DE RHAM COHOMOLOGY AND $\mathcal{D}$-MODULES IN MIXED AND POSITIVE CHARACTERISTIC

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1. Lecture 1, 02/02

We begin by defining algebraic de Rham cohomology. Let X be a smooth scheme over a commutative ring R. By definition, this means that the morphism $X \to \operatorname{Spec} R$ is locally of finite presentation, and the sheaf of Kähler differentials $\Omega^1_{X/R}$ is locally free such that its rank equals the relative dimension of X over R, on every connected component of X.

Recall that for an affine open $U \subset X$ the section $\Omega^1_{X/R}(U)$ of the sheaf of differentials is the $\mathcal{O}_X(U)$-module given by the following generators and relations:

$$\Omega^1_{X/R}(U) = \mathcal{O}_X(U)\langle df | f \in \mathcal{O}_X(U)\rangle/(d(f_1+f_2) = df_1+df_2, d(f\cdot g) = f\cdot dg + g\cdot df) \tag{1.1}$$

By definition, it comes with an R-linear derivation $d : \mathcal{O}_X \to \Omega^1_{X/R}$. We denote by $\Omega^i_{X/R}$ the i-th exterior power $\Lambda^i\Omega^1_{X/R}$. De Rham differential to the unique collection of maps $d : \Omega^i_{X/R} \to \Omega^{i+1}/R$ such that $d(\omega_i \wedge \omega_j) = d(\omega_i) \wedge \omega_j + (-1)^i\omega_i \wedge d(\omega_j)$ and $d^2 = 0$. We then can form the following complex of sheaves of R-modules in the Zariski topology on X, placing $\Omega^i_{X/R}$ in cohomological degree i:

$$\Omega^\bullet_{X/R} := \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/R} \xrightarrow{d} \Omega^2_{X/R} \xrightarrow{d} \dots \tag{1.2}$$

which we refer to as *algebraic de Rham complex* of X relative to R. We then define de Rham cohomology modules of X to be the values of the derived functor of global sections on this complex:

$$H^n_{\mathrm{dR}}(X/R) := \mathbb{H}^n_{\mathrm{Zar}}(X, \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/R} \xrightarrow{d} \dots) \tag{1.3}$$

By definition, to compute this cohomology groups we replace $\Omega^\bullet_{X/R}$ by a quasi-isomorphic complex of sheaves of R-modules $\mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ such that each of the terms $\mathcal{F}^i$ is acyclic for the global sections functor. De Rham cohomology $H^n_{\mathrm{dR}}(X/R)$ in degree n is then the n-th cohomology of the complex of R-modules $\mathcal{F}^0(X) \to \mathcal{F}^1(X) \to \dots$.

Example 1.1. If X is affine, each sheaf $\Omega^i_{X/R}$ is already acyclic, so algebraic de Rham cohomology of a smooth affine scheme is computed by the complex

$$\mathcal{O}_X(X) \xrightarrow{d} \Omega^1_{X/R}(X) \xrightarrow{d} \dots \tag{1.4}$$

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Exercise 1. Let F be a field of characteristic zero. Consider the affine curve $X = \operatorname{Spec} F[x,y]/(y^2-(x^3+ax+b)) \subset \mathbb{A}^2_{x,y}$ for some $a,b \in F$. If a,b are chosen so that X is smooth, calculate $H^1_{\mathrm{dR}}(X/F)$ by finding explicit 1-forms representing a basis in cohomology.

For a general smooth scheme X, one choice of a term-wise acyclic replacement for the de Rham complex is provided by the Čech complex. Let $X = \bigcup\limits_i U_i$ be a cover of X by affine open subschemes, such that all intersections of any finite subset of these affines is affine as well[1]. Then de Rham cohomology of X can be computed as by the total complex of the following bicomplex

$$
\begin{array}{ccccc}
\dots & & \dots & & \\
\uparrow & & \uparrow & & \\
\prod\limits_{i,j\in I} \mathcal{O}_X(U_i\cap U_j) & \xrightarrow{d} & \prod\limits_{i,j\in I} \Omega^1_{X/R}(U_i\cap U_j) & \xrightarrow{d} & \dots \\
\uparrow & & \uparrow & & \\
\prod\limits_{i\in I} \mathcal{O}_X(U_i) & \xrightarrow{d} & \prod\limits_{i\in I} \Omega^1_{X/R}(U_i) & \xrightarrow{d} & \dots
\end{array}
\tag{1.5}
$$

whose columns are Čech resolutions computing cohomology of sheaves $\Omega^i_{X/R}$ on X, and all horizontal arrows are given by de Rham differentials.

Exercise 2. Let F be a field of any characteristic. Calculated algebraic de Rham cohomology of $\mathbb{P}^1_F$ relative to F.

Hint: Use bicomplex (1.5). A convenient device for organizing this computation is the 'spectral sequence for a double complex' that expresses cohomology of the total complex in terms of the cohomology of the columns. If you haven't worked with such spectral sequence before, doing this computation from scratch for a cover $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$ is a nice way to rediscover this spectral sequence.

The definition of algebraic de Rham complex is of course motivated by the de RHam complex of a smooth manifold, though in differential geometry terms of the smooth de Rham complex are already acyclic. Remarkably, algebraic de Rham cohomology over $\mathbb{C}$ recovers de Rham cohomology of the underlying smooth manifold, and consequently, by de Rham theorem, identified with singular cohomology:

Theorem 1.2 ([Gro66, Theorem 1])**.** *Let X be a smooth separated scheme over $\mathbb{C}$. There is a natural isomorphism*

$$H^n_{\mathrm{dR}}(X/\mathbb{C}) \simeq H^n(X(\mathbb{C}),\mathbb{C}) \tag{1.6}$$

between algebraic de Rham cohomology and singular cohomology of the underlying topological space of $\mathbb{C}$-points of X.

Example 1.3. Let $X = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = \operatorname{Spec} \mathbb{C}[t,t^{-1}]$. This smooth scheme is affine, so its algebraic de Rham cohomology is computed by the following two-term complex:

$$\mathbb{C}[t,t^{-1}] \xrightarrow{d} \mathbb{C}[t,t^{-1}]dt \tag{1.7}$$

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[1] the latter condition is automatically satisfied if X is quasi-separated, that is the diagonal morphism $X \to X \times X$ is affine

with d given by $d(t^n) = nt^{n-1}dt$ for all $n \in \mathbb{Z}$. Then $H^0_{\mathrm{dR}}(X/\mathbb{C}) = \ker d = \mathbb{C} \cdot t^0$ and $H^1_{\mathrm{dR}}(X/\mathbb{C}) = \operatorname{coker} d = \mathbb{C} \cdot [\frac{dt}{t}]$, in accordance with the fact that $X(\mathbb{C})$ is homotopy equivalent to a circle, hence its singular cohomology is 1-dimensional in degrees $0, 1$ and is zero in all other degrees.

Formally, it is notable that algebraic differential forms are sufficient to calculate the cohomology of the C^∞-de Rham complex. To appreciate this, suppose that X is a smooth equidimensional affine scheme over $\mathbb{C}$ and consider the maps between algebraic, holomorphic, and the C^∞-de Rham complexes:

(1.8)

$$\begin{array}{ccccccccccc}
\mathcal{O}_X(X) & \xrightarrow{d} & \Omega^1_{X/\mathbb{C}}(X) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{\dim X}_{X/\mathbb{C}}(X) & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & & & \\
\mathcal{O}^{\mathrm{an}}_X(X(\mathbb{C})) & \xrightarrow{d} & \Omega^{1,\mathrm{an}}_X(X(\mathbb{C})) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{\dim X,\mathrm{an}}_X(X(\mathbb{C})) & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & & & \\
\mathcal{O}^{\mathrm{sm}}(X(\mathbb{C}),\mathbb{C}) & \xrightarrow{d} & \Omega^{1,\mathrm{sm}}(X(\mathbb{C})) & \xrightarrow{d} & \dots & & \Omega^{\dim X,\mathrm{sm}}(X(\mathbb{C}),\mathbb{C}) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{2\dim X,\mathrm{sm}}(X(\mathbb{C}),\mathbb{C})
\end{array}$$

Here the middle row is the holomorphic de Rham complex of the Stein complex manifold $X(\mathbb{C})$, and the bottom row is the de Rham complex (tensored with $\mathbb{C}$ over $\mathbb{R}$) of the $2\dim X$-dimensional smooth manifold $X(\mathbb{C})$ – the complex structure on $X(\mathbb{C})$ plays no role in its definition.

Remark 1.4. Let X be a smooth equidimensional affine scheme over X. Its algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X/\mathbb{C})$ is zero for $i > \dim X$, even though, from the differential geometric point of view, it is only evident to us that $H^i(X(\mathbb{C}), \mathbb{C})$ vanishes for $i > \dim X$. This is a special case of the Artin vanishing theorem.

Remark 1.5. For a smooth affine X of positive dimension, terms of the de Rham complex are infinite-dimensional $\mathbb{C}$-vector spaces, yet its cohomology groups are always finite-dimensional, because $X(\mathbb{C})$ admits a structure of a fintie CW complex. As Pavel remarked, this finite-dimensionality also has an algebraic proof [HTT08, Theorem 6.1.5(ii)] that proceeds by compactifying the variety and showing that cohomology of the compactification, and cohomology with support on the boundary are both finite-dimensional, by finite-dimensionality of cohomology of coherent sheaves.

Theorem 1.2 indicates that singular cohomology of a smooth algebraic variety, taken with $\mathbb{C}$-coefficients, has a purely algebraic description. In particular, it makes perfect sense to consider algebraic de Rham cohomology over other base rings. One of our goals will be to see that de Rham cohomology over $\mathbb{F}_p$ and $\mathbb{Z}_p$ possesses interesting additional structures.

We now turn to discussing de Rham cohomology in positive characteristic. Let k be a field of characteristic $p > 0$ which we assume to be perfect, that is the Frobenius endomorphism $a \mapsto a^p$ is a bijection on k. One reason this assumption is relevant when studying de Rham cohomology and differential forms is that the module of differentials $\Omega^1_{k/\mathbb{F}_p}$ of a perfect field k vanishes, hence for any scheme X over k we have $\Omega^1_{X/k} = \Omega^1_{X/\mathbb{F}_p}$, and most constructions we will perform do not seriously depend on the base field.

Example 1.6. Consider de Rham cohomology of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ over k. It is computed by the two-term complex

$$k[t] \xrightarrow{d} k[t]dt \tag{1.9}$$

with d given by the usual formula $d(t^n) = nt^{n-1}$. Using the grading on the polynomial ring $k[t]$ which gives t^n degree n, we note that $\ker d$ and $\operatorname{coker} d$ have to be given by a direct sum of their homogeneous parts, and one computes

$$H^0_{\mathrm{dR}}(\mathbb{A}^1_k) \simeq \bigoplus_{i\geq 0} k \cdot t^{pi} \qquad H^1_{\mathrm{dR}}(\mathbb{A}^1_k) \simeq \bigoplus_{i\geq 1} k \cdot [t^{pi-1}dt] \tag{1.10}$$

In particular, contrary to characteristic zero situation, $H^0_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \ker d$ and $H^1_{\mathrm{dR}}(\mathbb{A}^1_k/k)$ are infinite-dimensional k-vector spaces.

Exercise 3. Let X be a smooth and proper scheme over a field F of any characteristic. Prove that de Rham cohomology groups $H^n_{\mathrm{dR}}(X/F)$ are finite-dimensional F-vector spaces even if F has positive characteristic.

Infinite-dimensionality of cohomology of $\mathbb{A}^1$ is a shadow of a certain additional structure present on the de Rham complex in characteristic p. Recall that, despite the terms of the de Rham complex of a smooth scheme X over a base ring R being vector bundles (in particular, $\mathcal{O}_X$-modules), the algebraic de Rham complex is just a complex of sheaves of R-modules, because de Rham differential is not $\mathcal{O}_X$-linear. In characteristic p, however, de Rham complex has more structure.

For a smooth scheme X over k denote by $\mathcal{O}_X^p \subset \mathcal{O}_X$ the image of the endomorphism $f \mapsto f^p$ of the sheaf $\mathcal{O}_X$. We observe that algebraic de Rham complex

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \dots \tag{1.11}$$

is a complex of $\mathcal{O}_X^p$-modules. Indeed, for a section $f \in \mathcal{O}_X(U)$ on some open $U \subset X$ multiplication by f^p commutes with de Rham differential: for a form $\omega \in \Omega^i_{X/k}(U)$ we have $d(f^p\omega) = pf^{p-1}df \wedge \omega + f^p d\omega = f^p d\omega$ because $p = 0$ in $\Omega^{i+1}_{X/k}(U)$.

It is convenient to rephrase this observation in terms of an auxiliary scheme whose structure sheaf can be identified with $\mathcal{O}_X^p$. Recall that any $\mathbb{F}_p$-scheme Y has a natural *absolute Frobenius* endomorphism $F_{\mathrm{abs}} : Y \to Y$, uniquely characterized by the fact that it is functorial in Y and is induced by the ring map $a \mapsto a^p$ for every affine scheme $Y = \operatorname{Spec} A$.

Unless k is $\mathbb{F}_p$, absolute Frobenius endomorphism of X is not compatible with the structure map $\pi_X : X \to \operatorname{Spec} k$. Let us consider another copy of X with modified k-structure to be able to view F_{abs} as a k-linear map. Let $X^{(1)}$ be the k-scheme defined as the fiber product of the following square

$$\begin{array}{ccc} X^{(1)} & \xrightarrow{\nu_X} & X \\ \downarrow{\scriptstyle \pi_{X^{(1)}}} & & \downarrow{\scriptstyle \pi_X} \\ \operatorname{Spec} k & \xrightarrow{F_{\mathrm{abs}}} & \operatorname{Spec} k \end{array} \tag{1.12}$$

By the universal property of the fiber product, we have a map $F_{X/k} : X \to X^{(1)}$ such that $\pi_{X^{(1)}} \circ F_{X/k} = \pi_X$ and $\nu_X \circ F_{X/k} = F_{\mathrm{abs}}$, and we refer to this map as *relative Frobenius*, sometimes dropping subscript 'X/k' from the notation.

The map $F : X \to X^{(1)}$ is a bijection on the underlying topological spaces, and rephrasing the above observation about $\mathcal{O}_X^p$-linearity of the de Rham complex, we can form the complex

$$F_*\Omega_X^\bullet := F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega_X^1 \xrightarrow{d} \dots \tag{1.13}$$

which is now a complex of quasi-coherent $\mathcal{O}_{X^{(1)}}$-modules on $X^{(1)}$.

Example 1.7. For $X = \mathbb{A}_k^1$, this complex corresponds to the complex of $\mathcal{O}(X^{(1)}) = k[t^p]$-modules $k[t] \xrightarrow{d} k[t]dt$, terms being free modules of rank p. Its cohomology groups, computed in Example 1.6 are then free modules of rank 1.

Reducing to the case of an affine space, one can show:

Exercise 4. Let X be a smooth equidimensional scheme over k. Then morphism $F : X \to X^{(1)}$ is finite flat of degree $p^{\dim X}$.

In particular, each $F_*\Omega_X^i$ is a vector bundle on $X^{(1)}$.

Exercise 5. Let $X = \mathbb{P}_k^1$ be the projective line over a perfect field k of characteristic p. Describe vector bundles $F_*\mathcal{O}_X, F_*\Omega_X^1$ on $X^{(1)} \simeq \mathbb{P}_k^1$ as direct sums of line bundles, and calculate the map $d : F_*\mathcal{O}_X \to F_*\Omega_X^1$ in terms of this description.

Replacing the de Rham complex $\Omega_X^\bullet$ with $F_*\Omega_X^\bullet$ does not affect the global cohomology:

$$H^n_{\mathrm{dR}}(X/k) \simeq \mathbb{H}^n_{\mathrm{Zar}}(X^{(1)}, F_*\Omega_X^\bullet) \tag{1.14}$$

Indeed, cohomology of a complex of quasi-coherent sheaves, such as $F_*\Omega_X^\bullet$ can be computed on the level of underlying complexes of k-vector spaces and applying F_* does not affect this cohomology because $F : X \to X^{(1)}$ is a homeomorphism.

It is then natural to try to understand the complex $F_*\Omega_X^\bullet$ up to quasi-isomorphism, that is as an object of the derived category of quasi-coherent sheaves on $X^{(1)}$. We can first ask what its cohomology sheaves are, and remarkably the answer involved differential forms again:

Theorem 1.8 (Cartier isomorphism)**.** *For a smooth scheme X over k there is a natural isomorphism $C : \mathcal{H}^i(F_*\Omega_X^\bullet) \simeq \Omega^i_{X^{(1)}}$.*

Here $\mathcal{H}^i(F_*\Omega_X^\bullet)$ is the cohomology sheaf of $F_*\Omega_X^\bullet$, that is the coherent sheaf $\ker(d : F_*\Omega_X^i \to F_*\Omega_X^{i+1}) / \operatorname{Im}(F_*\Omega_X^{i-1} \to F_*\Omega_X^i)$. We could have considered cohomology sheaves of the de Rham complex of a variety in characteristic zero as well, except that it would only be a sheaf of vector spaces, rather than an $\mathcal{O}_X$-module. These sheaves are in several ways more mysterious that their characteristic p counterparts, here is one example of this discrepancy:

Exercise 6. Let X be a smooth scheme over any field F. Consider the presheaf on X sending $U \subset X$ to the F-vector space $H^i_{\mathrm{dR}}(U/F)$, denote by $\mathcal{H}^i_{\mathrm{dR}}$ the sheafification of this presheaf.

(1) If F is a perfect field of characteristic p, prove that the natural map $H^i_{\mathrm{dR}}(U/F) \to \mathcal{H}^i_{\mathrm{dR}}(U)$ is an isomorphism for every *affine* open $U \subset X$.
(2) If $F = \mathbb{C}$, given an example of a smooth X over $\mathbb{C}$ and an affine open $U \subset X$ such that the map $H^2_{\mathrm{dR}}(U/\mathbb{C}) \to \mathcal{H}^2_{\mathrm{dR}}(U)$ is not an isomorphism.

There are several ways to arrive at the Cartier isomorphism, and we will discuss a way of constructing it out of the Bockstein map coming from crystalline cohomology. Crystalline cohomology is a cohomology theory of algebraic varieties over k, lifting, in an appropriate sense de Rham cohomology.

So far we have been discussing individual de Rham cohomology groups, but it is convenient to introduce the relevant cohomology complex, as an object of the derived category. For a smooth scheme X over a ring R denote by $\mathrm{R}\Gamma_{\mathrm{dR}}(X/R) \in D(R)$ the object of the derived category of R-modules computing de Rham cohomology: it can be represented by the complex $\mathcal{F}^0(X) \to \mathcal{F}^1(X) \to \dots$ for any choice of a term-wise acyclic resolution $\mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ for the de Rham complex $\Omega^\bullet_{X/R}$.

For a perfect field k of characteristic p, crystalline cohomology is a cohomology theory with coefficients in the ring $W(k)$ of p-typical Witt vectors of k. It is the unique p-torsion-free ring equipped with an isomorphism $W(k)/p \simeq k$ which is p-complete, that is the natural map $W(k) \to \lim\limits_n W(k)/p^n$ is an isomorphism. For example, if k is $\mathbb{F}_p$, the ring $W(k)$ is simply the ring $\mathbb{Z}_p$ of p-adic integers. We denote $W(k)/p^n$ by $W_n(k)$, it is given by $\mathbb{Z}/p^n$ for $k = \mathbb{F}_p$.

Remark 1.9. One way to construct $W(k)$ is to use deformation theory. For a flat $\mathbb{Z}/p^n$-algebra R_n the obstruction to lifting it to a flat $\mathbb{Z}/p^{n+1}$-algebra R_{n+1} lies in the group $\mathrm{Ext}^2_{R_1}(L_{R_1/\mathbb{F}_p}, R_1)$ and if it vanishes, lifts form a torsor over the group $\mathrm{Ext}^1_{R_1}(L_{R_1/\mathbb{F}_p}, R_1)$. Here R_1 is R_n/p and $L_{R_1/\mathbb{F}_p}$ refers to the cotangent complex of R_1 relative to $\mathbb{F}_p$. For a perfect $\mathbb{F}_p$-algebra R cotangent complex vanishes, so applying this observation iteratively we can produce a unique $\mathbb{Z}_p$-flat p-complete algebra $W(R)$ with an isomorphism $W(R)/p \simeq R$.

2. Lecture 2, 02/04

Recall that a p-adic formal scheme over $W(k)$ is a collection $\{X_n\}_{n\geq 1}$ of schemes X_n over $W_n(k)$ together with isomorphisms $X_n \simeq X_{n+1} \times_{W_{n+1}(k)} W_n(k)$. A formal scheme is called *smooth* if each X_n is smooth over $W_n(k)$.

Exercise 7. Suppose that for a flat $W_n(k)$-scheme X_n the k-scheme $X_1 := X_n \times_{W_n(k)} k$ is smooth. Prove that X_n is smooth over $W_n(k)$.

Given a scheme Y over $W(k)$ we can create the formal scheme $\widehat{Y}$ with $\widehat{Y}_n$ given by $Y \times_{W(k)} W_n(k)$.

For each $i \geq 0$, we let $\widehat{\Omega}^i_{X/W(k)}$ be the sheaf of $W(k)$ modules on the Zariski site of X_1 given by $\lim\limits_n \widehat{\Omega}^i_{X_n/W_n(k)}$. We can form the de Rham complex $\widehat{\mathcal{O}}_X \xrightarrow{d} \widehat{\Omega}^1_X \xrightarrow{d} \dots$ and take ist hypercohomology with respect to the Zariski topology, to define de Rham cohomology of a formal scheme X:

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/W(k)) := \mathrm{R}\Gamma_{\mathrm{Zar}}(X_1, \widehat{\mathcal{O}}_X \xrightarrow{d} \widehat{\Omega}^1_X \xrightarrow{d} \dots) \in D(W(k)) \tag{2.1}$$

Given an inverse system of acyclic sheaves on a topological space, with surjective transition maps, the inverse limit is acyclic as well. In particular, for a smooth affine formal scheme X de Rham cohomology is computed by the following complex of $W(k)$-modules:

$$\widehat{\mathcal{O}}(X) \xrightarrow{d} \widehat{\Omega}^1(X) \xrightarrow{d} \dots \tag{2.2}$$

Example 2.1. De Rham cohomology of the formal affine line $X = \widehat{\mathbb{A}}^1_{W(k)}$ is computed by the two-term complex

$$W\langle t\rangle \xrightarrow{d} W\langle t\rangle dt \tag{2.3}$$

where $W\langle t\rangle = \lim\limits_n W_n[t]$ is the p-adic completion of the polynomial ring.

Exercise 8. Calculate $M := H^1_{\mathrm{dR}}(\widehat{\mathbb{A}}^1_{W(k)})$. Is this module p-adically complete? That is, is the natural map $M \to \lim M/p^n$ an isomorphism?

For a formal scheme $\widehat{Y}$ arising from a smooth algebraic scheme Y over $W(k)$ de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k))$ can be recovered as the *derived completion* of the algebraic de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$. Explicitly, if an object $M \in D(W(k))$ is represented by a complex $\ldots \to M^i \to M^{i+1} \to \ldots$ with p-torsion-free terms then its derived completion is given by completing this complex term-wise.

Example 2.2. If Y is a smooth proper scheme over $W(k)$, passing to the associated formal scheme does not change the de Rham cohomology. Indeed, $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ is a *perfect* complex: it can be represented by a finite complex of finitely generated projective $W(k)$-modules. In particular, it is quasi-isomorphic to its derived completion.

Theorem 2.3. *There exists a contravariant functor from smooth varieties over k to the derived category of $W(k)$-modules*

$$X \mapsto \mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \in D(W(k)) \tag{2.4}$$

satisfying two properties:

(1) $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes^L_{W(k)} k \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X/k) \in D(k)$

(2) For any smooth formal scheme $\widetilde{X}$ over $W(k)$ such that $\widetilde{X} \times_{W(k)} k \simeq X$, there is a natural quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/W(k))$.

Remarkably, second property implies that two formal schemes $\widetilde{X}, \widetilde{X}'$ with isomorphic mod p reductions have isomorphic de Rham cohomology. Moreover, naturality of the quasi-isomorphism in (2) implies:

Corollary 2.4. *If f is an automorphism of a smooth formal scheme $\widetilde{X}$ over $W(k)$ such that $f \times_{W(k)} k$ on $X := \widetilde{X} \times_{W(k)} k$ is the identity map, then the induced map $f^* : H^n_{\mathrm{dR}}(\widetilde{X}/W(k)) \to H^n_{\mathrm{dR}}(\widetilde{X}/W(k))$ equals to identity for all n.*

Remark 2.5. Theorem 2.3 has a precursor in characteristic zero, replacing the pair of rings $W(k) \to k$ with $\mathbb{C}[[t]] \to \mathbb{C}$. If X is a smooth t-adic formal scheme over $\mathbb{C}[[t]]$, then $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C}[[t]])$ can be recovered from the special fiber $X_{t=0}$: there is a natural quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C}[[t]]) \simeq \mathrm{R}\Gamma(X_{t=0}/\mathbb{C})\widehat{\otimes}_{\mathbb{C}}\mathbb{C}[[t]] \tag{2.5}$$

where $\widehat{\otimes}$ refers to the (derived) t-adic completion of the tensor product. Indeed, as an intermediate object we can consider the cohomology of X relative to $\mathbb{C}$:

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_{t=0}/\mathbb{C}) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C}) \to \mathrm{R}\Gamma(X/\mathbb{C}[[t]]) \tag{2.6}$$

Here the first map is induced by restriction along $X_{t=0} \hookrightarrow X$ and is a quasi-isomorphism by an algebraic version of Poincaré lemma. Composing the second map from the absolute de Rham cohomology to relative de Rham cohomology with the inverse of this quasi-isomorphism induced the desired map from RHS to LHS of (2.5).

In this argument we've used crucially that the surjection $\mathbb{C}[[t]] \to \mathbb{C}$ has a section, in particular every variety over $\mathbb{C}$ admits a trivial deformation over $\mathbb{C}[[t]]$. Another difference with the situation of Theorem 2.3 is that over a field of positive characteristic inclusion $X_{t=0} \hookrightarrow X$ no longer induced an isomorphism on cohomology, as (t-completed version of) Example 1.6 shows. Nonetheless, the core of the proof of Theorem 2.3 is an appropriate version of Poincaré lemma.

Let us give a direct proof of this corollary, it will then serve as an ingredient in our construction of crystalline cohomology

Proof of Corollary 2.4. For the duration of this proof, we assume that $p > 2$, and will later discuss how to cover the case $p = 2$ as well. Let us form the following endomorphism of the structure sheaf $\mathcal{O}_{\widetilde{X}}$ (viewed as a sheaf of $W(k)$-modules):

$$\log f := \sum_{i\geq 1}(-1)^{i+1}\frac{(f-1)^{\circ i}}{i} \tag{2.7}$$

Each term is indeed a well-defined endomorphism of $\mathcal{O}_{\widetilde{X}}$ because $f - \mathrm{Id}$ is divisible by p, and p^i is divisible by i in $\mathbb{Z}_p$, for all i. This series converges in the p-adic topology because, moreover, p-adic valuation of $\frac{p^i}{i}$ tends to infinity as i goes to infinity.

Additionally, each term of the sum is divisible by p, so $\log f$ can be written as $p \cdot D$ for some endomorphism $D : \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$. The key observation is that D is a derivation.

Exercise 9. Check that D is a $W(k)$-linear derivation of $\mathcal{O}_{\widetilde{X}}$.

Conversely, automorphism f can be recovered from D via the formula

$$f = \exp(\log f) := \sum_{i\geq 0}\frac{p^i}{i!}D^{\circ i} \tag{2.8}$$

For $p > 2$ this series clearly converges: p-adic valuation of $\frac{p^i}{i!}$ tends to infinity.

Let us view D as a vector field v_D on $\widetilde{X}$, and consider the Lie derivative $L_{v_D} : \Omega^m_{\widetilde{X}} \to \Omega^m_{\widetilde{X}}$ on differential forms. Then f^* is given by $\sum\limits_{i\geq 0}\frac{p^i}{i!}L^{\circ i}_{v_D}$, in particular $f^* - \mathrm{Id}$ on the de Rham complex factors through the map L_{v_D}. However, by Cartan homotopy formula $L_v = d\iota_v + \iota_v d$ the endomorphism L_{v_D} is homotopic to zero on the de Rham complex

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \Omega^i_{\widetilde{X}} & \xrightarrow{d} & \Omega^{i+1}_{\widetilde{X}} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle L_{v_D}} & \swarrow{\scriptstyle \iota_{v_D}} & \downarrow{\scriptstyle L_{v_D}} & & \\ \ldots & \longrightarrow & \Omega^i_{\widetilde{X}} & \xrightarrow{d} & \Omega^{i+1}_{\widetilde{X}} & \longrightarrow & \ldots \end{array} \tag{2.9}$$

Hence $f^* - \mathrm{Id}$ is homotopic to zero on the de Rham complex as well, and $f^* - \mathrm{Id}$ induces the zero map on $H^i_{\mathrm{dR}}(\widetilde{X}/W(k))$. □

Let us return to Cartier isomorphism and use crystalline cohomology to construct it. Given a smooth variety X over k our goal is to produce an isomorphism

$$C : \mathcal{H}^i(F_*\Omega^i_{X/k}) \simeq \Omega^i_{X^{(1)}/k} \tag{2.10}$$

For $i=0$ this is asserting that $\ker(F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_X)$ is isomorphic $\mathcal{O}_{X^{(1)}}$. There is clearly a map

$$\mathcal{O}_{X^{(1)}} \xrightarrow{f\mapsto f^p} \ker(F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_X) \tag{2.11}$$

because $d(f^p)=0$ for any $f\in\mathcal{O}_{X^{(1)}}$. We will check that it is an isomorphism in local coordinates: as is customary in algebraic geometry, this becomes possible after localizing in the étale topology on X. For any $x\in X$ we can find an open $U\subset X$ containing x with an étale map $f: U\to\mathbb{A}^n_k$. Then we have $(F_*\Omega^\bullet_X)|_{U^{(1)}}\simeq f^{*(1)}F_*\Omega^\bullet_{\mathbb{A}^n}$, so to check that (2.11) is an isomorphism over U it suffices to consider the case $X=\mathbb{A}^n_k$. This can be done by an explicit computation, as in Example 1.6, thanks to the grading on the polynomial ring $\mathcal{O}(\mathbb{A}^n_k)$.

Let us now describe a computation-free construction of the Cartier isomorphism, relying on mod p^2 crystalline cohomology. We may and do assume that $X=\operatorname{Spec}A$ is an affine scheme: if the isomorphism we construct in this case is canonical, it will give rise to the desired isomorphism of sheaves for an arbitrary X. We have the mod p^2 crystalline cohomology complex $\mathrm{R}\Gamma_{\mathrm{cris}}(\operatorname{Spec}A/W_2(k)) := \mathrm{R}\Gamma_{\mathrm{cris}}(\operatorname{Spec}A/W(k))\otimes_{W(k)}W_2(k)$, which can be computed by choosing a flat lift $\operatorname{Spec}\widetilde{A}$ of A over $W_2(k)$:

$$\mathrm{R}\Gamma_{\mathrm{cris}}(\operatorname{Spec}A/W_2(k))\simeq \widetilde{A}\xrightarrow{d}\Omega^1_{\widetilde{A}/W_2(k)}\xrightarrow{d}\dots$$

In particular, there is a distinguished triangle $\mathrm{R}\Gamma_{\mathrm{dR}}(A/k)\to\mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k))\to\mathrm{R}\Gamma_{\mathrm{dR}}(A/k)$ inducing a long exact sequence

$$\dots\to H^i_{\mathrm{cris}}(A/W_2(k))\to H^i_{\mathrm{dR}}(A/k)\xrightarrow{\beta_i} H^{i+1}_{\mathrm{dR}}(A/k)\to H^{i+1}_{\mathrm{cris}}(A/W_2(k))\to\dots \tag{2.12}$$

Explicitly, the connecting homomorphism β_i is given by sending the class $[\omega]\in H^i_{\mathrm{dR}}(A/k)$ of a closed i-form $\omega\in\Omega^i_{A/k}$ to the class $[\frac{d\widetilde{\omega}}{p}]$ where $\widetilde{\omega}\in\Omega^{i+1}_{\widetilde{A}/W_2(k)}$ is any lift of ω. Non-vanishing of β_i reflects the fact that it may not be possible to lift a closed form to a closed form modulo p^2.

Multiplication of differential forms induces a graded k-algebra structure on $\bigoplus\limits_{i\geq 0} H^i_{\mathrm{dR}}(A/k)$. The Bockstein maps β_i turn out to form a derivation of this ring:

Exercise 10. (1) $\beta_{i+j}(x\wedge y)=\beta_i(x)\wedge y+(-1)^i x\wedge\beta_i(y)$ for any pair of classes $x\in H^i_{\mathrm{dR}}(A), y\in H^j_{\mathrm{dR}}(A)$.
(2) $\beta_{i+1}\circ\beta_i=0$ for all $i\geq 0$.

Hint: You may need to appeal to de Rham cohomology modulo p^3 for assertion (2).

Together with Cartier isomorphism in degree 1 that we have already established, this now formally gives rise to map $\Omega^i_{A^{(1)}/k}\to H^i_{\mathrm{dR}}(A/k)$ thanks to the following universal property of the de Rham complex:

Lemma 2.6. *Let R be any algebra over a base ring k. Suppose that $\bigoplus\limits_{i\geq 0}A^i$ is a graded commutative k-algebra equipped with maps $\beta_i: A^i\to A^{i+1}$ satisfying properties (1),(2) from Exercise 10. Any k-algebra R map $f: R\to A^0$ extends uniquely to a graded algebra* *map* $\bigoplus\limits_{i\geq 0}\Omega^i_{R/k}\to\bigoplus\limits_{i\geq 0}A^i$ *making the diargam*

$$\begin{array}{ccccccc} A^0 & \xrightarrow{\beta_0} & A^1 & \xrightarrow{\beta_1} & A^2 & \xrightarrow{\beta_2} & \dots \\ {\scriptstyle f}\uparrow & & \uparrow & & \uparrow & & \\ R & \xrightarrow{d} & \Omega^1_R & \xrightarrow{d} & \Omega^2_R & \xrightarrow{d} & \dots \end{array} \tag{2.13}$$

commutative.

Indeed, we define a map $\Omega^1_R\to A^1$ by $dr\mapsto\beta_0(f(r))$ and then extend it to higher degree forms by multiplicativity.

We can apply this Lemma to $R=A^{(1)}$ and the graded de Rham cohomology ring $\bigoplus\limits_{i\geq 0}H^i_{\mathrm{dR}}(A/k)$ to get maps

$$C^{-1}:\Omega^i_{A^{(1)}/k}\to H^i_{\mathrm{dR}}(A/k) \tag{2.14}$$

extending the map $a\mapsto[a^p]\in H^0_{\mathrm{dR}}(A/k)$ for $i=0$.

Exercise 11. Give an explicit formula for the map C^{-1} just constructed, and prove that it is an isomorphism (for all $i\geq 0$) for every smooth k-algebra A, by reducing to the case $A=k[x_1,\dots,x_n]$.

Exercise 12. Let R be a smooth $\mathbb{Z}/p^n$-algebra. The goal of this problem is to calculate $H^0_{\mathrm{dR}}(R/\mathbb{Z}/p^n)$. Consider the map of sets $\prod\limits_{i=0}^{n-1}R/p\to H^0_{\mathrm{dR}}(R/\mathbb{Z}/p^n)$ given by

$$(r_0,\dots,r_{n-1})\mapsto\widetilde{r}_0^{p^n}+p\widetilde{r}_1^{p^{n-1}}+\dots+p^{n-1}\widetilde{r}_{n-1}\in\ker(R\to\Omega^1_R) \tag{2.15}$$

where $\widetilde{r}_0,\dots,\widetilde{r}_{n-1}\in R$ are arbitrary lifts of $r_0,\dots,r_{n-1}$. Prove that this expression is independent of the choice of lifts, that it indeed lies in the kernel of d, and that the resulting map $\prod\limits_{i=0}^{n-1}R/p\to H^0_{\mathrm{dR}}(R/p^n)$ is a bijection.

Prove that this map gives rise to a ring isomorphism $W_n(R/p)\simeq H^0_{\mathrm{dR}}(R/\mathbb{Z}/p^n)$ where $W_n(R/p)$ is the ring of length n Witt vectors of R/p.

3. Selected problems

Here is a selection of exercises appearing in the above notes.

Exercise I. Let F be a field of characteristic zero. Consider the affine curve $X=\operatorname{Spec}F[x,y]/(y^2-(x^3+ax+b))\subset\mathbb{A}^2_{x,y}$ for some $a,b\in F$. If a,b are chosen so that X is smooth, calculate $H^1_{\mathrm{dR}}(X/F)$ by finding explicit 1-forms representing a basis in cohomology.

Exercise II. Let F be a field of any characteristic. Calculated algebraic de Rham cohomology of $\mathbb{P}^1_F$ relative to F.

Hint: Use bicomplex (1.5). A convenient device for organizing this computation is the 'spectral sequence for a double complex' that expresses cohomology of the total complex in terms of the cohomology of the columns. If you haven't worked with such spectral sequence before, doing this computation from scratch for a cover $\mathbb{P}^1=\mathbb{P}^1\setminus\{0\}\cup\mathbb{P}^1\setminus\{\infty\}$ is a nice way to rediscover this spectral sequence.

Exercise III. Let $X = \mathbb{P}^1_k$ be the projective line over k. Describe vector bundles $F_*\mathcal{O}_X, F_*\Omega^1_X$ on $X^{(1)} \simeq \mathbb{P}^1_k$ as direct sums of line bundles, and calculate the map $d : F_*\mathcal{O}_X \to F_*\Omega^1_X$ in terms of this description.

Exercise IV. Calculate $M := H^1_{\mathrm{dR}}(\widehat{\mathbb{A}}^1_{W(k)})$. Is this module p-adically complete? That is, is the natural map $M \to \lim M/p^n$ an isomorphism?

Exercise V. Give an explicit formula for the map C^{-1} constructed in (2.14), and prove that it is an isomorphism (for all $i \geq 0$) for every smooth k-algebra A, by reducing to the case $A = k[x_1, \ldots, x_n]$.

Exercise VI. Let A be any commutative ring with an ideal $I \subset A$ such that $I^2 = 0$. Let Y be a smooth scheme over A. Consider the group of automorphisms $\mathrm{Aut}^0(Y)$ of Y over A such that the induced automorphism of $Y_0 := Y \times_A A/I$ is the identity. Construct an isomorphism between $\mathrm{Aut}^0(Y)$ and $H^0(Y_0, T_{Y_0/(A/I)} \otimes_{A/I} I)$, where $T_{Y_0/(A/I)}$ is the tangent bundle of Y_0.

Exercise VII. Recall that a $\mathbb{Z}_p$-module T is called *p-adically complete* if the natural map $T \to \lim\limits_n T/p^n$ is an isomorphism. Let $(M^\bullet, d_M)$ and $(N^\bullet, d_N)$ be complexes of $\mathbb{Z}_p$-modules, such that all terms M^i, N^i are p-adically complete and are torsion-free. Suppose that a map of complexes $f : (M^\bullet, d_M) \to (N^\bullet, d_N)$ induces a quasi-isomorphism $\overline{f} : (M^\bullet/p, d_M) \to (N^\bullet/p, d_N)$ of the mod p reductions of these complexes. Prove that f itself is a quasi-isomorphism.

Exercise VIII. Recall that if Y is a smooth proper scheme over $\mathbb{Z}_p$ then there is a natural isomorphism $H^n_{\mathrm{cris}}(Y_{\mathbb{F}_p}/\mathbb{Z}_p) \simeq H^n_{\mathrm{dR}}(Y/\mathbb{Z}_p)$ between crystalline cohomology of the special fiber $Y_{\mathbb{F}_p}$ and algebraic de Rham cohomology of Y. Moreover, if Y admits an endomorphism $\widetilde{F} : Y \to Y$ such that $\widetilde{F}$ equals the Frobenius endomorphism $F_{Y_{\mathbb{F}_p}}$ of $Y_{\mathbb{F}_p}$, then the map $\widetilde{F}^*$ on $H^n_{\mathrm{dR}}(Y/\mathbb{Z}_p)$ coincides with the map on crystalline cohomology induced by $F_{Y_{\mathbb{F}_p}}$. Hence in this case we have Lefschetz fixed point formula

$$\#Y_{\mathbb{F}_p}(\mathbb{F}_{p^r}) = \sum_{n \geq 0} \mathrm{Tr}(\widetilde{F}^{*r} : H^n_{\mathrm{dR}}(Y/\mathbb{Z}_p)[\frac{1}{p}]) \tag{3.1}$$

for each $r \geq 1$.

Assume that $Y = \mathbb{P}^n_{\mathbb{Z}_p}$ with $\widetilde{F}$ given by $[x_0 : \ldots : x_n] \mapsto [x_0^p : \ldots : x_n^p]$ in homogeneous coordinates. Give a direct proof of (3.1) in this case.

Exercise IX. Denote by $\mathbb{G}_{a,\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}[t]$ the additive group scheme over $\mathbb{Z}$. Consider the scheme $\mathbb{G}^\sharp_{a,\mathbb{Z}} := \mathrm{Spec}\,\mathbb{Z}[t, \frac{t^n}{n!} | n \in \mathbb{N}]$, where $\mathbb{Z}[t, \frac{t^n}{n!} | n \in \mathbb{N}]$ is the subring of $\mathbb{Q}[t]$ generated by polynomials $\frac{t^n}{n!}$ for all $n \in \mathbb{N}$. Prove that $\mathbb{G}^\sharp_{a,\mathbb{Z}}$ has a unique group scheme structure such that the natural map $\mathbb{G}^\sharp_{a,\mathbb{Z}} \to \mathbb{G}_{a,\mathbb{Z}}$ is a homomorphism of group schemes.

Exercise X. Let R be any ring, denote the R-group scheme $\mathbb{G}^\sharp_{a,\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,R$ by $\mathbb{G}^\sharp_{a,R}$. Construct an equivalence between the category of representations of $\mathbb{G}^\sharp_{a,R}$ on R-modules (that is, comodules over the ring of functions $\mathcal{O}(\mathbb{G}^\sharp_{a,R})$), and R-modules M equipped with a locally nilpotent endomorphism (that is, an R-linear endomorphism $f : M \to M$ such that $\bigcup\limits_{n \geq 1} \ker f^{\circ n} = M$).

Exercise XI. Let F be any field of characteristic zero. Consider the action of the formal group scheme $\widehat{\mathbb{G}}_a$ on $\mathbb{A}^1_F$ via translations. Construct an equivalence between the category of $\mathcal{D}_{\mathbb{A}^1}$-modules on $\mathbb{A}^1_F$ and $\widehat{\mathbb{G}}_a$-equivariant sheaves on $\mathbb{A}^1_F$.

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