Abstracts

Characteristic classes of étale local systems

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(joint work with Lue Pan)

To a local system $\mathbb{L}$ of $\mathbb{C}$-vector spaces on a smooth manifold M one can attach Cheeger-Chern-Simons characteristic classes $\widehat{c}_i(\mathbb{L}) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$ (cf. [1, Théorème 1]). They refine Chern classes of the complex vector bundle on M associated to $\mathbb{L}$: the image of $\widehat{c}_i(\mathbb{L})$ under the connecting homomorphism $H^{2i-1}(M, \mathbb{C}/\mathbb{Z}) \to H^{2i}(M, \mathbb{Z})$ is equal to the class $c_i(\mathbb{L} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_M)$.

The data of a rank n local system $\mathbb{L}$ is equivalent to the data of a representation $\rho_{\mathbb{L}}\colon \pi_1(M) \to GL_n(\mathbb{C})$ of the fundamental group (if M is connected), and class $\widehat{c}_i(\mathbb{L})$ arises as the image of the universal class $\widehat{c}_i \in H^{2i-1}_{\mathrm{grp}}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{Z})$ in group cohomology under the map $H^{2i-1}_{\mathrm{grp}}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \xrightarrow{\rho_{\mathbb{L}}^*} H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$

We investigate a p-adic analog of this theory. A crucial difference in the scope of it is that local systems with pro-finite coefficients can be considered not only on manifolds or topological spaces, but also on arithmetic objects such as varieties over non-algebraically closed fields.

For a connected scheme X consider an étale $\mathbb{Z}_p$-local system $\mathbb{L}$ of rank n on X. The data of $\mathbb{L}$ is equivalent to the data of a continuous representation $\rho_{\mathbb{L}}$: $\pi_1^{\text{ét}}(X) \to GL_n(\mathbb{Z}_p)$ of the étale fundamental group of X. This representation defines a map from the continuous cohomology of the group $GL_n(\mathbb{Z}_p)$ to the étale cohomology of X:

$$\rho_{\mathbb{L}}^* : H^\bullet_{\mathrm{cont}}(GL_n(\mathbb{Z}_p), \mathbb{Q}_p) \to H^\bullet_{\text{ét}}(X, \mathbb{Q}_p)$$

By a theorem of Lazard [2, Théorème V.2.4.9] continuous cohomology of $GL_n(\mathbb{Z}_p)$ is the free exterior algebra $\Lambda^\bullet_{\mathbb{Q}_p}(\ell_1, \dots, \ell_n)$ on n generators in degrees $\deg \ell_i = 2i-1$.

Definition. *Characteristic classes $\ell_i(\mathbb{L}) \in H^{2i-1}_{\text{ét}}(X, \mathbb{Q}_p)$ of a local system $\mathbb{L}$ on X are defined as the images of the classes ℓ_i under the map $\rho_{\mathbb{L}}^*$.*

This definition was introduced by Pappas [3, 4.4.2], and closely related constructions of characteristic classes of Galois representations have been considered previously by Kim [4].

The degree 1 class $\ell_1(\mathbb{L}) \in H^1_{\text{ét}}(X, \mathbb{Q}_p)$ is simply the result of applying the p-adic logarithm map $\mathbb{Z}_p^\times \to \mathbb{Q}_p$ to the determinant $\det \rho_{\mathbb{L}} \in H^1_{\text{ét}}(X, \mathbb{Z}_p^\times)$ of the representation $\rho_{\mathbb{L}}$. Our first main result is a partial calculation of characteristic classes for $\mathbb{Z}_p$-local systems on varieties over $\mathbb{Q}_p$:

Theorem 1. *Let X be a smooth proper geometrically connected variety over $\mathbb{Q}_p$ of dimension d. For a Hodge-Tate $\mathbb{Z}_p$-local system $\mathbb{L}$ on X its top degree characteristic class $\ell_{d+1}(\mathbb{L}) \in H^{2d+1}_{\text{ét}}(X, \mathbb{Q}_p) \simeq H^1(G_{\mathbb{Q}_p}, H^{2d}_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \simeq H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(-d)) \simeq$*

$\mathbb{Q}_p$ is equal to the following integer:

$$d! \sum_{m \in \mathbb{Z}} m \cdot \mathrm{ch}_d(\mathrm{gr}^m D_{\mathrm{HT}}(\mathbb{L})) \in \mathbb{Z} \subset \mathbb{Q}_p$$

where $D_{\mathrm{HT}}(\mathbb{L}) \simeq \bigoplus_m \mathrm{gr}^m D_{\mathrm{HT}}(\mathbb{L})$ is the graded Higgs bundle associated to $\mathbb{L}$, and $\mathrm{ch}_d(E) \in \frac{1}{d!}\mathbb{Z}$ for a vector bundle E denote its top degree Chern character.

One source of Hodge-Tate local systems is cohomology of families of varieties: for any smooth proper morphism $f : Y \to X$ the local system $\mathbb{L} = R^i f_* \mathbb{Z}_p$ of relative étale cohomology is Hodge-Tate with $D_{\mathrm{HT}}(\mathbb{L}) \simeq \bigoplus_{m \geq 0} R^{i-m} f_* \Omega^m_{Y/X}$. On the contrary, for local systems on varieties over an algebraically closed field the characteristic classes are zero in degrees > 1:

Theorem 2. *Let X be a smooth variety over an algebraically closed field $k = \overline{k}$ of characteristic zero. For a fixed rank n there exists a constant $c(n)$ such that for all primes $p > c(n)$ the class $\ell_i(\mathbb{L}) \in H^{2i-1}_{\text{ét}}(X, \mathbb{Q}_p)$ vanishes for $i > 1$ for all $\mathbb{Z}_p$-local systems $\mathbb{L}$ of rank n.*

This is a p-adic analog of a result of Reznikov [5] asserting that characteristic classes $\widehat{c}_i$ of all complex local systems on a smooth proper algebraic variety X over $\mathbb{C}$ vanish in $H^{2i-1}(X(\mathbb{C}), \mathbb{C}/\mathbb{Q})$ for $i > 1$.

The proof of Theorem 1 relies on the notion of Chern classes for pro-étale vector bundles on X that we introduce. Given a $\mathbb{Z}_p$-local system on a rigid-analytic variety X over a p-adic field K we can form the associated pro-étale vector bundle $\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X$ on the pro-étale site of X. As a consequence of the work of Huber-Kings [6], we prove that characteristic classes of $\mathbb{L}$ are related to Chern classes $c_i(\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X) \in H^{2i}_{\text{ét}}(X, \mathbb{Q}_p(i))$ of the corresponding pro-étale vector bundle via the formula:

$$c_i(\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X) = \ell_i(\mathbb{L}) \cdot \kappa_i$$

where $\kappa_i \in H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(i))$, for each $i \geq 0$, is a certain class in Galois cohomology (independent of X and $\mathbb{L}$). It can be described explicitly as the image of $(-1)^i \in \mathbb{Q}_p = (B^+_{\mathrm{dR}}/t^i B^+_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$ under the Bloch-Kato exponential map, which is the connecting homomorphism arising from the exact sequence of Galois modules $0 \to \mathbb{Q}_p(i) \to B^{+,\varphi=p^i}_{\mathrm{cris}} \to B^+_{\mathrm{dR}}/t^i B^+_{\mathrm{dR}} \to 0$.

For a Hodge-Tate local system $\mathbb{L}$, classes $c_i(\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X)$ can be calculated using the Hodge-Tate filtration on this pro-étale vector bundle. In the setting of Theorem 1 the class $\ell_{d+1}(\mathbb{L})$ can be recovered from the product $\ell_{d+1}(\mathbb{L}) \cdot \kappa_{d+1}$. However, in general more information than just the Chern classes of $\mathbb{L} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X$ is needed to recover the classes $\ell_i(\mathbb{L})$. One could ask if an analog of Theorem 1 nonetheless holds in the following sense.

For a smooth algebraic variety X over a finite extension K of $\mathbb{Q}_p$ we have a natural map

$$\alpha_X : H^n_{\text{ét}}(X, \mathbb{Q}_p) \to H^1(G_K, H^{n-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \simeq H^{n-1}_{\mathrm{dR}}(X/K)$$

If X has good reduction over $\mathcal{O}_K$, this map is an isomorphism for $n > 1$.

Question. For a Hodge-Tate local system $\mathbb{L}$ on X, is it true that the image of the characteristic class $\ell_i(\mathbb{L}) \in H^{2i-1}_{\text{ét}}(X, \mathbb{Q}_p)$ under the map α_X equals

$$(i-1)! \sum_{m \in \mathbb{Z}} m \cdot \mathrm{ch}_{i-1}(\mathrm{gr}^m D_{\mathrm{HT}}(\mathbb{L}))?$$

Here ch_{i-1} denotes the degree $2(i-1)$ component of the Chern character in de Rham cohomology.

References

[1] C. Soulé, Classes caractéristiques secondaires des fibrés plats, Astérisque No. 241 (1997), Exp. No. 819, 5, 411–424

[2] M. Lazard, Groupes analytiques p-adiques, Inst. Hautes Études Sci. Publ. Math. No. 26 (1965), 389–603

[3] G. Pappas, Volume and symplectic structure for ℓ-adic local systems, Adv. Math. **387** (2021), Paper No. 107836, 70 pp.

[4] M. Kim, Arithmetic Chern-Simons theory I, in *Galois covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants*, 155–180, Springer Proc. Math. Stat., 330, Springer, Cham,

[5] A. G. Reznikov, All regulators of flat bundles are torsion, Ann. of Math. (2) **141** (1995), no. 2, 373–386

[6] A. Huber-Klawitter and G. Kings, A p-adic analogue of the Borel regulator and the Bloch-Kato exponential map, J. Inst. Math. Jussieu **10** (2011), no. 1, 149–190

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