A higher-dimensional example in any characteristic

Let p be an arbitrary prime number. Denote $\overline{\mathbb{F}}_p$ by k.

Theorem 1. *There exist smooth projective derived equivalent varieties X_1, X_2 over k such that*

$$h^{0,3}(X_1) \neq h^{0,3}(X_2)$$

Moreover, for both $i = 1, 2$ the variety X_i satisfies the following properties:

(a) X_i can be lifted to a smooth formal scheme $\mathfrak{X}_i$ over $W(k)$ such that Hodge cohomology groups $H^r(\mathfrak{X}_i, \Omega^s_{\mathfrak{X}_i/W(k)})$ are torsion-free for all r, s.

(b) The Hodge-to-de Rham spectral sequence for X_i degenerates at the first page.

(c) The crystalline cohomology groups $H^n_{\mathrm{cris}}(X_i/W(k))$ are torsion-free for all n.

(d) The Hochschild-Kostant-Rosenberg spectral sequence for X_i degenerates at the second page. That is, there exists an isomorphism $\mathrm{HH}_n(X_i/k) \simeq \bigoplus_s H^s(X_i, \Omega^{n+s}_{X_i/k})$ for every n.

(e) X_i cannot be lifted to a smooth algebraic scheme over $W(k)$.

The varieties X_1, X_2 are both obtained as approximations of the quotient stack associated to a finite group acting on an abelian variety. The key to the construction is the appropriate choice of such finite group action that relies on complex multiplication and Honda-Tate theory.

Let $G = \mathbb{Z}/l\mathbb{Z}$ be the cyclic group of order l where l is an arbitrary odd prime divisor of a number of the form $p^{2r} + 1$, for an arbitrary $r \geq 1$.

Proposition 0.1. *There exists an abelian variety A over k equipped with an action of G by endomorphisms of A such that*

$$\dim_k H^3(A, \mathcal{O}_A)^G \neq \dim_k H^3(\widehat{A}, \mathcal{O}_{\widehat{A}})^G \tag{1}$$

Here $\widehat{A}$ denotes the dual abelian variety. Moreover, A can be lifted to a formal abelian scheme $\mathfrak{A}$ over $W(k)$ together with an action of G.

Proof. Take $A = \mathfrak{Z} \times_{W(k')} k$ with $\mathfrak{Z}$, k' provided by [Pet21], Proposition 3.1. The inequality (1) follows because there are G-equivariant isomorphisms $H^3(\widehat{A}, \mathcal{O}) \simeq \Lambda^3 H^1(\widehat{A}, \mathcal{O}_{\widehat{A}}) \simeq \Lambda^3(H^0(A, \Omega^1_{A/k})^\vee) \simeq H^0(A, \Omega^3_{A/k})^\vee$ (the last isomorphism exists even if $p = 3$) and $\dim_k H^0(A, \Omega^3_{A/k})^G = \dim_k (H^0(A, \Omega^3_{A/k})^\vee)^G$ as the order of G is prime to p. □

This proposition is specific to positive characteristic. For an abelian variety B equipped with an action of a finite group Γ over a field F of characteristic zero there must exist Γ-equivariant isomorphisms $H^i(B, \Omega^j_{B/F}) \simeq H^i(\widehat{B}, \Omega^j_{\widehat{B}/F})^\vee$ for all i, j as follows either from Hodge theory or thanks to the existence of a separable Γ-invariant polarization on B.

A more subtle feature of this construction is that it is impossible to find an abelian variety B with an action of a finite group Γ with $p \nmid |\Gamma|$ that would have $\dim_k H^i(B, \mathcal{O}_B)^\Gamma \neq \dim_k H^i(\widehat{B}, \mathcal{O}_{\widehat{B}})^\Gamma$ for $i = 1$ or $i = 2$. This can be deduced from Corollary 2.2 of [Pet21] applied to an approximation of the stack $[\mathfrak{B}/G]$ where $\mathfrak{B}$ is a formal Γ-equivariant lift of B that exists by Grothendieck-Messing theory combined with the fact that the order of Γ is prime to p.

Proof of Theorem 1. Let A be the abelian variety provided by Proposition 0.1. By Proposition 15 of [Ser58] there exists a smooth complete intersection Y of dimension 4 over k equipped with a free action of G. The diagonal action of G on the product of $A \times Y$ is free as well.

Define $X_1 = (A \times Y)/G$ and $X_2 = (\widehat{A} \times Y)/G$ where $\widehat{A}$ is the dual abelian variety of A equipped with the induced action of G. In both cases the quotient is taken with respect to the free diagonal action. The equivalence of $D^b(X_1)$ and $D^b(X_2)$ will follow from the Mukai equivalence between derived categories of an abelian scheme and its dual. Indeed, consider X_1 and X_2 as abelian schemes over Y/G. The base changes of both $\mathrm{Pic}^0_{Y/G}(X_1)$ and X_2 along $Y \to Y/G$ are isomorphic to $\widehat{A} \times Y$ compatibly with the G-action. By étale descent, $\mathrm{Pic}^0_{Y/G}(X_1) \simeq X_2$ as abelian schemes over Y/G. Proposition 6.7 of [BBHR09] implies that $D^b(X_1) \simeq D^b(X_2)$.

Next, we compare the Hodge numbers of X_1 and X_2. By Théorème 1.1 of Exposé XI [SGA73] we have $H^i(Y,\mathcal{O}_Y)=0$ for $1\leq i\leq 3$. Hence, there are G-equivariant identifications $H^3(A\times Y,\mathcal{O}_{A\times Y})\simeq H^3(A,\mathcal{O}_A)$ and $H^3(\widehat{A}\times Y,\mathcal{O}_{\widehat{A}\times Y})\simeq H^3(\widehat{A},\mathcal{O}_{\widehat{A}})$. Since G acts freely on both $A\times Y$ and $\widehat{A}\times Y$, the projections $A\times Y\to X_1$ and $\widehat{A}\times Y\to X_2$ are étale G-torsors and, since the order of G is prime to p, we have $H^3(X_1,\mathcal{O}_{X_1})\simeq H^3(A\times Y,\mathcal{O}_{A\times Y})^G$ and $H^3(X_2,\mathcal{O}_{X_2})\simeq H^3(\widehat{A}\times Y,\mathcal{O}_{\widehat{A}\times Y})^G$.

The inequality (1) therefore says that $h^{0,3}(X_1)\neq h^{0,3}(X_2)$. Condition (a) can be fulfilled as it is possible to choose Y that lifts to a smooth projective scheme over $W(k)$ together with an action of G, by Proposition 4.2.3 of [Ray79]. Denote by $\mathfrak{X}_1$ and $\mathfrak{X}_2$ the resulting formal schemes over $W(k)$ lifting X_1 and X_2. Since $\mathfrak{X}_i$ for $i=1,2$ can be presented as a quotient by a free action of G of a product of an abelian scheme with a complete intersection, the Hodge cohomology modules $H^r(\mathfrak{X}_i,\Omega^s_{\mathfrak{X}_i/W(k)})$ are free for all r,s.

Both properties (b) and (d) would be immediate if we had $\dim_k X_i\leq p$ but this is not always possible to achieve. Instead, we can argue using the lifts $\mathfrak{X}_i$. For (b), consider the Hodge-Tate complex $\overline{\Delta}_{\mathfrak{X}_i/W(k)[[u]]}$. By Proposition 4.15 of [BS21] there is a morphism $s:\Omega^1_{\mathfrak{X}_i/W(k)}[-1]\to\overline{\Delta}_{\mathfrak{X}_i/W(k)[[u]]}$ in the derived category of $\mathfrak{X}_i$ that induces an isomorphism on first cohomology. Taking n-th tensor power of s and precomposing it with the antisymmetrization map $\Omega^n_{\mathfrak{X}_i/W(k)}\to(\Omega^1_{\mathfrak{X}_i/W(k)})^{\otimes n}$ we obtain maps $\Omega^n_{\mathfrak{X}_i/W(k)}[-n]\to\overline{\Delta}_{\mathfrak{X}_i/W(k)[[u]]}$ that induce a quasi-isomorphism $\overline{\Delta}_{\mathfrak{X}_i/W(k)[[u]]}[\frac{1}{p}]\simeq\bigoplus_{n\geq 0}\Omega^n_{\mathfrak{X}_i/W(k)}[-n]\otimes_{W(k)}W(k)[\frac{1}{p}]$. In particular, the differentials in the Hodge-Tate spectral sequence $H^s(\mathfrak{X}_i,\Omega^r_{\mathfrak{X}_i/W(k)})\Rightarrow H^{s+r}_{\overline{\Delta}}(\mathfrak{X}_i/W(k)[[u]])$ vanish modulo torsion. But, as we established above, the Hodge cohomology of $\mathfrak{X}_i$ has no torsion, so the Hodge-Tate spectral sequence degenerates at the second page. Therefore the conjugate spectral sequence for X_i degenerates at the second page as well and, equivalently, the Hodge-to-de Rham spectral sequence degenerates at the first page.

Similarly, for (d) consider the Hochschild-Kostant-Rosenberg spectral sequence $E_2^{r,s}=H^r(\mathfrak{X}_i,\Omega^{-s}_{\mathfrak{X}_i/W(k)})$ converging to $\mathrm{HH}_{-r-s}(\mathfrak{X}_i/W(k))$. There exist maps $\varepsilon_n:\Omega^n_{\mathfrak{X}_i/W(k)}[n]\to\mathrm{HH}(\mathfrak{X}_i/W(k))$ into the Hochschild complex inducing multiplication by $n!$ on the n-th cohomology: $\varepsilon_n=n!:\Omega^n_{\mathfrak{X}_i/W(k)}\to\mathcal{H}^{-n}(\mathrm{HH}(\mathfrak{X}_i/W(k)))\simeq\Omega^n_{\mathfrak{X}_i/W(k)}$. Therefore, the HKR spectral sequence always degenerates modulo torsion, hence degenerates at the second page in our case. Passing to the mod p reduction gives (d).

The property (c) follows from (a) and (b) as $H^n_{\mathrm{cris}}(X_i/W(k))\simeq H^n_{\mathrm{dR}}(\mathfrak{X}_i/W(k))$.

Finally, to prove (e), note that by the same computation as above one sees that $h^{0,3}(X_1)=h^{3,0}(X_2)\neq h^{0,3}(X_2)=h^{3,0}(X_1)$ so both X_1 and X_2 violate Hodge symmetry. Denote by K the fraction field of $W(k)$. If $\mathcal{X}_i$ is a smooth scheme over $W(k)$ lifting X_i then we have

$$\dim_K H^r(\mathcal{X}_{i,K},\Omega^s_{\mathcal{X}_{i,K}/K})\leq\dim_k H^r(X_i,\Omega^s_{X_i/k}) \tag{2}$$

for all r,s by semi-continuity while $\dim_K H^n_{\mathrm{dR}}(\mathcal{X}_{i,K}/K)=\dim_k H^n_{\mathrm{dR}}(X_i/k)$ because $H^n_{\mathrm{dR}}(\mathcal{X}_i/W(k))\simeq H^n_{\mathrm{cris}}(X_i/W(k))$ is torsion-free for all n. Since Hodge-to-de Rham spectral sequences for X_i and $\mathcal{X}_{i,K}$ degenerate at the first page, we deduce that $\sum\limits_{r,s}\dim_K H^r(\mathcal{X}_{i,K},\Omega^s_{\mathcal{X}_{i,K}/K})=\sum\limits_{r,s}\dim_k H^r(X_i,\Omega^s_{X_i/k})$ so (2) is in fact equality for all r,s. But this means that the smooth proper algebraic variety $\mathcal{X}_{i,K}$ over a field of characteristic zero violates Hodge symmetry which is impossible.

□

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