1. Transmutation

1. Consider the R-algebra scheme D over R defined by $D(A) = A[\varepsilon]/\varepsilon^2$ (as a scheme, D is therefore isomorphic to $\mathbb{A}^2_R$). For a smooth R-scheme X describe the transmutation X^D.

2. Consider the Witt vector scheme W_2 as a $W_2(k)$-algebra scheme over k, for a perfect field k of characteristic p. For a smooth scheme X over $W_2(k)$ prove that transmutation X^{W_2} is represented by a smooth k-scheme and describe it in more familiar terms.

3. Consider the ring groupoid $\mathrm{cone}(R \xrightarrow{0} R)$ over a base ring R. It defines a stack $\mathcal{R}$ of R-algebras over R sending A to $A \otimes_R \mathrm{cone}(R \xrightarrow{0} R) \simeq \mathrm{cone}(A \xrightarrow{0} A)$. Describe the fpqc sheafification of the stack $X^{\mathcal{R}}$.

2. Cartier duality

1. Construct an equivalence of abelian categories $\mathrm{QCoh}(B\widehat{\mathbb{G}_a^\sharp}) \simeq \mathrm{QCoh}(\mathbb{A}^1)$. Apply this to finish the proof of the fact that $\mathcal{D}_{\mathbb{A}^1}$-modules are equivalent to $\widehat{\mathbb{G}_a^\sharp}$-equivariant sheaves on $\mathbb{A}^1$.

2. Working over a perfect field of characteristic p, calculate the kernel and cokernel of the group scheme map $\widehat{\mathbb{G}_a^\sharp} \to \mathbb{G}_a$.

3. Construct an equivalence of formal group schemes $\widehat{\mathbb{G}_a^\sharp} \simeq \widehat{\mathbb{G}_m^\sharp}$ over $\mathbb{Z}$.

4. Construct a surjective homorphism of group schemes $\mathbb{G}_m^\sharp \to \mathbb{G}_a^\sharp$ over $\operatorname{Spf} \mathbb{Z}_p$ and calculate its kernel.

3. Simplicial rings

1. Let R be any base ring. Construct an equivalence of simplicial R-algebras $\mathrm{cone}(R \xrightarrow{0} R) \simeq \mathrm{Sym}_R R[1]$.

2. For each $i \geq 0$, calculate the Frobenius on homotopy groups of the simplicial $\mathbb{F}_p$-algebra $\mathrm{Sym}\, \mathbb{F}_p[i]$.

3. Let A be any commutative $\mathbb{F}_p$-algebra. Calculate the homotopy groups of $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Construct an equivalence between sections of the natural map of simplicial $\mathbb{F}_p$-algebras $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p \to A$ and flat $\mathbb{Z}/p^2$-algebras $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/p \simeq A$.

4. de Rham stacks

1. For a smooth scheme X over a base ring R identify the stack $X^{\widehat{\mathrm{dR}}} := X^{\widehat{\mathbb{G}_a^{\mathrm{dR}}}}$ with the quotient of X by the relation given by the completed divided power envelope $(X \times X)^{\widehat{\sharp}}_{\Delta}$ of the diagonal in $X \times X$.

2. Let G be a smooth group scheme over R. Construct an isomorphism between $G^{\widehat{\mathrm{dR}}}$ and the quotient stack $[G/\widehat{G^\sharp}]$ of G by the divided power envelope of the unit in G.

3. Working over $R = k$ a perfect field of characteristic p describe the root stack of $\widehat{\mathbb{G}_a^{\mathrm{dR}}}$. Apply this to calculate the center of the algebra $\mathcal{D}_{\mathbb{A}^1_k}$ of differential operators on the affine line: via the equivalence $D_{\mathbb{A}^1_k}\text{-mod} \simeq \mathrm{QCoh}(\widehat{\mathbb{G}_a^{\mathrm{dR}}})$ this is equivalent to describing the center of the category of quasi-coherent sheaves on $\widehat{\mathbb{G}_a^{\mathrm{dR}}}$.

4. For any smooth scheme X over k calculate the root stack of $X^{\widehat{\mathrm{dR}}}$. Use this to define, for every $\mathcal{D}_X$-module M a natural map $F^*T_X \to \mathrm{End}(M)$. Show that your definition coincides with the classically defined p-curvature map.

5. Variations on $\mathcal{D}$-modules

1. Let X be a smooth scheme over $W(k)$. Consider the algebra $\mathcal{D}^1_X := \mathcal{D}_X\langle \frac{\partial_v^p - \partial_{v^{\widetilde{[p]}}}}{p} | v \in T_X\rangle \subset \mathcal{D}_X \otimes \mathbb{Q}$ where we divided by p all elements in the vertical part of the center of $\mathcal{D}_X/p$. Construct a ring stack $\mathbb{G}_a^{\widehat{\mathrm{dR}},1}$ such that $\mathrm{QCoh}(X^{\mathbb{G}_a^{\widehat{\mathrm{dR}},1}})$ is equivalent to $\mathcal{D}^1_X$-modules.

2. Let R be any base ring, equipped with an element $h \in R$. Consider the R-algebra stack $\mathbb{G}_a^{\widehat{\mathrm{dR}},h} := \mathrm{cone}(\mathbb{G}_a^{\widehat{\sharp}} \xrightarrow{h} \mathbb{G}_a)$ over R. Prove that for a smooth R-scheme X the category $\mathrm{QCoh}(X^{\mathbb{G}_a^{\widehat{\mathrm{dR}},h}})$ is equivalent to the category of $h-D$-modules on X.

3. Let X be a smooth formal scheme over $\mathrm{Spf}\,\mathbb{C}[[t]]$. Prove that the category of D-modules on X (relative to $\mathbb{C}[[t]]$) naturally depends only on $X_{\mathbb{C}[[t]]/t}$.

4. Let X be a smooth formal scheme over $\mathrm{Spf}\,\mathbb{C}[[t]]$. Prove that the category of t-D-modules on X naturally depends only on $X_{\mathbb{C}[[t]]/t^2}$.

5. Let X be a smooth scheme over a $\mathbb{F}_p$. Consider the $\mathbb{F}_p$-algebra stack $\mathbb{G}_a \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p$ where the $\mathbb{F}_p$-algebra structure is defined via the second tensor functor.

Prove that the map of stacks $X^{\mathbb{G}_a \otimes_{\mathbb{Z}_p} \mathbb{F}_p} \to X^{\mathbb{G}_a} = X$ has a section if and only if X admits a lift over $\mathbb{Z}/p^2$.