Boundedness of the p-primary torsion of the Brauer group of products of varieties

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Abstract

Let k be a field finitely generated over its prime subfield. We prove that the quotient of the Brauer group of a product of varieties over k by the Brauer groups of factors has finite exponent. The bulk of the proof concerns p-primary torsion in characteristic p. Our approach gives a more direct proof of the boundedness of the p-primary torsion of the Brauer group of an abelian variety, as recently proved by D'Addezio. We show that the transcendental Brauer group of a Kummer surface over k has finite exponent, but can be infinite when k is an infinite field of positive characteristic. This answers a question of Zarhin and the author.

Introduction

Let k be a field of characteristic exponent p. Thus $p = 1$ if $\mathrm{char}(k) = 0$, otherwise $p = \mathrm{char}(k)$. Let $\bar{k}$ be an algebraic closure of k, let k^{s} be the separable closure of k in $\bar{k}$, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$. For an abelian group A and a prime number ℓ we denote by $A\{\ell\}$ the ℓ-primary torsion subgroup of A. We write $A(p')$ for the direct sum of $A\{\ell\}$ over all primes $\ell \neq p$.

Assume that k is finitely generated over its prime subfield. Relation between the Tate conjecture for divisors for a smooth and projective variety X over k and finiteness properties of the Brauer group of X is well known, at least for torsion coprime to p. Indeed, the validity of the Tate conjecture for X at a prime $\ell \neq p$ is equivalent to the finiteness of $\mathrm{Br}(X_{k^{\mathrm{s}}})^\Gamma\{\ell\}$, and is also equivalent to the finiteness of the image of the natural map $\mathrm{Br}(X)\{\ell\} \to \mathrm{Br}(X_{k^{\mathrm{s}}})\{\ell\}$, see [CTS21, Thm. 16.1.1]. In particular, this holds for abelian varieties and K3 surfaces. Moreover, in these two cases $\mathrm{Br}(X_{k^{\mathrm{s}}})^\Gamma(p')$ is finite [SZ08, SZ15, Ito18], see also [CTS21, Ch. 16]. In [SZ08, Questions 1, 2] the authors asked whether $\mathrm{Br}(X_{k^{\mathrm{s}}})^\Gamma\{p\}$, or at least the image of $\mathrm{Br}(X)\{p\}$ in $\mathrm{Br}(X_{k^{\mathrm{s}}})\{p\}$, is finite when X is an abelian variety or a K3 surface and $p > 1$. In a recent paper, D'Addezio observed that for the self-product of a supersingular elliptic curve this image is infinite when k is infinite [D'Ad, Cor. 5.4]. On the positive side, he proved that $\mathrm{Br}(X_{k^{\mathrm{s}}})^{\Gamma}\{p\}$ has *finite exponent* when X is an abelian variety, see [D'Ad, Thm. 1.1]. (As pointed out in [D'Ad, Cor. 6.7], this may fail if k^{s} is replaced by $\bar{k}$.) One can hope that this result should be true in more generality, for example for K3 surfaces. For $p \neq 2$, we note that D'Addezio's examples descend to the associated Kummer surfaces. Thus the questions raised in [SZ08] have negative answers for K3 surfaces over infinite finitely generated fields of characteristic at least 3.

The main result of this note is the following

Theorem A *Let X and Y be smooth, projective, geometrically integral varieties over a finitely generated field k. Then the cokernel of the natural map*

$$\mathrm{Br}(X) \oplus \mathrm{Br}(Y) \to \mathrm{Br}(X \times_k Y)$$

has finite exponent.

For the prime-to-p torsion this easily follows from [SZ14, Thm. B] which says[1] that the cokernel of $\mathrm{Br}(X)(p') \oplus \mathrm{Br}(Y)(p') \to \mathrm{Br}(X \times_k Y)(p')$ is finite when $X \times_k Y$ has a k-point or $\mathrm{H}^3(k, (k^{\mathrm{s}})^{\times}) = 0$. In this paper we deal with the p-primary torsion. Our proof is inspired by [D'Ad] and crucially uses the crystalline Tate conjecture proved by de Jong [dJ98, Thm. 2.6]. As a consequence we obtain a more transparent proof of [D'Ad, Thm. 1.1]. Combined with the previous results of Zarhin and the author, it gives that $\mathrm{Br}(X_{k^{\mathrm{s}}})^{\Gamma}$ is a direct sum of a finite group and a p-group of finite exponent, when X is an abelian variety over a finitely generated field k, see Theorem 3.2. Using similar ideas, we also give a simplified proof of the flat version of the Tate conjecture for divisors on abelian varieties [D'Ad, Thm. 5.1], see Theorem 3.4.

The prime-to-p torsion part of the next result was obtained in [SZ14, Thm. 3.1].

Theorem B *Let X and Y be smooth, projective, geometrically integral varieties over a finitely generated field k of characteristic exponent p. Then the cokernel of the natural map $\mathrm{Br}(X_{k^{\mathrm{s}}})^{\Gamma} \oplus \mathrm{Br}(Y_{k^{\mathrm{s}}})^{\Gamma} \to \mathrm{Br}(X_{k^{\mathrm{s}}} \times_{k^{\mathrm{s}}} Y_{k^{\mathrm{s}}})^{\Gamma}$ is a direct sum of a finite group and a p-group of finite exponent.*

In particular, Theorem B implies that if X is a surface dominated by a product of curves, then $\mathrm{Br}(X_{k^{\mathrm{s}}})^{\Gamma}$ is a direct sum of a finite group and a p-group of finite exponent. This holds, for example, for the smooth surfaces in $\mathbb{P}^3_k$ given by the equation $f(x_0, x_1) = g(x_2, x_3)$, see Corollary 2.4.

Our approach is based on the systematic use of *pointed varieties*, i.e. varieties over k with a distinguished k-point. In Section 1 we obtain a version of the Künneth formula for the second flat cohomology group of the product of pointed varieties, see Theorem 1.3 and Corollary 1.4. Similarly to the ℓ-adic case, the embedding of the 'primitive' part of cohomology can be interpreted in terms of pairing with classes of

[1] In *loc. cit.* one assumes that $\mathrm{char}(k) = 0$, but the proof goes through for the prime-to-p torsion when $\mathrm{char}(k) = p > 0$.

certain natural torsors. In Section 2 we first prove Theorem A for pointed varieties (Theorem 2.1) from which we obtain the general case, see Theorem 2.2. We then deduce Theorem B, see Corollary 2.3. Applications to abelian varieties can be found in Section 3 and applications to Kummer surfaces in Section 4. The appendix by Alexander Petrov contains a structure theorem for the Brauer group of a smooth and proper variety over an algebraically closed field of positive characteristic p: this group is a direct sum of finitely many copies of $\mathbb{Q}_p/\mathbb{Z}_p$ and an abelian p-group of finite exponent, see Theorem A.1.

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1 Cohomology of the product

Let k be a field. Let F be a contravariant functor from the category of schemes over k to the category of abelian groups. We shall refer to a pair (X, x_0), where X is a k-scheme and $x_0 \in X(k)$, as a *pointed k-scheme*. For a pointed k-scheme (X, x_0) we define

$$F(X)_e := \mathrm{Ker}[x_0^* : F(X) \to F(k)].$$

Then we have $F(X) \cong F(k) \oplus F(X)_e$. For k-schemes X and Y we have an obvious commutative diagram

$$\begin{tikzcd} Y & X \times_k Y \arrow[l, "p_X"'] \arrow[d, "p_Y"] \\ \mathrm{Spec}(k) \arrow[from=u, "\pi_Y"'] & X \arrow[l, "\pi_X"'] \end{tikzcd}$$

When (X, x_0) and (Y, y_0) are pointed k-schemes, the k-points x_0 and y_0 give rise to sections to the four morphisms in this diagram. Thus $F(k)$, $F(X)$, $F(Y)$ are direct summands of $F(X \times_k Y)$ such that $F(X) \cap F(Y) = F(k)$. Therefore, $F(X)_e$ and $F(Y)_e$ are direct summands of $F(X \times_k Y)_e$ such that $F(X)_e \cap F(Y)_e = 0$. It follows that $F(X)_e \oplus F(Y)_e$ is a direct summand of $F(X \times_k Y)_e$. Define

$$F(X \times_k Y)_{\mathrm{prim}} := \mathrm{Ker}[F(X \times_k Y)_e \to F(X)_e \oplus F(Y)_e],$$

where the map $F(X \times_k Y)_e \to F(X)_e$ is the specialisation at y_0 and the map $F(X \times_k Y)_e \to F(Y)_e$ is the specialisation at x_0. This gives rise to a direct sum decomposition of abelian groups

$$F(X \times_k Y)_e \cong F(X)_e \oplus F(Y)_e \oplus F(X \times_k Y)_{\mathrm{prim}}, \tag{1}$$

which is functorial with respect to morphisms of pointed k-schemes.

For a field extension K/k we define the functor $F(X_K)^k := \mathrm{Im}[F(X) \to F(X_K)]$. The group $\mathrm{Br}(X_{k^s})^k$ is called the *transcendental Brauer group*.

Recall that by a theorem of Grothendieck, the Picard scheme $\mathbf{Pic}_{X/k}$ exists when X is proper over k, see the references in [CTS21, Thm. 2.5.7]. The Picard variety of a smooth, projective, geometrically integral variety X is the abelian variety $\mathbf{Pic}^0_{X/k,\mathrm{red}}$, where $\mathbf{Pic}^0_{X/k}$ is the connected component of 0. The Albanese variety A is defined as the dual abelian variety of the Picard variety of X so that $\mathbf{Pic}^0_{X/k,\mathrm{red}} \cong A^\vee$.

From now on we assume that X is a projective variety over a field k, and that p is a prime number that may or may not be equal to the characteristic of k, unless explicitly stated otherwise. Throughout the paper we consider fppf-cohomology, so we drop fppf from notation. We also write $\mathrm{H}^i(X) := \mathrm{H}^i(X_{\mathrm{fppf}}, \mu_{p^n})$.

Let S_X be the finite commutative group k-scheme whose Cartier dual $S_X^\vee$ is the subgroup k-scheme $\mathbf{Pic}_{X/k}[p^n] := \mathrm{Ker}[\mathbf{Pic}_{X/k} \xrightarrow{p^n} \mathbf{Pic}_{X/k}]$.

Proposition 1.1 *Let X and Y be pointed projective, geometrically reduced and geometrically connected varieties over a field k. Then there is a natural isomorphism*

$$\mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}} \cong \mathrm{H}^1(X, S_Y^\vee)_e.$$

Proof. For a proper, geometrically reduced and geometrically connected k-variety $\pi_Y\colon Y \to \mathrm{Spec}(k)$ the natural map $\mathcal{O}_{\mathrm{Spec}(k)} \to \pi_{Y*}\mathcal{O}_Y$ is an isomorphism. This implies that every k-morphism from Y to an affine k-scheme must be constant. In particular, the sheaf $\pi_{Y*}\mu_{p^n,Y}$ on $\mathrm{Spec}(k)_{\mathrm{fppf}}$ is μ_{p^n}. The Kummer sequence

$$1 \to \mu_{p^n} \to \mathbb{G}_{m,k} \xrightarrow{p^n} \mathbb{G}_{m,k} \to 1$$

is an exact sequence of sheaves on $\mathrm{Spec}(k)_{\mathrm{fppf}}$. Using that the natural morphism $\mathbb{G}_{m,k} \to \pi_{Y*}\mathbb{G}_{m,Y}$ is an isomorphism, we see that the group k-scheme $S_X^\vee$ represents the sheaf $R^1\pi_{X*}\mu_{p^n}$ on $\mathrm{Spec}(k)_{\mathrm{fppf}}$. By a theorem of Bragg and Olsson [BO, Cor. 1.4], since Y is projective, there is an affine group k-scheme G_n of finite type that represents the sheaf $R^2\pi_{Y*}\mu_{p^n}$ on $\mathrm{Spec}(k)_{\mathrm{fppf}}$.

Consider the spectral sequence attached to $p_Y\colon X \times_k Y \to X$:

$$E_2^{p,q} = \mathrm{H}^p(X, R^q p_{Y*}\mu_{p^n}) \Rightarrow \mathrm{H}^{p+q}(X \times_k Y).$$

Since (id, y_0) is a section of p_Y, the canonical map

$$\mathrm{H}^i(X) \cong \mathrm{H}^i(X, p_{Y*}\mu_{p^n}) \to \mathrm{H}^i(X \times_k Y)$$

is split injective for any $i \geq 0$. This implies that the differentials of any page of this spectral sequence with target $\mathrm{H}^i(X)$ are zero for any $i \geq 0$. It follows that we have an exact sequence

$$0 \to \mathrm{H}^1(X, S_Y^\vee) \to \mathrm{H}^2(X \times_k Y)/\mathrm{H}^2(X) \to \mathrm{H}^0(X, G_n) \to \mathrm{H}^2(X, S_Y^\vee).$$

When $X = \mathrm{Spec}(k)$ there is a compatible exact sequence giving rise to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(X, S_Y^\vee) \arrow[r] & \mathrm{H}^2(X \times_k Y)_e / \mathrm{H}^2(X)_e \arrow[r] & \mathrm{H}^0(X, G_n) \arrow[r] & \mathrm{H}^2(X, S_Y^\vee) \\
0 \arrow[r] & \mathrm{H}^1(k, S_Y^\vee) \arrow[r] \arrow[u, hook] & \mathrm{H}^2(Y)_e \arrow[r] \arrow[u, hook] & \mathrm{H}^0(k, G_n) \arrow[r] \arrow[u, "\cong"] & \mathrm{H}^2(k, S_Y^\vee) \arrow[u, hook]
\end{tikzcd}$$

All vertical maps are split injective, with splittings defined by the base point $x_0 \in X(k)$. The map $\mathrm{H}^0(k, G_n) \to \mathrm{H}^0(X, G_n)$ is an isomorphism since X is proper, geometrically reduced and geometrically connected, and G_n is affine. By diagram chase we obtain a natural isomorphism

$$\mathrm{H}^2(X \times_k Y)_e / \big(\mathrm{H}^2(X)_e \oplus \mathrm{H}^2(Y)_e\big) \cong \mathrm{H}^1(X, S_Y^\vee)_e.$$

This proves the proposition. $\square$

The following statement can be compared to [HS13, Prop. 1.1].

Proposition 1.2 *Let X be a pointed projective, geometrically reduced and geometrically connected variety over a field k. For any finite commutative group k-scheme $\mathcal{G}$ we have a functorial isomorphism*

$$\tau \colon \mathrm{H}^1(X, \mathcal{G})_e \xrightarrow{\sim} \mathrm{Hom}_k(\mathcal{G}^\vee, \mathbf{Pic}_{X/k}).$$

Proof. We adapt the method of proof of [CTSa87, Thm. 1.5.1].

There is the following spectral sequence for the fppf topology:

$$\mathrm{Ext}^p_k(A, R^q\pi_{X*}B) \Rightarrow \mathrm{Ext}^{p+q}_X(\pi_X^* A, B),$$

where A is a sheaf on $\mathrm{Spec}(k)_{\mathrm{fppf}}$ and B is a sheaf on X_{fppf}. This is a particular case of the spectral sequence of composed functors, namely $\Gamma(X, -)$ and $\mathrm{Hom}_k(A, -)$, using that π_X^* is a left adjoint to π_{X*}, and that π_{X*} sends injective sheaves on X_{fppf} to injective sheaves on $\mathrm{Spec}(k)_{\mathrm{fppf}}$. The last property is a consequence of the fact that π_X^* is exact, see [Mil80, Remark III.1.20] which refers to [Mil80, Prop. II.2.6]. See also [CTS21, §2.1.3] for a summary.

Since $\pi_X^*(\mathcal{G}^\vee) = \mathcal{G}_X^\vee$, we have the spectral sequence

$$\mathrm{Ext}^p_k(\mathcal{G}^\vee, R^q\pi_{X*}\mathbb{G}_{m,X}) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{G}_X^\vee, \mathbb{G}_{m,X}).$$

Since X is proper, geometrically reduced and geometrically connected, the natural morphism $\mathbb{G}_{m,k} \to \pi_{X*}\mathbb{G}_{m,X}$ is an isomorphism. Thus the exact sequence of terms of low degree of our spectral sequence can be written as follows:

$$0 \to \mathrm{Ext}^1_k(\mathcal{G}^\vee, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1_X(\mathcal{G}_X^\vee, \mathbb{G}_{m,X}) \to \mathrm{Hom}_k(\mathcal{G}^\vee, \mathbf{Pic}_{X/k})$$

$$\to \mathrm{Ext}^2_k(\mathcal{G}^\vee, \mathbb{G}_{m,k}) \to \mathrm{Ext}^2_X(\mathcal{G}^\vee_X, \mathbb{G}_{m,X}).$$

Using $x_0 \in X(k)$ we obtain that the second and fifth arrows here are split injective.

To calculate the terms of this sequence we consider the local-to-global spectral sequence of Ext-groups, see SGA 4, Exp. V, (6.1.3):

$$\mathrm{H}^p(X, \mathcal{E}xt^q_X(\mathcal{G}^\vee_X, \mathbb{G}_{m,k})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{G}^\vee_X, \mathbb{G}_{m,X}).$$

By SGA 7, Exp. VIII, Prop. 3.3.1, we have $\mathcal{E}xt^1_X(\mathcal{G}^\vee_X, \mathbb{G}_{m,k}) = 0$, from which we obtain

$$\mathrm{Ext}^1_k(\mathcal{G}^\vee, \mathbb{G}_{m,k}) \cong \mathrm{H}^1(k, \mathcal{G}), \quad \mathrm{Ext}^1_X(\mathcal{G}^\vee_X, \mathbb{G}_{m,k}) \cong \mathrm{H}^1(X, \mathcal{G}).$$

Specialising at the base point x_0 we deduce the required isomorphism τ. □

It follows that if $p^n\mathcal{G} = 0$, then τ is an isomorphism $\mathrm{H}^1(X, \mathcal{G})_e \xrightarrow{\sim} \mathrm{Hom}_k(\mathcal{G}^\vee, S^\vee_X)$.

Let $S_X \otimes S_Y$ be the fppf sheaf of abelian groups on $\mathrm{Spec}(k)$ given by the tensor product of sheaves associated to the commutative group k-schemes S_X and S_Y.

Theorem 1.3 *Let X and Y be pointed projective, geometrically reduced and geometrically connected varieties over a field k. Then there is an isomorphism*

$$\mathrm{Hom}_k(S_X \otimes S_Y, \mu_{p^n}) \cong \mathrm{Hom}_k(S_X, S^\vee_Y) \xrightarrow{\sim} \mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}}. \tag{2}$$

Proof. This follows from Proposition 1.1 and the natural isomorphisms

$$\mathrm{H}^1(X, S^\vee_Y)_e \cong \mathrm{Hom}_k(S_Y, S^\vee_X) \cong \mathrm{Hom}_k(S_X, S^\vee_Y) \cong \mathrm{Hom}_k(S_X \otimes S_Y, \mu_{p^n}).$$

The first isomorphism is τ of Proposition 1.2 for $\mathcal{G} = S^\vee_Y$. The second isomorphism is due to Cartier duality. The third isomorphism is obtained by applying the functor of sections to the canonical isomorphism

$$\mathrm{Hom}(A, \mathrm{Hom}(B, C)) \cong \mathrm{Hom}(A \otimes B, C)$$

in the category of fppf sheaves of abelian groups on $\mathrm{Spec}(k)$, and noticing that $\mathrm{Hom}(S_Y, \mu_{p^n}) \cong S^\vee_Y$ since S_Y is finite. □

Let us define $\mathrm{H}^i(X, \mathbb{Z}_p(1))$ as $\varprojlim \mathrm{H}^i(X, \mu_{p^n})$ as $n \to \infty$.

Corollary 1.4 *Let X and Y be pointed smooth, projective, geometrically integral varieties over a field k of characteristic $p > 0$. Then there is an isomorphism*

$$\mathrm{H}^2(X \times_k Y, \mathbb{Z}_p(1))_{\mathrm{prim}} \cong \mathrm{Hom}_k(A[p^\infty], B^\vee[p^\infty]), \tag{3}$$

where $A[p^\infty]$ is the p-divisible group of the Albanese variety A of X, and $B^\vee[p^\infty]$ is the p-divisible group of the Picard variety $B^\vee$ of Y.

Proof. We have an exact sequence of group k-schemes

$$0 \to \mathbf{Pic}^0_{X/k} \to \mathbf{Pic}_{X/k} \to \mathbf{NS}_{X/k} \to 0,$$

which is the definition of $\mathbf{NS}_{X/k}$, cf. [CTS21, §5.1]. The k-scheme $\mathbf{NS}_{X/k}$ is étale, see SGA 3, IV$_A$, Prop. 5.5.1. The group $\mathbf{NS}_{X/k}(k^{\mathrm{s}}) = \mathbf{NS}_{X/k}(\bar{k}) = \mathrm{NS}\,(X_{\bar{k}})$ is finitely generated by a theorem of Néron and Severi. Thus the cokernel of the map of group k-schemes $\mathbf{Pic}^0_{X/k}[p^n] \to \mathbf{Pic}_{X/k}[p^n]$ is finite. Next, by Grothendieck [FGA6, §3], the Picard variety $A^\vee = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ is a group subscheme of $\mathbf{Pic}^0_{X/k}$ with finite cokernel, see [CTS21, §5.1.1]. We conclude that there is an exact sequence of finite commutative group k-schemes

$$0 \to A^\vee[p^n] \to S_X^\vee \to F_X \to 0,$$

where F_X is finite. From the dual of this exact sequence and a similar sequence for Y we obtain the following exact sequence of abelian groups:

$$0 \to \mathrm{Hom}_k(A[p^n], B^\vee[p^n]) \to \mathrm{Hom}_k(S_X, S_Y^\vee) \to \mathrm{Hom}_k(S_X, F_Y) \oplus \mathrm{Hom}_k(F_X^\vee, S_Y),$$

where homomorphisms are taken in the category of finite commutative k-groups. We note that the last term in this sequence is annihilated by the maximum of the orders of F_X and F_Y. This gives an isomorphism

$$\varprojlim \mathrm{Hom}_k(A[p^n], B^\vee[p^n]) \cong \varprojlim \mathrm{Hom}_k(S_X, S_Y^\vee).$$

Thus passing to the projective limit in (2) we obtain (3). □

We finish this section by interpreting the isomorphism (2) of Theorem 1.3 in terms of certain canonical torsors on X and Y.

For any $n \geq 1$ define a *universal p^n-torsor*[2] $\mathcal{T}_{X,p^n} \to X$ as an fppf X-torsor with structure group S_X and trivial fibre at x_0 such that the map τ from Proposition 1.2 sends the class $[\mathcal{T}_{X,p^n}] \in \mathrm{H}^1(X, S_X)_e$ to the natural injective map

$$S_X^\vee = \mathbf{Pic}_{X/k}[p^n] \hookrightarrow \mathbf{Pic}_{X/k}.$$

It is clear that $\mathcal{T}_{X,p^n}$ is unique up to isomorphism.

The isomorphism (2) in Theorem 1.3 can be made explicit in terms of $\mathcal{T}_{X,p^n}$ and $\mathcal{T}_{Y,p^n}$, as follows. The cup-product pairing

$$\mathrm{H}^1(X \times Y, S_X) \times \mathrm{H}^1(X \times Y, S_Y) \to \mathrm{H}^2(X \times Y, S_X \otimes S_Y)$$

gives rise to the pairing

$$\mathrm{H}^1(X, S_X) \times \mathrm{H}^1(Y, S_Y) \to \mathrm{H}^2(X \times Y, S_X \otimes S_Y).$$

[2]This notion was introduced by Yang Cao in [Cao20] (for the étale topology and for varieties without a distinguished rational point), inspired by universal torsors of Colliot-Thélène and Sansuc [CTSa87] and by calculations in [SZ14].

Let us denote by

$$[\mathcal{T}_{X,p^n}] \boxtimes [\mathcal{T}_{X,p^n}] \in \mathrm{H}^2(X \times Y, S_X \otimes S_Y)_{\mathrm{prim}}$$

the value of the last pairing on the classes $[\mathcal{T}_{X,p^n}]$ and $[\mathcal{T}_{Y,p^n}]$. Define

$$\varepsilon\colon \mathrm{Hom}_k(S_X \otimes S_Y, \mu_{p^n}) \to \mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}}$$

as the map sending a homomorphism $\psi\colon S_X \otimes S_Y \to \mu_{p^n}$ of sheaves on $\mathrm{Spec}(k)_{\mathrm{fppf}}$ to $\psi_*\big([\mathcal{T}_{X,p^n}] \boxtimes [\mathcal{T}_{X,p^n}]\big)$.

Proposition 1.5 *Let X and Y be pointed projective, geometrically reduced and geometrically connected varieties over a field k. The isomorphism (2) is given by the map ε.*

Proof. The second proof of [CTS21, Thm. 5.7.7 (ii)] on pp. 161–162 works in our situation. We reproduce this argument for the convenience of the reader.

For a finite commutative group k-scheme $\mathcal{G}$ such that $p^n\mathcal{G} = 0$ we have a commutative diagram of pairings:

$$\begin{array}{ccccc}
\mathrm{H}^1(X, \mathcal{G}^\vee)_e & \times & \mathrm{H}^1(Y, \mathcal{G})_e & \to & \mathrm{H}^2(X \times_k Y, \mu_{p^n}) \\
|| & & \downarrow & & || \\
\mathrm{H}^1(X, \mathcal{G}^\vee)_e & \times & \mathrm{Ext}^1_Y(\mathcal{G}^\vee, \mu_{p^n}) & \to & \mathrm{H}^2(X \times_k Y, \mu_{p^n}) \\
|| & & || & & \uparrow \\
\mathrm{H}^1(X, \mathcal{G}^\vee)_e & \times & \mathrm{Ext}^1_k(\mathcal{G}^\vee, \tau_{\leq 1}\mathbf{R}\pi_{Y*}\mu_{p^n}) & \to & \mathrm{H}^2(X, \tau_{\leq 1}\mathbf{R}p_{Y*}\mu_{p^n}) \\
|| & & \downarrow & & \downarrow \\
\mathrm{H}^1(X, \mathcal{G}^\vee)_e & \times & \mathrm{Hom}_k(\mathcal{G}^\vee, S_Y^\vee) & \to & \mathrm{H}^1(X, S_Y^\vee)
\end{array}$$

The vertical map $\mathrm{H}^1(Y, \mathcal{G}) \to \mathrm{Ext}^1_Y(\mathcal{G}^\vee, \mu_{p^n})$ comes from the local-to-global spectral sequence (SGA 4, Exp. V, (6.1.3))

$$\mathrm{H}^p(Y, \mathcal{E}xt^q_Y(\mathcal{G}^\vee, \mu_{p^n})) \Rightarrow \mathrm{Ext}^{p+q}_Y(\mathcal{G}^\vee, \mu_{p^n}).$$

The first two pairings are compatible by [Mil80, Prop. V.1.20]. The two lower pairings are natural, and the compatibility of the rest of the diagram is clear. The composition of maps in the second column is the isomorphism τ.

Since Y is a pointed proper, geometrically reduced and geometrically connected variety over k, the object $\tau_{\leq 1}\mathbf{R}p_{Y*}\mu_{p^n}$ of the bounded derived category of sheaves on X_{fppf} is the direct sum of μ_{p^n} in degree 0 and $S_Y^\vee$ in degree 1. Thus $\mathrm{H}^1(X, S_Y^\vee)$ is a direct summand of $\mathrm{H}^2(X, \tau_{\leq 1}\mathbf{R}p_{Y*}\mu_{p^n})$. Taking $\mathcal{G} = S_Y$, the previous diagram gives rise to a commutative diagram of pairings

$$\begin{array}{ccccc}
\mathrm{H}^1(X, S_Y^\vee)_e & \times & \mathrm{H}^1(Y, S_Y)_e & \to & \mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}} \\
|| & & \tau \downarrow & & \uparrow \\
\mathrm{H}^1(X, S_Y^\vee)_e & \times & \mathrm{Hom}_k(S_Y^\vee, S_Y^\vee) & \to & \mathrm{H}^1(X, S_Y^\vee)_e
\end{array}$$

where both vertical arrows are isomorphisms of Propositions 1.1 and 1.2.

Let $\psi \in \mathrm{Hom}_k(S_X \otimes S_Y, \mu_{p^n})$. Let φ be the corresponding element in $\mathrm{Hom}(S_X, S_Y^\vee)$, and let $\varphi^\vee \in \mathrm{Hom}(S_Y, S_X^\vee)$ be its dual. By construction, the isomorphism (2) sends ψ to the image of $\tau^{-1}(\varphi^\vee) \in \mathrm{H}^1(X, S_Y^\vee)_e$ in $\mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}}$. On the other hand, $\varepsilon(\psi)$ is the value of the top pairing of the last diagram on $\varphi_*[\mathcal{T}_{X,p^n}] \in \mathrm{H}^1_{\text{ét}}(X, S_Y^\vee)_e$ and $[\mathcal{T}_{Y,p^n}] \in \mathrm{H}^1_{\text{ét}}(Y, S_Y)_e$. Since $\tau([\mathcal{T}_{Y,p^n}]) = \mathrm{id} \in \mathrm{Hom}(S_Y^\vee, S_Y^\vee)$, the commutativity of the diagram shows that $\varepsilon(\psi) \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\mathrm{prim}}$ comes from $\varphi_*[\mathcal{T}_{X,p^n}] \in \mathrm{H}^1_{\text{ét}}(X, S_Y^\vee)$. Since $\tau(\varphi_*[\mathcal{T}_{X,p^n}])$ is the precomposition of $\tau([\mathcal{T}_X]) = \mathrm{id} \in \mathrm{Hom}_k(S_X^\vee, S_X^\vee)$ with $\varphi^\vee \colon S_Y \to S_X^\vee$, we have $\tau(\varphi_*[\mathcal{T}_{X,p^n}]) = \varphi^\vee$. Thus (2) coincides with ε. □

2 Brauer group of the product

For an abelian group A the p-adic Tate module $T_p(A)$ is defined as the projective limit $\varprojlim A[p^n]$ when $n \to \infty$. It is easy to see that $T_p(\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}_p$ and that $T_p(M) = 0$ if the abelian group M has finite exponent.

Theorem 2.1 *Let X and Y be pointed smooth, projective, geometrically integral varieties over a finitely generated field k of characteristic $p > 0$. Then we have the following statements.*

(i) *The first Chern class gives an isomorphism*

$$\mathrm{Hom}_k(A, B^\vee) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{H}^2(X \times_k Y, \mathbb{Z}_p(1))_{\mathrm{prim}}.$$

(ii) *We have $T_p(\mathrm{Br}(X \times_k Y)_{\mathrm{prim}}) = 0$.*

(iii) *The abelian group $\mathrm{Br}(X \times_k Y)\{p\}_{\mathrm{prim}}$ has finite exponent.*

Proof. By a theorem of Chow (see [Con06, Thm. 3.19]), the natural map

$$\mathrm{Hom}_{k^s}(A_{k^s}, B_{k^s}^\vee) \to \mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, B_{\bar{k}}^\vee)$$

is an isomorphism. Hence we have natural isomorphisms:

$$\mathrm{Hom}_k(A, B^\vee) \xrightarrow{\sim} \mathrm{Hom}_{k^s}(A_{k^s}, B_{k^s}^\vee)^\Gamma \xrightarrow{\sim} \mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, B_{\bar{k}}^\vee)^\Gamma.$$

For a pointed projective, geometrically integral variety (X, x_0) the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})^\Gamma$ is an isomorphism [CTS21, Remark 5.4.3 (1)]. Thus we obtain from [CTS21, Prop. 5.7.3] an isomorphism of abelian groups

$$\mathrm{Pic}(X \times_k Y)_{\mathrm{prim}} \cong \mathrm{Hom}_k(A, B^\vee).$$

Thus the primitive part of the Kummer exact sequence can be written as

$$0 \to \mathrm{Hom}_k(A, B^\vee)/p^n \xrightarrow{c_1} \mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}} \to \mathrm{Br}(X \times_k Y)[p^n]_{\mathrm{prim}} \to 0.$$

The arrow marked c_1 is given by the first Chern class. Since $\mathrm{Hom}_k(A, B^\vee)$ is a finitely generated free abelian group, passing to the limit in n and using Corollary 1.4 we obtain an exact sequence

$$0 \to \mathrm{Hom}_k(A, B^\vee) \otimes \mathbb{Z}_p \xrightarrow{c_1} \mathrm{Hom}_k(A[p^\infty], B^\vee[p^\infty]) \to T_p(\mathrm{Br}(X \times_k Y)_{\mathrm{prim}}) \to 0. \quad (4)$$

De Jong's theorem (the crystalline Tate conjecture) [dJ98, Thm. 2.6] says that the natural action of morphisms of abelian varieties on torsion points induces an isomorphism

$$\mathrm{Hom}_k(A, B^\vee) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{Hom}_k(A[p^\infty], B^\vee[p^\infty]).$$

This implies that the source and the target of the map c_1 are finitely generated $\mathbb{Z}_p$-modules of the same rank. Since $T_p(\mathrm{Br}(X \times_k Y)_{\mathrm{prim}})$ is torsion-free, the map c_1 must be an isomorphism, so $T_p(\mathrm{Br}(X \times_k Y)_{\mathrm{prim}}) = 0$. This proves (i) and (ii).

Let us prove (iii). For a finite extension k'/k a standard restriction-corestriction argument [CTS21, Prop. 3.8.4] shows that the kernel of the natural map

$$\mathrm{Br}(X \times_k Y)_{\mathrm{prim}} \to \mathrm{Br}(X_{k'} \times_{k'} Y_{k'})_{\mathrm{prim}}$$

is annihilated by $[k' : k]$. Thus it is enough to prove (iii) after replacing k by a finite field extension. In particular, we can assume that we have an isomorphism

$$\mathrm{Hom}_k(A, B^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, B^\vee_{\bar{k}}).$$

Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\bar{k}}(A_{\bar{k}}, B^\vee_{\bar{k}})/p^n \arrow[r] & \mathrm{H}^2(X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}}, \mu_{p^n})_{\mathrm{prim}} \arrow[r] & \mathrm{Br}(X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}})[p^n]_{\mathrm{prim}} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_k(A, B^\vee)/p^n \arrow[r] \arrow[u, "\cong"] & \mathrm{H}^2(X \times_k Y, \mu_{p^n})_{\mathrm{prim}} \arrow[r] \arrow[u] & \mathrm{Br}(X \times_k Y)[p^n]_{\mathrm{prim}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Comparing isomorphisms (2) for k and $\bar{k}$, we see that the middle vertical map is injective. Now the snake lemma gives the injectivity of the right-hand map, hence $\mathrm{Br}(X \times_k Y)\{p\}_{\mathrm{prim}}$ is a subgroup of $\mathrm{Br}(X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}})\{p\}_{\mathrm{prim}}$. By Theorem A.1 of the appendix, the group $\mathrm{Br}(X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}})\{p\}$ is the direct sum of an abelian p-group of finite exponent and finitely many copies of $\mathbb{Q}_p/\mathbb{Z}_p$, hence the same is true for $\mathrm{Br}(X \times_k Y)\{p\}_{\mathrm{prim}}$. Thus (ii) implies (iii). □

Theorem 2.2 *Let X and Y be smooth, projective, geometrically integral varieties over a finitely generated field k. Then the cokernel of the natural map*

$$\mathrm{Br}(X) \oplus \mathrm{Br}(Y) \to \mathrm{Br}(X \times_k Y)$$

has finite exponent.

Proof. Since X and Y are smooth, there is a finite separable field extension $k \subset k'$ such that $X(k') \neq \emptyset$ and $Y(k') \neq \emptyset$. Moreover, we can assume that k'/k is Galois with Galois group G. There is an obvious commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X_{k'})^G \oplus \mathrm{Br}(Y_{k'})^G \arrow[r] & \mathrm{Br}(X_{k'} \times_{k'} Y_{k'})^G \\ \mathrm{Br}(X) \oplus \mathrm{Br}(Y) \arrow[r] \arrow[u] & \mathrm{Br}(X \times_k Y) \arrow[u] \end{tikzcd}$$

The kernels and cokernels of both vertical maps are annihilated by the order of G, see [CTS21, Prop. 3.8.4, Thm. 3.8.5]. By the commutativity of the diagram, it remains to show that the cokernel of the top horizontal map has finite exponent.

Since $X(k') \neq \emptyset$ and $Y(k') \neq \emptyset$, the direct sum decomposition of abelian groups (1) shows that we have an exact sequence of G-modules

$$0 \to \mathrm{Br}(k') \to \mathrm{Br}(X_{k'}) \oplus \mathrm{Br}(Y_{k'}) \to \mathrm{Br}(X_{k'} \times_{k'} Y_{k'}) \to \mathrm{Br}(X_{k'} \times_{k'} Y_{k'})_{\mathrm{prim}} \to 0.$$

Let B be the image of the middle arrow in this sequence. By Theorem 2.1 (for the p-primary part) and [SZ14, Thm. B] (for the prime-to-p part), there is a positive integer m that annihilates $\mathrm{Br}(X_{k'} \times_{k'} Y_{k'})_{\mathrm{prim}}$. Thus the exact sequence of G-modules

$$0 \to B \to \mathrm{Br}(X_{k'} \times_{k'} Y_{k'}) \to \mathrm{Br}(X_{k'} \times_{k'} Y_{k'})_{\mathrm{prim}} \to 0$$

shows that $m\,\mathrm{Br}(X_{k'} \times_{k'} Y_{k'})^G$ comes from B^G. Next, the exact sequence of G-modules

$$0 \to \mathrm{Br}(k') \to \mathrm{Br}(X_{k'}) \oplus \mathrm{Br}(Y_{k'}) \to B \to 0$$

gives rise to the exact sequence of cohomology groups

$$\mathrm{Br}(X_{k'})^G \oplus \mathrm{Br}(Y_{k'})^G \to B^G \to \mathrm{H}^1(G, \mathrm{Br}(k')).$$

The last group is annihilated by the order of G. This finishes the proof. $\square$

Corollary 2.3 *Let X and Y be smooth, projective, geometrically integral varieties over a finitely generated field k of characteristic exponent p. Then the cokernel of each of the following natural maps is a direct sum of a finite group and a p-group of finite exponent:*

(i) $\mathrm{Br}(X_{k^s})^\Gamma \oplus \mathrm{Br}(Y_{k^s})^\Gamma \to \mathrm{Br}(X_{k^s} \times_{k^s} Y_{k^s})^\Gamma$;
(ii) $\mathrm{Br}(X_{k^s})^k \oplus \mathrm{Br}(Y_{k^s})^k \to \mathrm{Br}(X_{k^s} \times_{k^s} Y_{k^s})^k$;
(iii) $\mathrm{Br}(X_{\bar{k}})^k \oplus \mathrm{Br}(Y_{\bar{k}})^k \to \mathrm{Br}(X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}})^k$.

Proof. (i) For every positive integer n coprime to p the group $\mathrm{Br}(X_{k^s})[n]$ is finite, see, e.g., [CTS21, Cor. 5.2.8]. Thus it remains to bound the exponent of the cokernel of the map in (i). We have a commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X_{k^s})^\Gamma \oplus \mathrm{Br}(Y_{k^s})^\Gamma \arrow[r] & \mathrm{Br}(X_{k^s} \times_{k^s} Y_{k^s})^\Gamma \\ \mathrm{Br}(X) \oplus \mathrm{Br}(Y) \arrow[r] \arrow[u] & \mathrm{Br}(X \times_k Y) \arrow[u] \end{tikzcd}$$

By [CTS21, Thm. 5.4.12], the cokernel of right-hand vertical map has finite exponent. By Theorem 2.2 the cokernel of the lower horizontal map has finite exponent. Now (i) follows from the commutativity of the diagram.

(ii) As in (i), it is enough to prove that the cokernel has finite exponent. This immediately follows from Theorem 2.2. The same proof gives (iii). □

Corollary 2.4 *Let X be a smooth, projective, geometrically integral surface over a finitely generated field k of characteristic exponent p. If X is dominated by a product of curves, then $\mathrm{Br}(X_{k^s})^\Gamma$ is a direct sum of a finite abelian group and a p-group of finite exponent.*

Proof. Using resolution of singularities (available in dimension 2 by a theorem of Abhyankar), and the triviality of the Brauer groups of curves over algebraically closed fields, we can follow the proof of [CTS21, Thm. 16.3.3], see also [GS22, §2.1], to obtain that $\mathrm{Br}(X_{k^s})(p')^\Gamma$ is finite and $\mathrm{Br}(X_{k^s})\{p\}^\Gamma$ has finite exponent. □

In higher dimension, the same statement holds conditionally on resolution of singularities in characteristic p.

Corollary 2.4 can be applied to the surface $X \subset \mathbb{P}^3_k$ given by $f(x_0, x_1) = g(x_2, x_3)$, where f and g are homogeneous polynomials of the same degree $d \geq 1$ without multiple roots. In this case the group $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is finite, see [CTS21, Cor. 16.3.4], [GS22, §2.1]. Thus, when k is finitely generated, the group $\big(\mathrm{Br}(X)/\mathrm{Br}_0(X)\big)(p')$ is finite and the group $\big(\mathrm{Br}(X)/\mathrm{Br}_0(X)\big)\{p\}$ has finite exponent.

3 Abelian varieties

The following lemma may be well-known to the experts; we give a proof because we could not find it in the literature.

Lemma 3.1 *Let A be an abelian variety over an algebraically closed field k. Let p be a prime, possibly equal to $\mathrm{char}(k)$. For any integer m the endomorphism $[m]\colon A \to A$ acts on $\mathrm{H}^2_{\mathrm{fppf}}(A, \mu_{p^n})$ as m^2 for any $n \geq 1$.*

Proof. In the case $p \neq \mathrm{char}(k)$ we can replace fppf cohomology by étale cohomology. Since $[m]$ acts on $\mathrm{H}^1_{\text{ét}}(A, \mu_{p^n}) \cong A^\vee(k)[p^n]$ as m, it acts on $\mathrm{H}^2_{\text{ét}}(A, \mu_{p^n}) \cong \wedge^2\mathrm{H}^1_{\text{ét}}(A, \mu_{p^n})(-1)$ as m^2.

Now let $p = \mathrm{char}(k)$. Considering the map $[p^n]\colon \mathcal{O}_A^\times \to \mathcal{O}_A^\times$ in the fppf and étale topologies gives rise to a canonical isomorphism [Ill79, (II.5.1.4)]

$$\mathrm{H}^i_{\mathrm{fppf}}(A, \mu_{p^n}) \cong \mathrm{H}^{i-1}_{\text{ét}}(A, \mathcal{O}_A^\times/O_A^{\times p^n}).$$

There is a map of étale sheaves of abelian groups $d\log\colon \mathcal{O}_A^\times/O_A^{\times p^n} \to W_n\Omega^1_X$, see [Ill79, Prop. I.3.23.2]. By [Ill79, Thm. II.1.4, (II.1.3.3)] for each $i \geq 0$ we have a canonical isomorphism $\mathrm{H}^i_{\mathrm{cris}}(A/W_n) \cong \mathrm{H}^i_{\text{ét}}(A, W_n\Omega^\bullet_A)$. We claim that the resulting map

$$d\log\colon \mathrm{H}^1_{\text{ét}}(A, \mathcal{O}_A^\times/O_A^{\times p^n}) \to \mathrm{H}^2_{\mathrm{cris}}(A/W_n)$$

is injective. We sketch the proof referring to [YY] for details.

The case $n = 1$ is stated in [Ill79, Remarque II.5.17 (a)]. It is a consequence of the following two facts:

(1) the map $\mathrm{H}^0(A, Z^1_X) \to \mathrm{H}^0(A, \Omega^1_X)$ is surjective, where $Z^1_A := \mathrm{Ker}[d\colon \Omega^1_A \to \Omega^2_A]$;

(2) the map $\mathrm{H}^1(A, Z^1_A) \to \mathrm{H}^2_{\mathrm{dR}}(A/k)$ induced by the natural morphism of complexes $Z^1_A[-1] \to \Omega^\bullet_A$ is injective.

Property (1) is true for any commutative group scheme A. Indeed, for invariant vector fields X and Y and an invariant differential ω we have

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) + \omega([X, Y]) = 0,$$

because $\omega(Y)$ and $\omega(X)$ are in k, and the Lie algebra of A is abelian.

The map in (2) factors as $\mathrm{H}^1(A, Z^1_A) \to \mathbb{H}^2(A, \Omega^{\geq 1}_A) \to \mathbb{H}^2(A, \Omega^\bullet_A) = \mathrm{H}^2_{\mathrm{dR}}(A/k)$. The second arrow is injective because for abelian varieties the Hodge-de Rham spectral sequence degenerates at the first page, by a theorem of Oda [Oda69, Prop. 5.1]. The injectivity of the first map can be easily checked using Čech cohomology.

The case of $n \geq 2$ follows by induction in n from the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2_{\mathrm{fppf}}(A, \mu_{p^m}) \arrow[r] \arrow[d, "d\log"] & \mathrm{H}^2_{\mathrm{fppf}}(A, \mu_{p^{m+n}}) \arrow[r] \arrow[d, "d\log"] & \mathrm{H}^2_{\mathrm{fppf}}(A, \mu_{p^n}) \arrow[d, "d\log"] \\
0 \arrow[r] & \mathrm{H}^2_{\mathrm{cris}}(A/W_m) \arrow[r] & \mathrm{H}^2_{\mathrm{cris}}(A/W_{m+n}) \arrow[r] & \mathrm{H}^2_{\mathrm{cris}}(A/W_n)
\end{tikzcd}$$

The zero in the top row is due to the natural isomorphism $\mathrm{H}^1_{\mathrm{fppf}}(A, \mu_{p^n}) \cong A^\vee(k)[p^n]$ and the surjectivity of multiplication by p^m on $A^\vee(k)$. The zero in the bottom row follows from the isomorphisms $\mathrm{H}^i_{\mathrm{cris}}(A/W_n) \cong \mathrm{H}^i_{\mathrm{cris}}(A/W)/p^n$ which are consequences of the fact that the groups $\mathrm{H}^i_{\mathrm{cris}}(A/W)$ are torsion-free W-modules.

A canonical isomorphism $\mathrm{H}^i_{\mathrm{cris}}(X/W) \cong \wedge^i \mathrm{H}^1_{\mathrm{cris}}(X/W)$ shows that $[m]$ acts on $\mathrm{H}^i_{\mathrm{cris}}(X/W)$ as m^i. Thus the proposition follows from the claim. □

We can use Theorem 2.1 to give a shorter proof of a result of D'Addezio [D'Ad, Thm. 5.2].

Theorem 3.2 *Let A be an abelian variety over a finitely generated field k of characteristic exponent p. Then $\mathrm{Br}(A_{\bar{k}})^k$ is a direct sum of a finite group and a p-group of finite exponent.*

Proof. Let $m\colon A \times_k A \to A$ be the group law of A. Define $\delta\colon \mathrm{Br}(A) \to \mathrm{Br}(A \times A)$ as $m^* - \pi_1^* - \pi_2^*$. It is immediate to check that $\delta(\mathrm{Br}(A)_e) \subset \mathrm{Br}(A \times A)_{\mathrm{prim}}$. By [OSVZ22, Lemma 2.1, Prop. 2.2] we have an exact sequence

$$0 \to \mathrm{Br}(A)_e \cap \mathrm{Br}_A(A) \to \mathrm{Br}(A)_e \xrightarrow{\delta} \mathrm{Br}(A \times A)_{\mathrm{prim}}, \tag{5}$$

where $\mathrm{Br}_A(A)$ is the invariant Brauer group of A. The group $\mathrm{Br}(A\times A)_{\mathrm{prim}}$ has finite exponent by Theorem 2.2. The image of $\mathrm{Br}_A(A)$ in $\mathrm{Br}(A_{\bar{k}})$ is contained in $\mathrm{Br}_A(A_{\bar{k}})$, but $\mathrm{Br}_A(A_{\bar{k}})$ has exponent 2. Indeed, on the one hand, by Lemma 3.1 and the Kummer exact sequence, $[-1]^*$ acts on $\mathrm{Br}(A_{\bar{k}})$ trivially. On the other hand, $[-1]^*$ acts on $\mathrm{Br}_A(A_{\bar{k}})$ as -1, see [OSVZ22, Prop. 2.2]. We conclude from (5) that $\mathrm{Br}(A_{\bar{k}})^k$ has finite exponent. It remains to use the finiteness of $\mathrm{Br}(A_{\bar{k}})[n]$ where n is coprime to p, see [CTS21, Cor. 5.2.8]. $\square$

Remark 3.3 Since the Picard scheme of an abelian variety is smooth, the natural map $\mathrm{Br}(A_{k^s})\to\mathrm{Br}(A_{\bar{k}})$ is injective [CTS21, Thm. 5.2.5 (ii)], [D'Ad, Cor. 3.4], thus $\mathrm{Br}(A_{k^s})^k\cong\mathrm{Br}(A_{\bar{k}})^k$. By [CTS21, Thm. 5.4.12] we conclude from Theorem 3.2 that $\mathrm{Br}(A_{k^s})^\Gamma$ is a direct sum of a finite group and a p-group of finite exponent.

Using similar ideas, we can give a simplified proof of the flat version of the Tate conjecture for divisors proved by D'Addezio in [D'Ad, Thm. 5.1].

Theorem 3.4 *Let A be an abelian variety over a finitely generated field k of characteristic $p>0$. The image of $\mathrm{H}^2(A,\mathbb{Z}_p(1))$ in $\mathrm{H}^2(A_{\bar{k}},\mathbb{Z}_p(1))^\Gamma$ is contained in the image of the first Chern class map $\mathrm{c}_1\colon \mathrm{NS}\,(A_{\bar{k}})^\Gamma\otimes\mathbb{Z}_p\to\mathrm{H}^2(A_{\bar{k}},\mathbb{Z}_p(1))^\Gamma$. After replacing k with a finite separable extension, the two images become equal.*

Proof. We continue to write $\delta=m^*-\pi_1^*-\pi_2^*$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(A,\mathbb{Z}_p(1))_e \arrow[r] \arrow[d, "\delta"] & \mathrm{H}^2(A_{\bar{k}},\mathbb{Z}_p(1))^\Gamma \arrow[d, "\delta"] & \mathrm{NS}\,(A_{\bar{k}})^\Gamma\otimes\mathbb{Z}_p \arrow[l, "\mathrm{c}_1"'] \arrow[d, "\cong"', "\delta"] \\
\mathrm{H}^2(A\times_k A,\mathbb{Z}_p(1))^{\mathrm{sym}}_{\mathrm{prim}} \arrow[r] \arrow[urr, dotted] & \mathrm{H}^2(A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}},\mathbb{Z}_p(1))^{\mathrm{sym},\Gamma}_{\mathrm{prim}} & \mathrm{NS}\,(A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}})^{\mathrm{sym},\Gamma}_{\mathrm{prim}}\otimes\mathbb{Z}_p \arrow[l, "\mathrm{c}_1"']
\end{tikzcd}$$

where the superscript 'sym' stands for the elements fixed by the permutation of factors in $A\times_k A$ and $A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}}$. To prove the first statement it is enough to construct the dotted line such that the resulting diagram is still commutative.

Theorem 2.1 (i) gives an isomorphism

$$\mathrm{Hom}_k(A,A^\vee)\otimes\mathbb{Z}_p\xrightarrow{\sim}\mathrm{Pic}(A\times_k A)_{\mathrm{prim}}\otimes\mathbb{Z}_p\xrightarrow{\sim}\mathrm{H}^2(A\times_k A,\mathbb{Z}_p(1))_{\mathrm{prim}}. \tag{6}$$

Here the first arrow sends $f\in\mathrm{Hom}_k(A,A^\vee)$ to $(\mathrm{id},f)^*\mathcal{P}$, where $\mathcal{P}$ is the Poincaré line bundle on $A\times_k A^\vee$. The second arrow is the first Chern class c_1. If $f=f^\vee$, then the image of f lands in the symmetric subgroup of $\mathrm{H}^2(A\times_k A,\mathbb{Z}_p(1))_{\mathrm{prim}}$. The same construction over $\bar{k}$ gives an isomorphism of Γ-modules

$$\mathrm{Hom}(A_{\bar{k}},A_{\bar{k}}^\vee)\otimes\mathbb{Z}_p\xrightarrow{\sim}\mathrm{NS}\,(A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}})_{\mathrm{prim}}\otimes\mathbb{Z}_p\cong\mathrm{Pic}(A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}})_{\mathrm{prim}}\otimes\mathbb{Z}_p,$$

which is clearly compatible with the first map of (6) and which gives this map after restricting to the Γ-invariant subgroups. We finally note that the isomorphism of Γ-modules $\mathrm{Hom}(A_{\bar{k}},A_{\bar{k}}^\vee)^{\mathrm{sym}}\cong\mathrm{NS}\,(A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}})^{\mathrm{sym}}_{\mathrm{prim}}$ identifies the map of Γ-modules

$$\delta\colon\mathrm{NS}\,(A_{\bar{k}})\to\mathrm{NS}\,(A_{\bar{k}}\times_{\bar{k}} A_{\bar{k}})^{\mathrm{sym}}_{\mathrm{prim}}$$

with the isomorphism $\mathrm{NS}\,(A_{\bar{k}}) \xrightarrow{\sim} \mathrm{Hom}(A_{\bar{k}}, A_{\bar{k}}^\vee)^{\mathrm{sym}}$ sending L to φ_L. (This follows from $(\mathrm{id}, \varphi_L)^*\mathcal{P} = m^*L \otimes \pi_1^*L^{-1} \otimes \pi_2^*L^{-1}$, see [Mum74, Ch. 8], cf. [OSVZ22, Prop. 6.1].) Putting all of this together gives rise to a dotted line in the diagram, which is the identity map on $\mathrm{Hom}_k(A, A^\vee)^{\mathrm{sym}} \otimes \mathbb{Z}_p$ once the source and the target are identified with this group. The resulting diagram commutes.

Since $\mathrm{NS}\,(A_{\bar{k}})$ is finitely generated, replacing k by a finite separable extension we can ensure that the map $\mathrm{Pic}(A) \to \mathrm{NS}\,(A_{\bar{k}})^\Gamma$ is surjective. Then the image of the first Chern class map $\mathrm{c}_1 \colon \mathrm{NS}\,(A_{\bar{k}})^\Gamma \otimes \mathbb{Z}_p \to \mathrm{H}^2(A_{\bar{k}}, \mathbb{Z}_p(1))^\Gamma$ is contained in the image of $\mathrm{H}^2(A, \mathbb{Z}_p(1)) \to \mathrm{H}^2(A_{\bar{k}}, \mathbb{Z}_p(1))^\Gamma$. The second statement follows. □

4 Kummer surfaces

Recall that the Picard scheme of a K3 surface is smooth, hence the natural map $\mathrm{Br}(X_{k^{\mathrm{s}}}) \to \mathrm{Br}(X_{\bar{k}})$ is injective [CTS21, Thm. 5.2.5 (ii)], [D'Ad, Cor. 3.4]. This implies $\mathrm{Br}(X_{k^{\mathrm{s}}})^k \cong \mathrm{Br}(X_{\bar{k}})^k$.

Proposition 4.1 *Let k be a field of characteristic exponent $p \neq 2$. Let A be an abelian surface and let $X = \mathrm{Kum}(A)$ be the associated Kummer surface. Then there is a natural isomorphism of Γ-modules $\mathrm{Br}(X_{\bar{k}}) \xrightarrow{\sim} \mathrm{Br}(A_{\bar{k}})$.*

Proof. For all primes $\ell \neq p$ (including $\ell = 2$) the proof of [SZ12, Prop. 1.3] shows that $\mathrm{Br}(X_{\bar{k}})\{\ell\} \to \mathrm{Br}(A_{\bar{k}})\{\ell\}$ is an isomorphism. In fact, for any $\ell \neq 2$ (including $\ell = p$ if $p > 1$) the map $\mathrm{Br}(X_{\bar{k}})\{\ell\} \to \mathrm{Br}(A_{\bar{k}})\{\ell\}$ is injective with image $\mathrm{Br}(A_{\bar{k}})\{\ell\}^{[-1]^*}$ by [CTS21, Thm. 3.8.5]. In view of the Kummer sequence, it suffices to show that $[-1]$ acts on $\mathrm{H}^2_{\mathrm{fppf}}(A_{\bar{k}}, \mu_{\ell^n})$ trivially. This was proved in Lemma 3.1. □

Corollary 4.2 *Let k be a field of characteristic exponent $p \neq 2$. Let A be an abelian surface and let $X = \mathrm{Kum}(Y)$ be the associated Kummer surface. For all odd primes ℓ (including $\ell = p$ if $p > 1$) there are natural isomorphisms of abelian groups*

$$\mathrm{Br}(X_{\bar{k}})\{\ell\}^k \xrightarrow{\sim} \mathrm{Br}(A_{\bar{k}})\{\ell\}^k.$$

Proof. This is proved in [SZ12, Thm. 2.4]. Let us give this proof for the convenience of the reader. By Proposition 4.1, the map is injective, so it remains to prove that it is surjective. For any odd ℓ we have a direct sum decomposition

$$\mathrm{Br}(A)\{\ell\} = \mathrm{Br}(A)\{\ell\}^+ \oplus \mathrm{Br}(A)\{\ell\}^-,$$

where $\mathrm{Br}(A)\{\ell\}^+ = \mathrm{Br}(A)\{\ell\}^{[-1]^*}$ is the $[-1]^*$-invariant subgroup and $\mathrm{Br}(A)\{\ell\}^-$ is the $[-1]^*$-antiinvariant subgroup. By the proof of Proposition 4.1, the action of $[-1]$ on $\mathrm{Br}(A_{\bar{k}})$ is trivial, thus any element of $\mathrm{Br}(A_{\bar{k}})\{\ell\}^k$ lifts to an element of $\mathrm{Br}(A)\{\ell\}^+$. The last group is the image of $\mathrm{Br}(X)\{\ell\}$ by [CTS21, Thm. 3.8.5]. □

Corollary 4.3 *Let k be a finitely generated field of characteristic exponent $p \neq 2$. Let A be an abelian surface and let $X = \mathrm{Kum}(A)$ be the associated Kummer surface. Then each of the groups $\mathrm{Br}(X_{k^s})^\Gamma$ and $\mathrm{Br}(X_{k^s})^k \cong \mathrm{Br}(X_{\bar{k}})^k$ is a direct sum of a finite group and and a p-group of finite exponent.*

Proof. For any K3 surface X over k, the finiteness of $\mathrm{Br}(X_{k^s})(p')^\Gamma$ and $\mathrm{Br}(X_{k^s})(p')^k$ was proved in [SZ08, Thm. 1.2] when $p = 1$ and in [SZ15, Thm. 1.3] for $p > 2$. For $p > 2$ the statements for the p-primary torsion follow from Theorem 3.2 and Corollary 4.2. □

Example 1. The group $\mathrm{Br}(A_{k^s})[p]^k$ may well be infinite. Let us reproduce here the example in [D'Ad, Cor. 5.4]. Let E be a supersingular elliptic curve over an *infinite* finitely generated field k of characteristic $p > 0$, and let $A = E \times_k E$. The group scheme $E[p]$ is an extension of α_p by α_p, hence there is an injective map of abelian groups $\mathrm{End}_k(\alpha_p) \to \mathrm{End}_k(E[p])$ which sends an endomorphism $\phi\colon \alpha_p \to \alpha_p$ to the composition

$$E[p] \to \alpha_p \xrightarrow{\phi} \alpha_p \to E[p].$$

By Theorem 1.3 we have $\mathrm{H}^2(A, \mu_p)_{\mathrm{prim}} \cong \mathrm{End}_k(E[p])$, hence

$$\mathrm{Br}(A)[p]_{\mathrm{prim}} \cong \mathrm{End}_k(E[p])/\big(\mathrm{End}_k(E)/p\big).$$

Since $\mathrm{End}_k(\alpha_p) \cong k$, we have an injective homomorphism $k \to \mathrm{End}_k(E[p])$, hence compatible homomorphisms $k \to \mathrm{Br}(A)[p]$ and $\bar{k} \to \mathrm{Br}(A_{\bar{k}})[p]$. Since $\mathrm{End}_{\bar{k}}(E)/p$ is finite, we have infinitely many elements of $\mathrm{Br}(A)[p]$ surviving in $\mathrm{Br}(A_{\bar{k}})[p]$. Now let $p \neq 2$. Then we can consider the Kummer surface $X = \mathrm{Kum}(A)$ over k. Corollary 4.2 implies that $\mathrm{Br}(X_{k^s})[p]^k \cong \mathrm{Br}(X_{\bar{k}})[p]^k$ is infinite. This gives an example of a K3 surface with an infinite transcendental Brauer group, answering [SZ08, Questions 1, 2] in the negative.

Example 2. D'Addezio also gives an example to show that in the case of finite characteristic, the group $\mathrm{Br}(A_{\bar{k}})^\Gamma$ does not always have finite exponent [D'Ad, Cor. 6.7]. Take $A = E \times_k E$, where E is an elliptic curve over k whose j-invariant is transcendental over $\mathbb{F}_p$. Then E is ordinary and we have $\mathrm{End}_{\bar{k}}(E) \cong \mathbb{Z}$. Then $T_p(\mathrm{Br}(A_{\bar{k}}))$ contains the quotient of $\mathrm{End}(E_{\bar{k}}[p^\infty])$ by $\mathrm{End}(E_{\bar{k}}) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$. Taking Galois invariants we obtain that $T_p(\mathrm{Br}(A_{\bar{k}}))^\Gamma$ contains the quotient of $\mathrm{End}(E_{\bar{k}}[p^\infty])^\Gamma$ by $\mathbb{Z}_p$, so it is enough to show that the rank of the $\mathbb{Z}_p$-module $\mathrm{End}(E_{\bar{k}}[p^\infty])^\Gamma$ is at least 2. Since E is ordinary, the p-divisible group $E[p^\infty]$ has at least two slopes. By the Dieudonné–Manin classification, this implies that $E[p^\infty]$ is isogenous to the direct sum of two non-zero p-divisible groups, hence the rank of $\mathrm{End}(E_{\bar{k}}[p^\infty])^\Gamma$ is at least 2. As before, if $p \neq 2$, then for $X = \mathrm{Kum}(A)$ we obtain from Proposition 4.1 that $\mathrm{Br}(X_{\bar{k}})^\Gamma$ does not have finite exponent.

A Appendix, by Alexander Petrov

Let k be an algebraically closed field of characteristic $p > 0$. We write $W = W(k)$ for the ring of Witt vectors of k and K for the field of fractions of W.

For a smooth proper variety X over k we denote by $\rho = \dim_{\mathbb{Q}}(\mathrm{NS}\,(X) \otimes \mathbb{Q})$ the Picard number of X.

For $i \geq 0$, let r_i be the dimension of the $\mathbb{Q}_p$-vector space $(\mathrm{H}^i_{\mathrm{cris}}(X/W) \otimes K)^{F=p}$.

Consider the complex of weight 1 syntomic cohomology of X:

$$R\Gamma(X, \mathbb{Z}_p(1)) := R\varprojlim \left(R\Gamma_{\mathrm{fppf}}(X, \mu_{p^n})\right)$$

Here $R\varprojlim$ is the derived inverse limit [Stacks, 08TC] of the system of objects $R\Gamma_{\mathrm{fppf}}(X, \mu_{p^n}) \in D(\mathbb{Z}_p)$ of the derived category of $\mathbb{Z}_p$-modules. In fact, each individual cohomology group $\mathrm{H}^i(X, \mathbb{Z}_p(1)) := \mathrm{H}^i(R\Gamma(X, \mathbb{Z}_p(1)))$ is isomorphic to $\varprojlim \mathrm{H}^i_{\mathrm{fppf}}(X, \mu_{p^n})$ by the proof of [Ill79, Thm. II.5.5].

Theorem A.1 *Let X be a smooth proper variety over k. Then*

$$\mathrm{Br}(X)\{p\} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus(r_2-\rho)} \oplus \mathrm{H}^3(X, \mathbb{Z}_p(1))\{p\},$$

where the group $\mathrm{H}^3(X, \mathbb{Z}_p(1))\{p\}$ is annihilated by a power of p.

Syntomic cohomology modules $\mathrm{H}^i(X, \mathbb{Z}_p(1))$ are finitely generated over $\mathbb{Z}_p$ for $i \leq 2$, see [Ill79, Prop. II.5.9], but not always for $i \geq 3$ (cf. the example of a supersingular K3 surface in [Ill79, II.7.2]). However, they satisfy a weaker finiteness property that we will use to deduce Theorem A.1:

Lemma A.2 (Illusie–Raynaud) *For each $i \geq 0$, the $\mathbb{Z}_p$-module $\mathrm{H}^i(X, \mathbb{Z}_p(1))$ is isomorphic to $\mathbb{Z}_p^{\oplus r_i} \oplus \mathrm{H}^i(X, \mathbb{Z}_p(1))\{p\}$, where $\mathrm{H}^i(X, \mathbb{Z}_p(1))\{p\}$ is a $\mathbb{Z}_p$-module annihilated by a power of p.*

Proof. The statement is clear for $i = 0$, so we assume $i \geq 1$.

By [IR83, Cor. IV.3.5 (a)], for $i, j \geq 0$ we have a short exact sequence

$$0 \to \mathrm{H}^j(X, W\Omega^i_{X,\log}) \to \mathrm{H}^j_{\mathrm{Zar}}(X, W\Omega^i_X) \xrightarrow{1-F} \mathrm{H}^j_{\mathrm{Zar}}(X, W\Omega^i_X) \to 0,$$

where $W\Omega^i_X$ is the sheaf of de Rham–Witt differential forms, and $F : W\Omega^i_X \to W\Omega^i_X$ is its semi-linear Frobenius endomorphism [Ill79, I.2.E]. By definition,

$$\mathrm{H}^j(X, W\Omega^i_{X,\log}) := \varprojlim \mathrm{H}^j_{\mathrm{Zar}}(X, W_n\Omega^i_{X,\log}),$$

where $W_n\Omega^i_{X,\log}$ is a subsheaf of 'logarithmic' forms in $W_n\Omega^i_X$. In view of a canonical isomorphism $\mathrm{H}^i(X, \mathbb{Z}_p(1)) \cong \mathrm{H}^{i-1}(X, W\Omega^1_{X,\log})$ [Ill79, §II.5], we obtain an isomorphism

$$\mathrm{H}^i(X, \mathbb{Z}_p(1)) \cong \mathrm{H}^{i-1}(X, W\Omega^1_X)^{F=1}.$$

While $\mathrm{H}^j_{\mathrm{Zar}}(X, W\Omega^i_X)$ is not always finitely generated as a W-module, it is isomorphic to a direct sum of a finitely generated free W-module and a module annihilated by a power of p, by [Ill79, Thm. II.2.13].

If M is a finitely generated free W-module equipped with a Frobenius-linear endomorphism $F : M \to M$, then $M^{F=1} := \mathrm{Ker}[1 - F : M \to M]$ is a finitely generated free $\mathbb{Z}_p$-module because the natural map $M^{F=1} \otimes_{\mathbb{Z}_p} W \to M$ is an injection, as follows for example from the Dieudonné–Manin classification. This implies that $\mathrm{H}^i(X, \mathbb{Z}_p(1))$ is isomorphic to $\mathbb{Z}_p^{\oplus r} \oplus \mathrm{H}^i(X, \mathbb{Z}_p(1))\{p\}$ for some $r \geq 0$, and $\mathrm{H}^i(X, \mathbb{Z}_p(1))\{p\}$ is annihilated by a power of p.

By [Ill79, Thm. II.5.5 (5.5.3)] we have an isomorphism of $\mathbb{Q}_p$-vector spaces

$$\mathrm{H}^i(X, \mathbb{Z}_p(1)) \otimes \mathbb{Q}_p \cong (\mathrm{H}^i_{\mathrm{cris}}(X/W) \otimes K)^{F=p},$$

thus $r = r_i$. $\square$

Proof of Theorem A.1 For each n, we have a distinguished triangle

$$R\Gamma(X, \mathbb{Z}_p(1)) \xrightarrow{p^n} R\Gamma(X, \mathbb{Z}_p(1)) \to R\Gamma_{\mathrm{fppf}}(X, \mu_{p^n}) \tag{7}$$

obtained from the distinguished triangles

$$R\Gamma_{\mathrm{fppf}}(X, \mu_{p^m}) \xrightarrow{p^n} R\Gamma_{\mathrm{fppf}}(X, \mu_{p^{n+m}}) \to R\Gamma_{\mathrm{fppf}}(X, \mu_{p^n})$$

by passing to the inverse limit over all m. For all i, n the triangle (7) induces the short exact sequences

$$0 \to \mathrm{H}^i(X, \mathbb{Z}_p(1))/p^n \to \mathrm{H}^i(X, \mu_{p^n}) \to \mathrm{H}^{i+1}(X, \mathbb{Z}_p(1))[p^n] \to 0. \tag{8}$$

For each i, passing to the direct limit along the maps induced by $\mu_{p^n} \hookrightarrow \mu_{p^{n+1}}$ we get the short exact sequence

$$0 \to \mathrm{H}^i(X, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \varinjlim \mathrm{H}^i(X, \mu_{p^n}) \to \mathrm{H}^{i+1}(X, \mathbb{Z}_p(1))\{p\} \to 0. \tag{9}$$

By Lemma A.2 the group $\mathrm{H}^i(X, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus r_i}$, and the group $\mathrm{H}^{i+1}(X, \mathbb{Z}_p(1))\{p\}$ is annihilated by a power of p. The abelian group $\mathbb{Q}_p/\mathbb{Z}_p$ is divisible, hence injective, thus $\varinjlim \mathrm{H}^i(X, \mu_{p^n})$ is isomorphic to the direct sum of $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus r_i}$ and $\mathrm{H}^{i+1}(X, \mathbb{Z}_p(1))\{p\}$.

On the other hand, for all n we have short exact sequences

$$0 \to \mathrm{Pic}(X)/p^n \to \mathrm{H}^2(X, \mu_{p^n}) \to \mathrm{Br}(X)[p^n] \to 0 \tag{10}$$

which induce, after passing to the direct limit, a surjection $\varinjlim \mathrm{H}^2(X, \mu_{p^n}) \to \mathrm{Br}(X)\{p\}$ with kernel $\varinjlim \mathrm{Pic}(X)/p^n \cong (\mathbb{Q}_p/\mathbb{Z}_p)^\rho$. This proves the theorem. $\square$

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