lemma 13.11(1) the class $\alpha(V|_{T_p \ltimes A_p}) \in H^{p-1}(A_p, V^{(1)})^{T_p}$ is the image of some class in $H^{p-1}(A_p, \chi_1^p)^{T_p}$. Since $\alpha(V|_{T_p \ltimes A_p})$ is non-zero, and $H^{p-1}(A_p, \chi_1^p)^{T_p}$ is a one-dimensional vector space, the result follows. $\square$

Remark 13.13. We do not expect Lemma 13.10(2) to remain true for $q = p$. E.g. when $p = 2$ in Remark 12.3 we computed explicitly a cocycle representing the class $\alpha(V) \in H^1_{\mathrm{alg}}(SL_2, V^{(1)})$, and its restriction to $U_2 \simeq \mathbb{G}_a$ is the image of $\mathrm{Id} \in \mathrm{Hom}_{\mathrm{grp}}(U_2, \mathbb{G}_a) =$

$H^1_{\mathrm{alg}}(U_2, k)$ under the map in the long exact sequence

$$\cdots \to H^0_{\mathrm{alg}}(U_2, k) \xrightarrow{\delta} H^1_{\mathrm{alg}}(U_2, k) \to H^1_{\mathrm{alg}}(U_2, V^{(1)}) \to H^1_{\mathrm{alg}}(U_2, k) \to \dots \tag{13.15}$$

The connecting homomorphism δ sends $1 \in k = H^0_{\mathrm{alg}}(U_2, k)$ to the Frobenius map $\mathrm{Fr}_2 \in \mathrm{Hom}(U_2, \mathbb{G}_a) = H^1_{\mathrm{alg}}(U_2, k)$. Since Id coincides with Fr_2 when restricted to $U_2(\mathbb{F}_2)$, the image of $\alpha(V) \in H^1_{\mathrm{alg}}(SL_2, V^{(1)})$ in $H^1(U_2(\mathbb{F}_2), V^{(1)})$ (and consequently in $H^1(SL_2(\mathbb{F}_2), V^{(1)})$) is zero.

14. Cohomology of SL_p over rings of integers

We will enhance the results of the previous section by showing that for an appropriately chosen discrete group acting on the vector space V, the Bockstein homomorphism (associated to a lift of V over $W_2(\mathbb{F}_q)$) applied to the class $\alpha(V)$ is non-zero. We keep the notation of the previous section: we work over a finite field $k = \mathbb{F}_q$, and V is a p-dimensional $\mathbb{F}_q$-vector space.

Let $\widetilde{V}$ be a free $W_2(\mathbb{F}_q)$-module such that $\widetilde{V}/p \simeq V$, it is equipped with the tautological action of the discrete group $GL_p(W_2(\mathbb{F}_q))$. Denote by $\widetilde{V}^{(1)} := \widetilde{V} \otimes_{W_2(\mathbb{F}_q), W_2(\mathrm{Fr}_p)} W_2(\mathbb{F}_q)$ the twist of $\widetilde{V}$ by the Frobenius automorphism of $W_2(\mathbb{F}_q)$ induced by $\mathrm{Fr}_p : t \mapsto t^p$ on $\mathbb{F}_q$. The module $\widetilde{V}^{(1)}$ can be $GL_p(W_2(\mathbb{F}_q))$-equivariantly identified with $\widetilde{V}$ where the action on the latter is modified by precomposing with the Frobenius automorphism $GL_p(W_2(\mathbb{F}_q)) \xrightarrow{W_2(\mathrm{Fr}_p)} GL_p(W_2(\mathbb{F}_q))$. We have a short exact sequence

$$0 \to V^{(1)} \xrightarrow{\psi_1} \widetilde{V}^{(1)} \xrightarrow{\psi_2} V^{(1)} \to 0 \tag{14.1}$$

of $GL_p(W_2(\mathbb{F}_q))$-modules which gives rise to the connecting homomorphisms $\mathrm{Bock}^i : H^i(G, V^{(1)}) \to H^{i+1}(G, V^{(1)})$ for any group G mapping to $GL_p(W_2(\mathbb{F}_q))$.

Let now F be an arbitrary number field in which p is unramified and such that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with the residue field $\mathbb{F}_q$. Prime ideal $\mathfrak{p}$ gives rise to a surjection $\kappa : \mathcal{O}_F \twoheadrightarrow \mathbb{F}_q$, and there is a unique homomorphism of rings $\mathcal{O}_F \twoheadrightarrow W_2(\mathbb{F}_q)$ lifting κ. This homomorphism gives rise to a map $SL_p(\mathcal{O}_F) \to SL_p(W_2(\mathbb{F}_q))$ which defines an action of $SL_p(\mathcal{O}_F)$ on $V, V^{(1)}$, and $\widetilde{V}^{(1)}$, and hence defines Bockstein homomorphisms $\mathrm{Bock}^i : H^i(SL_p(\mathcal{O}_F), V^{(1)}) \to H^{i+1}(SL_p(\mathcal{O}_F), V^{(1)})$.

Proposition 14.1. *For every prime p there exists a quadratic extension $F/\mathbb{Q}$ in which p is not split, such that $\mathrm{Bock}^{p-1}(\alpha(V)) \in H^p(SL_p(\mathcal{O}_F), V^{(1)})$ is non-zero.*

We will prove this non-vanishing by a method similar to the one employed in previous section, using the technique of [CPSvdK77]. For the method to work we need the image of the reduction map $\mathcal{O}_F^\times \to \mathbb{F}_{p^2}^\times$ on groups of units to be large enough: the field F will be chosen appropriately in Lemma 14.2.

As in the previous section, we denote by $B_p \subset SL_p$ the subgroup of upper triangular matrices with respect to a given basis, $T_p \subset B_p$ is the diagonal torus, and $\mathbb{G}_a^{p-1} \simeq A_p \subset B_p$ is the subgroup of matrices that send the basis vector e_i to a vector of the form $a_i e_1 + e_i$, for all $i \geq 2$. By Lemmas 13.10 and 13.11(2), the image of the class $\alpha(V) \in H^{p-1}_{\mathrm{alg}}(SL(V), V^{(1)})$ in $H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)}$ is non-zero and moreover lies in the image of the homomorphism $H^{p-1}(A_p(\mathbb{F}_q), \chi_1^p)^{T_p(\mathbb{F}_q)} \to H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)}$. Therefore to prove that $\mathrm{Bock}(\alpha(V))$ is non-zero we may work with the cohomology of the group of $\mathcal{O}_F$-points of the subgroup $T_p \ltimes A_p \subset SL_p$ which is a fairly explicit object thanks to Proposition 13.4.

When restricted to $B_p(\mathcal{O}_F)$, the representation $\widetilde{V}$ admits a filtration with graded quotients $\widetilde{\chi}_1, \dots, \widetilde{\chi}_p$ that are characters factoring through $B_p(\mathcal{O}_F) \to T_p(\mathcal{O}_F)$, and lifting the characters $\chi_1, \dots, \chi_p$. We denote by $\widetilde{\chi}_i^{(1)}$ the character $B_p(\mathcal{O}_F) \to T_p(\mathcal{O}_F) \to W_2(\mathbb{F}_q)^\times$ obtained by composing $\widetilde{\chi}_i$ with the Frobenius automorphism $\mathrm{Fr}_p : W_2(\mathbb{F}_q)^\times \to W_2(\mathbb{F}_q)^\times$. The representation $\widetilde{V}^{(1)}$ of $B_p(\mathcal{O}_F)$ is likewise filtered with graded quotients isomorphic to $\widetilde{\chi}_i^{(1)}$. Note that $\widetilde{\chi}_i^{(1)}/p \simeq \chi_i^{(1)} \simeq \chi_i^p$, but $\widetilde{\chi}_i^{(1)}$ is generally *not* isomorphic to $\widetilde{\chi}_i^p$. The discrepancy between these two characters will be key for proving that the class $\alpha(V) \in H^{p-1}(SL_p(\mathcal{O}_F), V^{(1)})$ does not lift to a class in $H^{p-1}(SL_p(\mathcal{O}_F), \widetilde{V}^{(1)})$.

We treat separately cases $p = 2$ and $p > 2$ because the proof for $p > 2$ relies on the results of the previous section that were only proven away from the case $p = 2$. We also hope that reading the proof for $p = 2$ first makes the general argument clearer. In all of the cases, we choose an extension $F/\mathbb{Q}$ as directed by Lemma 14.2.

Lemma 14.2. *For every prime number p there exists an integer $N > 0$ such that p is not split in the real quadratic field $F = \mathbb{Q}(\sqrt{N})$, and the two conditions are satisfied:*

1. *The group of units $\mathcal{O}_F^\times$ surjects onto $\{x \in \mathbb{F}_{p^2}^\times | N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x) = \pm 1\}$ under the reduction map $\mathcal{O}_F \to \mathcal{O}_F/p \simeq \mathbb{F}_{p^2}$*
2. *There exists a unit $u \in \mathcal{O}_F^\times$ whose reduction $\overline{u}$ in $W_2(\mathbb{F}_{p^2})^\times$ satisfies $\mathrm{Fr}_p(\overline{u}) \neq \overline{u}^p$.*

Proof. We first treat the case $p > 2$. Let $u_0 \in \mathbb{F}_{p^2}^\times$ be a generator of the cyclic group $\{x \in \mathbb{F}_{p^2}^\times | N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x) = \pm 1\}$. It satisfies the equation $u_0^2 - 2d_0u_0 - 1 = 0$ where $d_0 = \frac{1}{2}\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(u_0) \in \mathbb{F}_p$, hence we can write u_0 as $d_0 + \sqrt{d_0^2 + 1}$. Let $d \in \mathbb{Z}$ be an arbitrary integer reducing to d_0 modulo p, and take F to be the field $\mathbb{Q}(\sqrt{d^2+1})$. By construction, $d^2 + 1$ is not a square modulo p, hence $d^2 + 1$ is not a square in $\mathbb{Q}$, and p is non-split in $\mathcal{O}_F$. The element $u = d + \sqrt{d^2+1} \in \mathcal{O}_F$ is invertible and it (or its conjugate) reduces to u_0 in $\mathbb{F}_{p^2}$, hence condition (1) is satisfied.

Next, let us check that we can choose the lift d of d_0 to ensure that condition (2) is satisfied for $u = d + \sqrt{d^2+1}$. The element $\mathrm{Fr}_p(\overline{u}) \in W_2(\mathbb{F}_{p^2})$ is the mod p^2 reduction of $d - \sqrt{d^2+1}$, hence to prove that $\mathrm{Fr}_p(\overline{u}) \neq \overline{u}^p$ it is enough to ensure that the integers $\mathrm{Tr}_{F/\mathbb{Q}}(d - \sqrt{d^2+1})$ and $\mathrm{Tr}_{F/\mathbb{Q}}((d + \sqrt{d^2+1})^p)$ are not congruent modulo p^2. The first one is equal to $2d$, and we can expand their difference as

$$\mathrm{Tr}_{F/\mathbb{Q}}((d + \sqrt{d^2+1})^p) - \mathrm{Tr}_{F/\mathbb{Q}}(d - \sqrt{d^2+1}) = 2(d^p + \binom{p}{2}d^{p-2}(d^2+1) + \ldots + \binom{p}{p-1}d(d^2+1)^{(p-1)/2}) - 2d \tag{14.2}$$

This is a polynomial of degree p in d that reduces to $2(d^p - d)$ modulo p. In particular, by Hensel's lemma, this polynomial has exactly one root in $\mathbb{Z}/p^2$ reducing to d_0, so we can choose the lift d of d_0 such that the integer (14.2) is not zero modulo p^2.

Finally, for $p = 2$ take $F = \mathbb{Q}(\sqrt{5})$. The unit $a = \frac{1+\sqrt{5}}{2} \in \mathcal{O}_F = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ has minimal polynomial $a^2 - a - 1 = 0$, hence 2 is not split in $\mathcal{O}_F$, and a reduces to an element of $\mathbb{F}_4 \setminus \mathbb{F}_2$ that necessarily generates $\mathbb{F}_4^\times$. Condition (2) is fulfilled simply by $u = -1$. $\square$

Proof of Proposition 14.1 for $p = 2$. First, we will prove that the class $\alpha(V) \in H^1(SL_2(\mathbb{F}_q), V^{(1)})$ survives under the map to $H^1(SL_2(\mathcal{O}_F), V^{(1)})$. When restricted to the

Borel subgroup B_2, the module $V^{(1)}$ fits into the extension $0 \to \chi_1^2 \to V^{(1)} \to \chi_1^{-2} \to 0$. The extension induced from $0 \to V^{(1)} \to S^2V \to \Lambda^2 V \to 0$ via the map $V^{(1)} \to \chi_1^{-2}$ is split by the map $S^2V \to S^2(\chi_1^{-1})$, hence the class $\alpha(V)|_{B_2} \in H^1_{\mathrm{alg}}(B_2, V^{(1)})$ is in the image of the map $H^1(B_2, \chi_1^2) \to H^1(B_2, V^{(1)})$.

In particular, the restriction $\alpha(V)|_{B_2(\mathbb{F}_q)}$ is in the image of the map $H^1(B_2(\mathbb{F}_q), \chi_1^2) \to H^1(B_2(\mathbb{F}_q), V^{(1)})$. Hence it is enough to show the following two facts:

(1) $H^1(B_2(\mathbb{F}_q), \chi_1^2) \to H^1(B_2(\mathcal{O}_F), \chi_1^2)$ is injective
(2) $H^1(B_2(\mathcal{O}_F), \chi_1^2) \to H^1(B_2(\mathcal{O}_F), V^{(1)})$ is injective

For (1), $H^1(B_2(\mathbb{F}_q), \chi_1^2) = H^1(A_2(\mathbb{F}_q), \chi_1^2)^{T_2(\mathbb{F}_q)}$ obviously injects into $H^1(A_2(\mathbb{F}_q), \chi_1^2)$, and the map $H^1(A_2(\mathbb{F}_q), \chi_1^2) \to H^1(A_2(\mathcal{O}_F), \chi_1^2)$ is an isomorphism because A_2 acts trivially on χ_1^2 here, and the natural map $A_2(\mathcal{O}_F) \to A_2(\mathbb{F}_q)$ induces an isomorphism $A_2(\mathcal{O}_F)/2 \simeq A_2(\mathbb{F}_q)$. This implies (1).

For (2), it is enough to show that $H^0(B_2(\mathcal{O}_F), \chi_1^{-2}) = 0$. The group $T_2(\mathcal{O}_F) = \mathcal{O}_F^\times$ maps surjectively onto $\mathbb{F}_q^\times = \mathbb{F}_4^\times$, hence χ_1^{-2} is a non-trivial character of $T_2(\mathcal{O}_F) \subset B_2(\mathcal{O}_F)$, and the space of invariants $H^0(B_2(\mathcal{O}_F), \chi_1^{-2}) = (\chi_1^{-2})^{B_2(\mathcal{O}_F)}$ vanishes. Therefore the image of $\alpha(V)$ in $H^1(SL_2(\mathcal{O}_F), V^{(1)})$ is indeed non-zero.

Next, we will check that the Bockstein map $\mathrm{Bock}^1 : H^1(SL_2(\mathcal{O}_F), V^{(1)}) \to H^2(SL_2(\mathcal{O}_F), V^{(1)})$ induced by the $W_2(k)$-module $\widetilde{V}^{(1)}$ is injective. The key input for this is that $H^1(SL_2(\mathcal{O}_F), \widetilde{V}^{(1)})$ is annihilated by multiplication by 2. Indeed, the central element $\mathrm{diag}(-1,-1) \in SL_2(\mathcal{O}_F)$ acts in the representation $\widetilde{V}^{(1)}$ via multiplication by (-1) so multiplication by (-1) on $H^i(SL_2(\mathcal{O}_F), \widetilde{V}^{(1)})$, for all i, is equal to identity, hence these cohomology groups are 2-torsion.

Injectivity of Bock^1 now follows by considering the long exact sequence

$$\ldots \to H^0(V^{(1)}) \xrightarrow{\mathrm{Bock}^0} H^1(V^{(1)}) \xrightarrow{\psi_1} H^1(\widetilde{V}^{(1)}) \xrightarrow{\psi_2} H^1(V^{(1)}) \xrightarrow{\mathrm{Bock}^1} H^2(V^{(1)}) \tag{14.3}$$

where H^i everywhere refers to the cohomology of $SL_2(\mathcal{O}_F)$. The first visible term $H^0(V^{(1)}) = (V^{(1)})^{SL_2(\mathcal{O}_F)}$ is zero, because $SL_2(\mathcal{O}_F) \to SL_2(\mathbb{F}_q)$ is a surjection, hence ψ_1 is injective. The composition $\psi_1 \circ \psi_2$ is the multiplication by 2 map on $H^1(\widetilde{V}^{(1)})$ which we know to be zero, so ψ_2 has to be zero, which is equivalent to injectivity of Bock^1. This finishes the proof of Proposition 14.1 for $p = 2$. ☐

Proof of Proposition 14.1 for $p > 2$. We will show the following vanishing results, and Proposition 14.1 will be deduced as a formal consequence of these.

Lemma 14.3. *Assume that $q = p^2$. There exists a real quadratic extension $F/\mathbb{Q}$ with $\mathcal{O}_F/p \simeq \mathbb{F}_q = \mathbb{F}_{p^2}$, such that*

(1) The map $H^{p-1}((T_p \ltimes A_p)(\mathbb{F}_q), \chi_1^p) \to H^{p-1}((T_p \ltimes A_p)(\mathcal{O}_F), \chi_1^p)$ is injective.
(2) The group $H^{p-1}((T_p \ltimes A_p)(\mathcal{O}_F), \chi_2^p \oplus \ldots \oplus \chi_p^p)$ vanishes.
(3) The group $H^{p-2}((T_p \ltimes A_p)(\mathcal{O}_F), \chi_1^p)$ vanishes.
(4) the $W_2(k)$-module $H^{p-1}((T_p \ltimes A_p)(\mathcal{O}_F), \widetilde{\chi}_1^{(1)})$ is annihilated by p.

Proof. We choose F satisfying the properties listed in Lemma 14.2. The reduction map $A_p(\mathcal{O}_F) \to A_p(\mathbb{F}_q)$ induces an isomorphism $A_p(\mathcal{O}_F)/p \simeq A_p(\mathbb{F}_q)$. Therefore the induced map $H^1(A_p(\mathbb{F}_q), k) \to H^1(A_p(\mathcal{O}_F), k)$ on cohomology in degree 1 is an isomorphism, and induces a $T_p(\mathcal{O}_F)$-equivariant surjection $H^i(A_p(\mathbb{F}_q), k) \to H^i(A_p(\mathcal{O}_F), k) \simeq \Lambda^i(\mathfrak{a}_k \cdot x_e \oplus \mathfrak{a}_k^{(1)} \cdot x_\tau)$ for $i \geq 1$, where we use the notation of Proposition 13.4, and e, τ are the elements of the Galois group $\mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbb{Z}/2$. The action of $T_p(\mathcal{O}_F)$ on $H^1(A_p(\mathcal{O}_F), k) \simeq \mathfrak{a}_k \cdot x_e \oplus \mathfrak{a}_k^{(1)} \cdot x_\tau$ factors through $T_p(\mathcal{O}_F) \to T_p(\mathbb{F}_q)$, and this module explicitly is given as the direct sum of inverses of the characters $\chi_1 - \chi_2, \ldots, \chi_1 - \chi_p, p(\chi_1 - \chi_2), \ldots, p(\chi_1 - \chi_p)$.

The reduction map $(\mathcal{O}_F^\times)^{p-1} \simeq T_p(\mathcal{O}_F) \to T_p(\mathbb{F}_q) \simeq (\mathbb{F}_{p^2}^\times)^{p-1}$ is not surjective (as soon as $p > 3$), but its image is equal to $(\{x | N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x) = \pm 1\})^{p-1}$, by our choice of the field F. We need the following partial refinement of Lemma 13.9 to nevertheless be able to bound the invariant subspaces $H^j(A_p(\mathcal{O}_F), \chi_i^p)^{T_p(\mathcal{O}_F)}$.

Lemma 14.4. *Denote by S the set $\{\chi_1 - \chi_2, \ldots, \chi_1 - \chi_p, p(\chi_1 - \chi_2), \ldots, p(\chi_1 - \chi_p)\} \subset X^*(T_p)$.*

(1) For $2 \le i \le p$ the character $p\chi_i$ is not congruent modulo $p+1$ to a sum of $\le p-1$ elements of S.
(2) The character $p\chi_1$ is not congruent modulo $p+1$ to a sum of $\le p-2$ elements of S.
(3) The only, up to permutation, congruence modulo $p+1$ between $p\chi_1$ and a sum of $\le p-1$ elements of S is the equality $p\chi_1 = (\chi_1 - \chi_2) + \ldots + (\chi_1 - \chi_p)$.

Proof of Lemma 14.4. We use $\chi_1, \ldots, \chi_{p-1}$ as a basis for $X^*(T_p)$, the character χ_p is expressed as $-(\chi_1 + \ldots + \chi_{p-1})$. Denote by $\sigma : X^*(T_p) \to \mathbb{Z}$ the map sending $a_1\chi_1 + \ldots + a_{p-1}\chi_{p-1}$ to $a_1 + \ldots + a_{p-1}$. We have $\sigma(\chi_1 - \chi_i) = 0$ for $i \le p-1$ and $\sigma(\chi_1 - \chi_p) = p$. In the rest of the proof, symbol $\equiv$ alway refers to congruence modulo $p+1$.

1) The only elements of S with a non-zero value of σ are $\chi_1 - \chi_p$ and $p(\chi_1 - \chi_p)$, with the values $p \equiv -1$ and $p^2 \equiv 1$, respectively. Suppose that we have a congruence $p\chi_i \equiv r_1 + \ldots + r_l \bmod p+1$ with $l \le p-1$, and all $r_1, \ldots, r_l$ from S.

Suppose first that $i \neq p$. If $\chi_1 - \chi_p$ appears in this sum a times, and $p(\chi_1 - \chi_p) \equiv \chi_p - \chi_1$ appears b times, then $a - b \equiv 1 \bmod p+1$ because $\sigma(p\chi_i) \equiv -1$. This forces b to be equal to $a-1$. Therefore the difference $p\chi_i - (a(\chi_1 - \chi_p) + (a-1)(\chi_p - \chi_1)) \equiv -\chi_i - \chi_1 + \chi_p = -2\chi_1 - \chi_2 - \ldots - 2\chi_i - \ldots - \chi_{p-1}$ is congruent to a sum of $\le p-2$ elements of the form $\pm(\chi_1 - \chi_j)$ for $j = 2, \ldots, p-1$. But such a congruence would have to use, for each $2 \le j \le p-1, j \neq i$, an element of the form $\pm(\chi_1 - \chi_j)$ at least once, and an element of the form $\pm(\chi_1 - \chi_i)$ at least twice (because $p+1 \ge 4$), so at least $p-1$ elements of S would be needed.

Next, let us rule out the possibility of a congruence $p\chi_p \equiv r_1 + \ldots + r_l$. We have $\sigma(p\chi_p) = -p(p-1) \equiv -2$. Hence if $\chi_1 - \chi_p$ appears a times in this congruence, then $p(\chi_1 - \chi_p)$ appears $a-2$ times, so $p\chi_p - 2(\chi_1 - \chi_p) \equiv -3\chi_1 - \chi_2 - \ldots - \chi_{p-1}$ is congruent to a sum of $\le p-3$ elements of the form $\pm(\chi_1 - \chi_j), j \le p-1$. But similarly to the previous case, such a sum would have to use at least $p-2$ such elements, and the original congruence cannot exist.

2) We have $\sigma(p\chi_1) = p \equiv -1$, hence a congruence $p\chi_1 \equiv r_1 + \ldots + r_l$ would have to use a instances of $\chi_1 - \chi_p$ and $a-1$ instances of $p(\chi_p - \chi_1)$, for some a. But this leaves us with $p\chi_1 - (\chi_1 - \chi_p) = (p-2)\chi_1 - \chi_2 - \ldots - \chi_{p-1}$ being congruent to a sum of $\le p-3$ elements of the form $\pm(\chi_1 - \chi_j), j \le p-1$, which is impossible.

3) As in part (2), such a congruence would induce a congruence between $(p-2)\chi_1 - \chi_2 - \ldots - \chi_{p-1}$ and a sum of $\le p-2$ elements of the form $\pm(\chi_1 - \chi_j), j \le p-1$. For each $j = 2, \ldots, p-1$, we have to use an element of the form $\pm(\chi_1 - \chi_j)$ at least once, hence exactly once, and this element must be $\chi_1 - \chi_j$, forcing $a = 1$ and implying the desired uniqueness. $\square$

We can now proceed with the proof of Lemma 14.3. In (1), we will prove that even the composition of this map with further restriction $H^{p-1}((T_p\ltimes A_p)(\mathcal{O}_F),\chi_1^p)\to H^{p-1}(A_p(\mathcal{O}_F),\chi_1^p)$ is injective. By Lemma 13.9(4), in the notation of Proposition 13.4, the invariant subspace $H^{p-1}(A_p(\mathbb{F}_q),\chi_1^p)^{T(\mathbb{F}_q)}\subset H^{p-1}(A_p(\mathbb{F}_q),\chi_1^p)$ is 1-dimensional and is equal to $\Lambda^{p-1}(\mathfrak{a}_k\cdot x_0)$. This shows the injectivity asserted in part (1), because $\mathfrak{a}_k\cdot x_0\subset H^1(A_p(\mathbb{F}_q),k)$ maps isomorphically onto $\mathfrak{a}_k\cdot x_e\subset H^1(A_p(\mathcal{O}_F),k)$.

By Lemma 14.5 below, to prove part (2) it is enough to show that every character of $T_p(\mathcal{O}_F)$ appearing as a subquotient of $H^i(A_p(\mathcal{O}_F),\chi_j^p)$ for $j\geq 2, i\leq p-1$ is non-trivial. The module $H^i(A_p(\mathcal{O}_F),k)$ is isomorphic to a direct sum of characters that are products of i elements of

$$\{-(\chi_1-\chi_2),\dots,-(\chi_1-\chi_p),-p(\chi_1-\chi_2),\dots,-p(\chi_1-\chi_p)\}\subset X^*(T_p)$$

Since the kernel of the restriction $X^*(T_p)\to\operatorname{Hom}(T_p(\mathcal{O}_F),k^\times)$ is contained in $(p+1)\cdot X^*(T_p)$ by our choice of the field F, the assertion follows from Lemma 14.4(1). Analogously, part (3) follows from Lemma 14.4(2).

We now turn to proving part (4). By part (3), we have that $H^i(A_p(\mathcal{O}_F),\chi_1^p)$ decomposes as a direct sum of non-trivial characters of $T_p(\mathcal{O}_F)$ for $i<p-1$, therefore $\mathrm{R}\Gamma(T_p(\mathcal{O}_F),H^i(A_p(\mathcal{O}_F),\chi_1^p))$ and $\mathrm{R}\Gamma(T_p(\mathcal{O}_F),H^i(A_p(\mathcal{O}_F),\widetilde{\chi}_1^{(1)}))$ are quasi-isomorphic to 0. Hence $H^{p-1}((T_p\ltimes A_p)(\mathcal{O}_F),\widetilde{\chi}_1^{(1)})$ injects into $H^{p-1}(A_p(\mathcal{O}_F),\widetilde{\chi}_1^{(1)})^{T_p(\mathcal{O}_F)}$, and it is enough to prove that the latter $W_2(k)$-module is annihilated by p.

We have a $T_p(\mathcal{O}_F)$-equivariant identification $H^{p-1}(A_p(\mathcal{O}_F),\widetilde{\chi}_1^{(1)})\simeq\widetilde{\chi}_1^{(1)}\otimes\Lambda^{p-1}(\mathfrak{a}_{W_2(k)}\oplus\mathfrak{a}_{W_2(k)}^{(1)})$, and this module decomposes as a direct sum of characters of the form $\widetilde{\chi}_1^{(1)}\otimes\eta$ where η is a product of $p-1$ characters of the form $-(\widetilde{\chi}_1-\widetilde{\chi}_j)$ or $-(\widetilde{\chi}_1^{(1)}-\widetilde{\chi}_j^{(1)})$ for $j=2,\dots,p$. By Lemma 14.4, even the mod p reduction of $\widetilde{\chi}_1^{(1)}\otimes\eta$ is a non-trivial character of $T_p(\mathcal{O}_F)$ unless $\eta=-(\widetilde{\chi}_1-\widetilde{\chi}_2)-\ldots-(\widetilde{\chi}_1-\widetilde{\chi}_{p-1})-(\widetilde{\chi}_1-\widetilde{\chi}_p)=-p\widetilde{\chi}_1$. Therefore $H^{p-1}(A_p(\mathcal{O}_F),\widetilde{\chi}_1^{(1)})^{T_p(\mathcal{O}_F)}=(\widetilde{\chi}_1^{(1)}\otimes\widetilde{\chi}_1^{-p})^{T_p(\mathcal{O}_F)}$.

Let now $u\in\mathcal{O}_F^\times$ be a unit such that its reduction $\overline{u}$ in $W_2(\mathbb{F}_{p^2})^\times$ satisfies $\mathrm{Fr}_p(\overline{u})\neq\overline{u}^p$, as provided by Lemma 14.2(2). Then the element $\operatorname{diag}(u,u^{-1},1,\dots,1)\in T_p(\mathcal{O}_F)$ acts in the character $\widetilde{\chi}_1^{(1)}\otimes\widetilde{\chi}_1^{-p}$ via multiplication by $\mathrm{Fr}_p(\overline{u})\overline{u}^{-p}$, therefore the $W_2(k)$-module of invariants $(\widetilde{\chi}_1^{(1)}\otimes\widetilde{\chi}_1^{-p})^{T_p(\mathcal{O}_F)}$ is isomorphic to k, which proves part (4). □

Lemma 14.5. *Suppose that F is a number field such that the group of units of $\mathcal{O}_F$ is infinite. For any split torus T over $\mathcal{O}_F$, if $\chi:T(\mathcal{O}_F)\to k^\times$ is a non-trivial character then* $\mathrm{R}\Gamma(T(\mathcal{O}_F),\chi)=0$.

Proof. Choose an isomorphism $T(\mathcal{O}_F)\simeq\mathbb{Z}^{\oplus N}\times T(\mathcal{O}_F)^{\mathrm{tors}}$ such that the restriction of χ to $\mathbb{Z}^{\oplus N}$ is still nontrivial. We have $\mathrm{R}\Gamma(T(\mathcal{O}_F),\chi)=\mathrm{R}\Gamma(T(\mathcal{O}_F)^{\mathrm{tors}},\mathrm{R}\Gamma(\mathbb{Z}^{\oplus N},\chi))$ so it is enough to show that $\mathrm{R}\Gamma(\mathbb{Z}^{\oplus N},\chi)=0$. By definition, $\mathrm{R}\Gamma(\mathbb{Z}^{\oplus N},\chi)=\mathrm{RHom}_{k[x_1^{\pm1},\dots,x_N^{\pm1}]}(k,\chi)$ where we denote by χ the module over the group algebra $k[x_1^{\pm1},\dots,x_N^{\pm1}]$ corresponding to the character $\chi|_{\mathbb{Z}^{\oplus N}}$, and k is the module on which all x_i act by 1. By the assumption that $\chi|_{\mathbb{Z}^{\oplus N}}$ is non-trivial, k and χ have disjoint supports in $\operatorname{Spec}k[x_1^{\pm1},\dots,x_N^{\pm1}]$ which implies the vanishing. □

Having proven Lemma 14.3, we will now deduce Proposition 14.1. Consider the long exact sequence induced by $0\to\chi_1^p\xrightarrow{\psi_1}\widetilde{\chi}_1^{(1)}\xrightarrow{\psi_2}\chi_1^p\to 0$:

(14.4)

$$\ldots \to H^{p-2}(\chi_1^p) \xrightarrow{\mathrm{Bock}^{p-2}} H^{p-1}(\chi_1^p) \xrightarrow{\psi_1} H^{p-1}(\widetilde{\chi}_1^{(1)}) \xrightarrow{\psi_2} H^{p-1}(\chi_1^p) \xrightarrow{\mathrm{Bock}^{p-1}} H^p(\chi_1^p) \to \ldots$$

where $H^i(M)$ is the abbreviation for $H^i((T_p \ltimes A_p)(\mathcal{O}_F), M)$. The multiplication by p map on $H^{p-1}(\widetilde{\chi}_1^{(1)})$ factors as the composition $H^{p-1}(\widetilde{\chi}_1^{(1)}) \xrightarrow{\psi_2} H^{p-1}(\chi_1^p) \xrightarrow{\psi_1} H^{p-1}(\widetilde{\chi}_1^{(1)})$. By Lemma 14.3(3) the map ψ_1 is injective, but Lemma 14.3(4) says that the composition $\psi_1 \circ \psi_2$ is zero, hence ψ_2 is zero itself and the Bockstein homomorphism $H^{p-1}((T_p \ltimes A_p)(\mathcal{O}_F), \chi_1^p) \to H^p((T_p \ltimes A_p)(\mathcal{O}_F), \chi_1^p)$ induced by the character $\widetilde{\chi}_1^{(1)}$ is injective.

Consider now the long exact sequence of cohomology associated with the sequence $0 \to \chi_1^p \to V^{(1)} \to \chi_2^p \oplus \ldots \oplus \chi_p^p \to 0$. By Lemma 14.3(2) we get that the map $H^p((T_p \ltimes A_p)(\mathcal{O}_F), \chi_1^p) \to H^p((T_p \ltimes A_p)(\mathcal{O}_F), V^{(1)})$ is injective. Combined with Lemma 14.3(1) this implies that the composition

$$H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)} \to H^{p-1}((T_p \ltimes A_p)(\mathcal{O}_F), V^{(1)}) \xrightarrow{\mathrm{Bock}^{p-1}_{\widetilde{V}^{(1)}}} H^p((T_p \ltimes A_p)(\mathcal{O}_F), V^{(1)})$$

is injective when restricted to the image of the map $H^{p-1}(A_p(\mathbb{F}_q), \chi_1^p)^{T_p(\mathbb{F}_q)} \to H^{p-1}(A_p(\mathbb{F}_q), V^{(1)})^{T_p(\mathbb{F}_q)}$. But that map is an isomorphism by Lemma 13.11 (2) so Proposition 14.1 is proven. $\square$

References

[ABM21] Benjamin Antieau, Bhargav Bhatt, and Akhil Mathew. Counterexamples to Hochschild-Kostant-Rosenberg in characteristic p. *Forum Math. Sigma*, 9:Paper No. e49, 26, 2021.

[AS20] Piotr Achinger and Junecue Suh. Some refinements of the Deligne-Illusie theorem. `arxiv:2003.09857v2`, to appear in Algebra & Number Theory, 2020.

[Beu19] Kevin Beuchot. Liftings to $\mathbb{Z}/p^2$ and splitting of the de Rham complex. SPUR Final Paper, `https://math.mit.edu/research/undergraduate/spur/documents/2019Beuchot.pdf`, 2019.

[BGMN21] Clark Barwick, Saul Glasman, Akhil Mathew, and Thomas Nikolaus. K-theory and polynomial functors. `arxiv:2102.00936v2`, 2021.

[Bha12] Bhargav Bhatt. p-adic derived de Rham cohomology. `arxiv:1204.6560v1`, 2012.

[BL22a] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. `arxiv:2201.06120v1`, 2022.

[BL22b] Bhargav Bhatt and Jacob Lurie. The prismatization of p-adic formal schemes. `arxiv:2201.06124v1`, 2022.

[BLM21] Bhargav Bhatt, Jacob Lurie, and Akhil Mathew. *Revisiting the de Rham-Witt complex*, volume 424 of *Astérisque*. Paris: Société Mathématique de France (SMF), 2021.

[BM19] Lukas Brantner and Akhil Mathew. Deformation theory and partition lie algebras. `arxiv:1904.07352v1`, 2019.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 128:219–397, 2018.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[BS22] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[CCO14] Ching-Li Chai, Brian Conrad, and Frans Oort. *Complex multiplication and lifting problems*, volume 195 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.

[CPS83] Edward T. Cline, Brian J. Parshall, and Leonard L. Scott. Detecting rational cohomology of algebraic groups. *J. London Math. Soc. (2)*, 28(2):293–300, 1983.

[CPSvdK77] E. Cline, B. Parshall, L. Scott, and Wilberd van der Kallen. Rational and generic cohomology. *Invent. Math.*, 39(2):143–163, 1977.

[DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[Dri21] Vladimir Drinfeld. Prismatization. `arxiv:2005.04746v5`, 2021.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergeb. Math. Grenzgeb., 3. Folge.* Berlin etc.: Springer-Verlag, 1990.

[FLS94] Vincent Franjou, Jean Lannes, and Lionel Schwartz. Autour de la cohomologie de Mac Lane des corps finis. *Invent. Math.*, 115(3):513–538, 1994.

[FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.

[Gab01] O. Gabber. On space filling curves and Albanese varieties. *Geom. Funct. Anal.*, 11(6):1192–1200, 2001.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[HM82] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67:23–86, 1982.

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[Ill20] Luc Illusie. Partial degeneration of Hodge to de Rham spectral sequences and Kodaira type vanishing theorems for locally complete intersections in positive characteristic. `https://www.imo.universite-paris-saclay.fr/~luc.illusie/DI-revisited2a.pdf`, 2020.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Kal18] D. Kaledin. Co-periodic cyclic homology. *Adv. Math.*, 334:81–150, 2018.

[Kem76] George R. Kempf. Linear systems on homogeneous spaces. *Ann. of Math. (2)*, 103(3):557–591, 1976.

[KP21] Dmitry Kubrak and Artem Prikhodko. p-adic Hodge theory for Artin stacks. `arxiv:2105.05319v1`, to appear in Memoirs of the AMS, 2021.

[KP22] Dmitry Kubrak and Artem Prikhodko. Hodge-to–de Rham degeneration for stacks. *Int. Math. Res. Not. IMRN*, (17):12852–12939, 2022.

[LM21] Shizhang Li and Shubhodip Mondal. On endomorphisms of the de Rham cohomology functor. `arxiv:2109.04303v2`, to appear in Geometry and Topology, 2021.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. `https://www.math.ias.edu/~lurie/papers/HA.pdf`, 2017.

[Lur22] Jacob Lurie. Kerodon. `https://kerodon.net/`, 2022.

[May70] J. Peter May. A general algebraic approach to Steenrod operations. In *The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)*, Lecture Notes in Mathematics, Vol. 168, pages 153–231. Springer, Berlin, 1970.

[MR23] Shubhodip Mondal and Emanuel Reinecke. Unipotent homotopy theory of schemes. `arxiv:2302.10703v1`, 2023.

[Oor74] Frans Oort. Subvarieties of moduli spaces. *Invent. Math.*, 24:95–119, 1974.

[Pri73] Stewart Priddy. Mod $-\,p$ right derived functor algebras of the symmetric algebra functor. *J. Pure Appl. Algebra*, 3:337–356, 1973.

[PVV18] Alexander Petrov, Dmitry Vaintrob, and Vadim Vologodsky. The Gauss-Manin connection on the periodic cyclic homology. *Selecta Math. (N.S.)*, 24(1):531–561, 2018.

[Rak20] Arpon Raksit. Hochschild homology and the derived de Rham complex revisited. `arxiv:2007.02576v2`, 2020.

[Ray83] M. Raynaud. Around the Mordell conjecture for function fields and a conjecture of Serge Lang. Algebraic geometry, Proc. Jap.-Fr. Conf., Tokyo and Kyoto 1982, Lect. Notes Math. 1016, 1-19 (1983)., 1983.

[Sca21] Federico Scavia. Steenrod operations on the de Rham cohomology of algebraic stacks. *Journal of the Institute of Mathematics of Jussieu*, pages 1–48, May 2021.

[Ser58] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[Sri90] V. Srinivas. Decomposition of the de Rham complex. *Proc. Indian Acad. Sci., Math. Sci.*, 100(2):103–106, 1990.

[Sta23] The Stacks Project Authors. The Stacks Project. `https://stacks.math.columbia.edu`, 2023.

[Ste53] N. E. Steenrod. Homology groups of symmetric groups and reduced power operations. *Proc. Nat. Acad. Sci. U.S.A.*, 39:213–217, 1953.

[Ste62] N. E. Steenrod. *Cohomology operations*. Annals of Mathematics Studies, No. 50. Princeton University Press, Princeton, N.J., 1962. Lectures by N. E. Steenrod written and revised by D. B. A. Epstein.

[Ste12] David I. Stewart. The second cohomology of simple SL_3-modules. *Comm. Algebra*, 40(12):4702–4716, 2012.

[Szp81] Lucien Szpiro. Propriétés numériques du faisceau dualisant relatif. Number 86, pages 44–78. 1981. Seminar on Pencils of Curves of Genus at Least Two.

Max Planck Institute for Mathematics, Bonn, Germany
Email address: `alexander.petrov.57@gmail.com`