On spectral sequences for semiabelian varieties over non-closed fields

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November 22, 2024

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Abstract

We give a new, short proof of the formula for the first potentially non-zero differential of the Hochschild–Serre spectral sequence for semiabelian varieties over non-closed fields. We show that this differential is non-zero for the Jacobian of a curve when the image of the torsor of theta-characteristics under the Bockstein map is non-zero. An explicit example is a curve of genus 2 whose Albanese torsor is not divisible by 2. When the Albanese torsor is trivial, we show that the Hochschild–Serre spectral sequence for the Jacobian degenerates at the second page. We give a formula for the differential of the Hochschild–Serre spectral sequence for a torus which computes its Brauer group. Finally, we describe the differentials of the Hochschild–Serre spectral sequence for a smooth projective curve, generalising a lemma of Suslin.

Introduction

For an algebraic variety X over a field k the action of the absolute Galois group $\Gamma = \mathrm{Gal}(k_s/k)$ on the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(X_{k_s}, \mathbb{Z}_\ell)$ contains arithmetic information about the variety X such as the number of rational points in the case when k is a finite field.

The action of Γ on individual étale cohomology groups comes from its action on the étale cohomology complex $\mathrm{R}\Gamma_{\text{ét}}(X_{k_{\mathrm{s}}}, \mathbb{Z}_\ell)$. The phenomenon that we study in this paper is that the quasi-isomorphism class of this complex in general contains more information than individual cohomology Galois modules.

Remarkably, for a smooth projective variety X over k (with $\ell \neq \mathrm{char}(k)$), by Deligne's decomposition theorem [Del68, Proposition 2.4], the rational étale cohomology complex $\mathrm{R}\Gamma_{\text{ét}}(X_{k_{\mathrm{s}}}, \mathbb{Q}_\ell)$ is quasi-isomorphic to the direct sum $\bigoplus_{i\geq 0} \mathrm{H}^i_{\text{ét}}(X_{k_{\mathrm{s}}}, \mathbb{Q}_\ell)[-i]$ of individual cohomology groups, as an object of the derived category of $\mathbb{Q}_\ell$-vector spaces equipped with an action of Γ.

This is no longer true for cohomology with integral coefficients, and is witnessed by the fact that the Hochschild–Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X_{k_{\mathrm{s}}}, \mathbb{Z}_\ell)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Z}_\ell) \tag{1}$$

in general has non-zero differentials. If X is a real curve with no real points and $\ell = 2$, then a differential on the 2nd (respectively, 3rd) page is non-zero if the genus of X is odd (respectively, even), see Examples 6.2 and 6.7.

Our first main result is that (1) may fail to degenerate even for abelian varieties. We denote by $D(\Gamma, \mathbb{Z}/n)$ the derived category of the abelian category of discrete $\mathbb{Z}/n$-modules with a continuous action of the Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$.

Theorem 1 (Theorem 3.4)**.** *There exists a principally polarised abelian surface A over $\mathbb{Q}$ such that $\mathrm{R}\Gamma_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Z}/2)$ is not quasi-isomorphic to $\bigoplus_{i\geq 0} \mathrm{H}^i_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Z}/2)[-i]$ in the derived category $D(\Gamma, \mathbb{Z}/2)$ of discrete $\mathbb{Z}/2$-modules with a continuous action of $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Specifically, the differential*

$$\delta_2^{0,2} \colon \mathrm{H}^0(\mathbb{Q}, \mathrm{H}^2_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Z}/2)) \to \mathrm{H}^2(\mathbb{Q}, \mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Z}/2))$$

on the second page of the Hochschild–Serre spectral sequence for A is non-zero.

By Theorem 3 below, the Hochschild-Serre spectral sequence degenerates for any abelian variety arising as a direct factor of a product of Jacobians of curves each with a rational divisor class of degree 1. In particular, A from Theorem 1 is an example of an abelian variety not of this form.

Let us sketch the proof of Theorem 1. Let C be a smooth proper curve of genus g over k and let J be the Jacobian of C. We identify J with the dual abelian variety using its canonical principal polarization. Using a general description of $\delta_2^{0,2}$ given in Theorem 2 below, we show that the mod 2 first Chern class of the canonical principal polarization is an element of $\mathrm{H}^2_{\text{ét}}(J_{k_{\mathrm{s}}}, \mathbb{Z}/2)^\Gamma$ which is sent by $\delta_2^{0,2}$ to the image of the class of the torsor of theta-characteristics under the Bockstein map $\mathrm{H}^1(k, J[2]) \to \mathrm{H}^2(k, J[2])$. Thus $\delta_2^{0,2} \neq 0$ whenever $[\mathbf{Pic}^{g-1}_{C/k}] \in \mathrm{H}^1(k, J)$ is not divisible by 2. A systematic method to construct such curves over number fields was found by B. Creutz [Cre13]. The idea is to exploit the fine arithmetic structure available in this case, namely, that $[\mathbf{Pic}^{g-1}_{C/k}]$ is automatically a 2-torsion element of the Tate–Shafarevich group $Ш(J)$. The class $[\mathbf{Pic}^{g-1}_{C/k}]$ is divisible by 2 in $\mathrm{H}^1(k,J)$ if and only if its image in $Ш^2(k,J[2])$ under the Bockstein map is orthogonal to $Ш^1(k,J[2])$ with respect to the Poitou–Tate duality pairing. Since the Poitou–Tate pairing is compatible with the Cassels–Tate pairing, $[\mathbf{Pic}^{g-1}_{C/k}]$ is divisible by 2 in $\mathrm{H}^1(k,J)$ if and only if $[\mathbf{Pic}^{g-1}_{C/k}]$ is orthogonal to the image of $Ш^1(k,J[2])$ in $Ш(J)$ with respect to the Cassels–Tate pairing. Following Manin, this can be interpreted in terms of the Brauer–Manin obstruction on the everywhere locally soluble variety $X=\mathbf{Pic}^{g-1}_{C/k}$: any element $t\in Ш(J)$ gives rise to an everywhere locally constant element $\mathcal{B}$ of the Brauer group $\mathrm{Br}(X)$ whose Brauer–Manin pairing with an arbitrary adelic point of X equals the Cassels–Tate pairing of t and $[X]$. When $g=2$, the variety X is the Albanese torsor of C, so C is a closed subvariety of X. If C itself is everywhere locally soluble, we can take an adelic point of X to be an adelic point of C and then we only need the restriction of $\mathcal{B}$ to C. The easiest way to arrange for this is to consider hyperelliptic curves C given by the equation $y^2=f(x)$, where the separable polynomial $f(x)\in k[x]$ of degree 6 is everywhere locally soluble. In this setting, one can construct elements of $\mathrm{Br}(C)$ coming from $\mathrm{H}^1(k,J[2])$ by an explicit formula, which allows one to compute the Brauer–Manin pairing with the adelic points of $f(x)=0$. A worked out example over $k=\mathbb{Q}$ when this pairing is non-trivial is $f(x)=3(x^2+1)(x^2+17)(x^2-17)$, see [Cre13, p. 941] and [CV15, Theorem 6.7], so in this case $[\mathbf{Pic}^{g-1}_{C/k}]$ is not divisible by 2 in $\mathrm{H}^1(k,J)$.

Let us now give a general description of the first potentially non-trivial extension in the étale cohomology complex of a semi-abelian variety over an arbitrary field. It is easy to see that all differentials on the i-th page of (1) are zero for $\ell>i$, see Corollary 1.3. For the study of the differentials on the second page we can thus assume that $\ell=2$. For a $\mathbb{Z}_2$-module M we denote by $Q^2(M)$ the module of quadratic functions on $M^\vee=\mathrm{Hom}_{\mathbb{Z}_2}(M,\mathbb{Z}_2)$ with values in $\mathbb{Z}_2$. It fits into the exact sequence

$$0\to M\to Q^2(M)\to M^{\otimes 2}\to \Lambda^2 M\to 0 \tag{2}$$

where the middle map sends a quadratic function $f: M^\vee\to\mathbb{Z}_2$ to the bilinear form $\langle x,y\rangle=f(x+y)-f(x)-f(y)$ which we view as an element of $\mathrm{Hom}(M^\vee\otimes M^\vee,\mathbb{Z}_2)\cong M^{\otimes 2}$. The above sequence is natural in M, so if M is equipped with an action of Γ then (2) becomes an exact sequence of Γ-modules.

Theorem 2 (Theorem 1.6)**.** *Let A be a semi-abelian variety over a field k of characteristic not equal to 2. The class in $\mathrm{Ext}^2_\Gamma(\mathrm{H}^2_{\text{ét}}(A_{k_s},\mathbb{Z}_2),\mathrm{H}^1_{\text{ét}}(A_{k_s},\mathbb{Z}_2))$ corresponding to $\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(A_{k_s},\mathbb{Z}_2)$ is equal to the class of the Yoneda extension* (2) *for $M=\mathrm{H}^1_{\text{ét}}(A_{k_s},\mathbb{Z}_2)$.*

In the case of abelian varieties this was proved in [P], but here we give a new shorter proof exhibiting an explicit quasi-isomorphism between (2) and the bar complex of the Tate module of A. Theorem 2, in particular, gives a description of the differentials $\mathrm{H}^i(k, \mathrm{H}^2_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)) \to \mathrm{H}^{i+2}(k, \mathrm{H}^1_{\text{ét}}(A_{k_s}, \mathbb{Z}_2))$ of (1), as well as the analogous differentials in the Hochschild–Serre spectral sequence with $\mathbb{Z}/2^n$-coefficients, for any n, see Theorem 1.10.

Theorem 1 shows that the class described in Theorem 2 is not always zero for abelian varieties, but at the moment we do not know if it is ever non-zero for tori, cf. Remark 4.2.

In contrast, for the Jacobians of curves with a rational divisor class of degree 1 the étale cohomology complex decomposes in all degrees. We give a proof of this result, which may be well-known to the experts, but does not seem to be available in the literature.

Theorem 3 (Theorem 2.1)**.** *Let $C_1, \ldots, C_m$ be geometrically connected smooth proper curves over k each admitting a k-rational divisor class of degree 1, for $i = 1, \ldots, m$. Let A be a direct factor of the product $\prod_{i=1}^m \mathrm{Jac}(C_i)$ of Jacobians of $C_1, \ldots, C_m$. Let $n \geq 0$ be an integer not divisible by $\mathrm{char}\, k$. Then $\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}/n)$ is quasi-isomorphic to $\bigoplus_{i\geq 0} \mathrm{H}^i_{\text{ét}}(A_{k_s}, \mathbb{Z}/n)[-i]$ in $D(\Gamma, \mathbb{Z}/n)$. In particular, the Hochschild–Serre spectral sequence*

$$E_2^{i,j} = \mathrm{H}^i(k, \mathrm{H}^j_{\text{ét}}(A_{k_s}, \mathbb{Z}/n)) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(A, \mathbb{Z}/n)$$

degenerates at the second page.

Another class of examples of non-zero differentials in the Hochschild–Serre spectral sequence comes from varieties without a rational point. If a geometrically connected k-variety X has a k-point, then $\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}_\ell)$ is quasi-isomorphic to $\mathbb{Z}_\ell \oplus \tau^{\geq 1}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}_\ell)$ in the derived category of Γ-modules. This is no longer true for varieties without a rational point, and we calculate the exact obstruction to splitting off $\mathrm{H}^0(X_{k_s}, \mathbb{Z}_\ell)$ from $\tau^{\leq 1}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}_\ell)$ when X is a torsor for a semiabelian variety. We denote by $\mathcal{A}(\Gamma, \mathbb{Z}/n)$ the abelian category of discrete $\mathbb{Z}/n$-modules equipped with a continuous action of Γ.

Theorem 4 (Theorem 5.1)**.** *Let X be a torsor for a semiabelian variety A over a field k. Let $n \geq 1$ be an integer not divisible by $\mathrm{char}\, k$. Then the class in*

$$\mathrm{Ext}^2_{\mathcal{A}(\Gamma,\mathbb{Z}/n)}(\mathrm{H}^1(X_{k_s}, \mathbb{Z}/n), \mathbb{Z}/n) \cong \mathrm{Ext}^2_{\mathcal{A}(\Gamma,\mathbb{Z}/n)}(\mathrm{H}^1(A_{k_s}, \mathbb{Z}/n), \mathbb{Z}/n) \cong \mathrm{H}^2(k, A[n])$$

corresponding to $\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}/n)$ is equal to the image of the class of X in $\mathrm{H}^1(k, A)$ under the Bockstein homomorphism $\mathrm{H}^1(k, A) \to \mathrm{H}^2(k, A[n])$.

As we point out in Remark 5.2, in the setting of Theorem 4, the obstruction to the existence of a rational 0-cycle of degree 1 coming from $\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \widehat{\mathbb{Z}})$ being non-split coincides with the obstruction coming from the failure of the abelianised fundamental exact sequence for the arithmetic fundamental group to have a section.

We give several complements of the above results, summarised below.

In Theorem 4.1 we describe a differential on the 3rd page of the Hochschild–Serre spectral sequence with $\mathbb{G}_m$-coefficients for a *torus*, which allows one to completely compute the Brauer group of any algebraic torus, answering a question raised in [CTS21, p. 220]. When the first version of this paper was completed, Julian Demeio told us about his result that for a torus T over a field k of characteristic zero the natural map $\mathrm{Br}(T) \to \mathrm{Br}(T_{k_s})^\Gamma$ is surjective when T is quasi-trivial or when k is a number field [Dem, Theorem 1.1]. This also follows from our Theorem 4.1, see Corollary 4.3. Demeio's proof is based on the elaborate analysis of differentials of the Hochschild–Serre spectral sequence by L.S. Charlap, A.T. Vasques, and C.H. Sah (see the references in [Dem]). The proof of Theorem 4.1 in this paper is self-contained: it reduces the computation of the differential with $\mathbb{G}_m$-coefficients to the case of finite coefficients which is handled in our explicit proof of Theorem 2.

In Proposition 6.1 we combine Theorem 4 with Poincaré duality to describe all differentials on the second page of the Hochschild–Serre spectral sequence with coefficients in μ_n for a smooth proper *curve*. When all differentials on the second page vanish, in Proposition 6.5 we describe the differentials on the 3rd page of this spectral sequence, generalising a result of Suslin in the case of genus 0.

Notation. For an object C of the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ its truncation $\tau^{[i,i+1]}C$ in any two consecutive degrees fits into a distinguished triangle

$$\mathrm{H}^i(C)[-i] \to \tau^{[i,i+1]}C \to \mathrm{H}^{i+1}(C)[-i-1] \xrightarrow{\delta_i} \mathrm{H}^i(C)[-i+1]$$

We refer to the map

$$\delta_i \in \mathrm{Hom}_{D(\mathcal{A})}(\mathrm{H}^{i+1}(C)[-i-1], \mathrm{H}^i(C)[-i+1]) \simeq \mathrm{Ext}^2_{\mathcal{A}}(\mathrm{H}^{i+1}(C), \mathrm{H}^i(C))$$

as the extension class of $\tau^{[i,i+1]}C$.

For a profinite group G and an integer n (possibly $n = 0$) we denote by $\mathcal{A}(G, \mathbb{Z}/n)$ the abelian category of discrete $\mathbb{Z}/n$-modules equipped with a continuous action of G, cf. [Wei94, 6.11]. We write $D(G, \mathbb{Z}/n)$ for the derived category of $\mathcal{A}(G, \mathbb{Z}/n)$. For a prime ℓ we denote by $D(G, \mathbb{Z}_\ell)$ the derived category of sheaves of $\mathbb{Z}_\ell$-modules on the pro-étale site BG_{proet}, where $\mathbb{Z}_\ell$ is the sheaf of rings obtained from the topological ring $\mathbb{Z}_\ell$ with a trivial action of G, cf. [BS15, 4.3]. Recall that an ℓ-adically complete $\mathbb{Z}_\ell$-module M equipped with a continuous action of G gives rise to an object of the triangulated category $D(G, \mathbb{Z}_\ell)$, and $\mathrm{Hom}_{D(G,\mathbb{Z}_\ell)}(\mathbb{Z}_\ell, M[i])$ is isomorphic to i-th continuous cohomology group $\mathrm{H}^i_{\mathrm{cont}}(G, M)$ defined using the complex of continuous cochains on G.

When $\Gamma = \mathrm{Gal}(k_s/k)$ is the absolute Galois group of a field k, we denote $\mathrm{H}^i_{\mathrm{cont}}(\Gamma, M)$ by $\mathrm{H}^i(k, M)$ for any continuous Γ-module M.

Acknowledgements. The first named author was supported by the Clay Research Fellowship and was a member at the Institute for Advanced Study in Princeton during the preparation of this paper. A part of this work was done during a visit of the second named author to the IAS, whose hospitality is gratefully acknowledged. We thank Jean-Louis Colliot-Thélène for telling us about Suslin's lemma [Sus82, Lemma 1] and Vadim Vologodsky for explaining to us the result obtained in Theorem 3.

1 Semiabelian varieties

Let A be a semiabelian variety over k. Let ℓ be a prime number, $\ell \neq \mathrm{char}(k)$. For all $i \geq 0$, $n \geq 1$, we have $\mathrm{H}^i_{\text{ét}}(A_{k_s}, \mathbb{Z}/\ell^n) = \wedge^i \mathrm{H}^1_{\text{ét}}(A_{k_s}, \mathbb{Z}/\ell^n)$ compatibly with the natural action of the Galois group $\Gamma = \mathrm{Gal}(k_s/k)$.

We would like to study the differentials $\delta_i^{p,q} \colon E_i^{p,q} \to E_i^{p+i,q-i+1}$ of the Hochschild–Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A_{k_s}, \mathbb{Z}/\ell^n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbb{Z}/\ell^n). \tag{3}$$

Let us make some simple remarks about these differentials.

Remark 1.1. The origin of the group law on A gives a section of the structure morphism $p \colon A \to \mathrm{Spec}(k)$, hence the natural maps $\mathrm{H}^r_{\text{ét}}(k, \mathbb{Z}/\ell^n) \to \mathrm{H}^r_{\text{ét}}(A, \mathbb{Z}/\ell^n)$ are injective for all $r \geq 0$. Therefore, we have $\delta_i^{p,i-1} = 0$ for $i \geq 2$.

Lemma 1.2. *Let $i \geq 2$, $p \geq 0$, $q \geq i-1$ be integers, and let ℓ be a prime. When $\ell - 1$ divides $i-1$ we define $n = \mathrm{val}_\ell\big((i-1)/(\ell-1)\big)$.*

(a) *Suppose $\ell \neq 2$. If $\ell - 1$ does not divide $i-1$, then $\delta_i^{p,q} = 0$. Otherwise, we have $\ell^{\min\{q-i+1,n+1\}}\delta_i^{p,q} = 0$.*

(b) *Suppose $\ell = 2$. If i is even, then $2^{\min\{q-i+1,1\}}\delta_i^{p,q} = 0$. If i is odd, then $2^{\min\{q-i+1,n+2\}}\delta_i^{p,q} = 0$.*

Proof. The spectral sequence (3) is functorial in A, so it is compatible with multiplication by m map $[m] \colon A \to A$, for any integer m. The induced map on $\mathrm{H}^r_{\text{ét}}(A_{k_s}, \mathbb{Z}/\ell^n) \cong \wedge^r \mathrm{H}^1_{\text{ét}}(A_{k_s}, \mathbb{Z}/\ell^n)$ is multiplication by m^r. Thus we have

$$(m^q - m^{q-i+1})\delta_i^{p,q} = m^{q-i+1}(m^{i-1} - 1)\delta_i^{p,q} = 0.$$

Taking $m = \ell$ we see that ℓ^{q-i+1} annihilates $\delta_i^{p,q}$. From now on we assume that $(m, \ell) = 1$. Then we have $(m^{i-1} - 1)\delta_i^{p,q} = 0$.

Suppose $\ell \neq 2$. If $\ell - 1$ does not divide $i-1$, then we can find an integer m such that $(m, \ell) = 1$ and $m^{i-1} - 1$ is non-zero modulo ℓ. Then $\delta_i^{p,q} = 0$.

If $\ell - 1$ divides $i-1$, we claim that the lowest value of $\mathrm{val}_\ell(m^{i-1} - 1)$, where $(m, \ell) = 1$, is $n+1$. If $(m, \ell) = 1$, then $m^{r\ell^n(\ell-1)} - 1$ is clearly divisible by ℓ^{n+1}, so it is enough to check that $\mathrm{val}_\ell((1+\ell)^{r\ell^n(\ell-1)} - 1) = n+1$ when $(r, \ell) = 1$. This is immediate for $n = 0$, and the general case follows by induction in n. This proves (a).

Suppose $\ell = 2$. If i is even, then taking $m = -1$ we prove the first statement of (b). If i odd, so that $n \geq 1$, we claim that the smallest value of $\mathrm{val}_2(m^{r2^n} - 1)$, where r and m are odd, is $n+2$. We have

$$m^{r2^n} - 1 = (m^{2^n} - 1)(m^{(r-1)2^n} + \ldots + 1).$$

Since r and m are odd, the second factor in the right hand side is odd. Thus we can assume that $r = 1$. For $n = 1$ the statement is obvious, and the general case immediately follows by induction. This proves (b). $\square$

Note that the statement of Remark 1.1 is a particular case of Lemma 1.2.

Corollary 1.3. *All differentials on the i-page of the spectral sequence* (3) *are zero for $\ell > i$.*

Thus the differentials on the second page can be non-zero only for $\ell = 2$. The first non-trivial case is the differential $\delta_2^{p,2}$, where $p \geq 0$. Explicit calculation of this differential is easier for the limit spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbb{Z}_2), \tag{4}$$

so we start with this case.

Let M be a free, finitely generated $\mathbb{Z}_2$-module. Recall that $S^2(M)$ is the quotient of $M^{\otimes 2}$ by the $\mathbb{Z}_2$-submodule generated by elements $x \otimes y - y \otimes x$ for all $x, y \in M$, and that $\wedge^2(M)$ is the quotient of $M^{\otimes 2}$ by the $\mathbb{Z}_2$-submodule generated by $x \otimes x$, for all $x \in M$.

Definition 1.4. *For a free finitely generated $\mathbb{Z}_2$-module M let $Q^2(M)$ be the module of quadratic functions on the dual module $M^\vee$. That is, $Q^2(M)$ is the module of maps of sets $f : M^\vee \to \mathbb{Z}_2$ such that the function $\langle x, y\rangle := f(x + y) - f(x) - f(y)$ is bilinear in x and y.*

For example, $Q^2(\mathbb{Z}_2)$ is a free rank 2 module consisting of functions of the form $a \cdot x + b \cdot \frac{x^2 - x}{2}$ with $a, b \in \mathbb{Z}_2$.

There is a natural injection $M \to Q^2(M)$ sending an element of M to the linear function on $M^\vee$ that it defines. The cokernel of this map is $\mathrm{Hom}_{\mathbb{Z}_2}(S^2(M^\vee), \mathbb{Z}_2)$. One immediately checks (for example, by choosing a basis of M and the dual basis of $M^\vee$) that under the natural pairing

$$M^{\otimes 2} \times (M^\vee)^{\otimes 2} \to \mathbb{Z}_2$$

the $\mathbb{Z}_2$-submodules $(M \otimes M)^{S_2} = \langle m \otimes m | m \in M\rangle$ and $\langle a \otimes b - b \otimes a | a, b \in M^\vee\rangle$ are exact annihilators of each other. Thus there is a canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}_2}(S^2(M^\vee), \mathbb{Z}_2) \cong (M \otimes M)^{S_2}$. We obtain a canonical exact sequence

$$0 \to M \to Q^2(M) \to M^{\otimes 2} \to \wedge^2 M \to 0. \tag{5}$$

The $\mathbb{Z}_2$-module $Q^2(M)$ contains the submodule of quadratic forms $M^\vee \to \mathbb{Z}_2$. The map $Q^2(M) \to M^{\otimes 2}$ sends a quadratic form to the associated bilinear form.

The extension defined by (5) is equivalent to an extension with smaller terms. Denote by $(M^{\otimes 2})_{S_2,\mathrm{sgn}} = (M^{\otimes 2})/\langle m_1 \otimes m_2 + m_2 \otimes m_1 | m_1, m_2 \in M\rangle$ the coinvariants of the involution $m_1 \otimes m_2 \mapsto -m_2 \otimes m_1$ acting on $M^{\otimes 2}$. This module fits into the following exact sequence:

$$0 \to M \xrightarrow{2} M \to (M^{\otimes 2})_{S_2,\mathrm{sgn}} \to \wedge^2 M \to 0, \tag{6}$$

where the map $M \to (M^{\otimes 2})_{S_2,\mathrm{sgn}}$ sends m to $m \otimes m$, and the rightmost map sends $m_1 \otimes m_2 \in (M^{\otimes 2})_{S_2,\mathrm{sgn}}$ to $m_1 \wedge m_2 \in \Lambda^2 M$. A direct computation shows:

Lemma 1.5. *There is a commutative diagram given by*

$$\begin{array}{ccccccc} M & \longrightarrow & Q^2(M) & \longrightarrow & M^{\otimes 2} & \longrightarrow & \Lambda^2 M \\ \| & & \downarrow & & \downarrow & & \| \\ M & \xrightarrow{2} & M & \longrightarrow & (M^{\otimes 2})_{S_2,\mathrm{sgn}} & \longrightarrow & \Lambda^2 M \end{array}$$

where the map $Q^2(M) \to M$ sends a quadratic function f on $M^\vee$ to the linear function $m \mapsto 4f(m) - f(2m)$.

The extension (6) is the Yoneda product of

$$0 \to M \xrightarrow{2} M \to M/2 \to 0 \tag{7}$$

and

$$0 \to M/2 \to (M^{\otimes 2})_{S_2,\mathrm{sgn}} \to \wedge^2 M \to 0. \tag{8}$$

All of the above constructions and exact sequences are compatible with the action of the automorphisms of the $\mathbb{Z}_2$-module M. In particular, if M is a $\mathbb{Z}_2$-module equipped with the 2-adic topology and a continuous action of Γ, all the sequences above are naturally sequence of Γ-modules.

The following statement is the particular case $p = 2$ of [P, Corollary 9.5 (1)]. Here we give a short elementary proof of this result.

Theorem 1.6. *Let k be a field of characteristic different from 2, and let A be a semiabelian variety over k. Write $M = \mathrm{H}^1_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)$ so that $\wedge^i M \cong \mathrm{H}^i_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)$ are continuous Γ-modules for $i \geq 0$. The differential*

$$\delta_2^{p,2}\colon \mathrm{H}^p(k, \wedge^2 M) \to \mathrm{H}^{p+2}(k, M)$$

of the spectral sequence of Γ-modules (4) equals the connecting map of the 2-extension of Γ-modules (5).

Proof. For any scheme X with a geometric base point $\overline{x}$ there is a natural map

$$\mathrm{R}\Gamma(\pi_1^{\text{ét}}(X, \overline{x}), \mathbb{Z}/n) \to \mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{Z}/n) \tag{9}$$

induced by the pullback along the map of sites $X_{\text{ét}} \to X_{\text{fet}}$, where the target site consists of schemes finite étale over X, cf. [Ols09, §5]. Moreover, if $X = Y_{k_s}$, where Y is a scheme over k, and $\overline{x}$ is lying above a k-point of Y, this map is naturally a map in $D(\Gamma, \mathbb{Z}/n)$.

The map (9) always induces an isomorphism on H^1. For $X = A_{k_s}$, where A is a semiabelian variety and char k does not divide n, the cohomology rings of both source and target are free exterior algebras on H^1, hence (9) is a quasi-isomorphism in this case. Moreover, $\pi_1^{\text{ét}}(A_{k_s})$ is the product of its 2-adic Tate module $T_2(A_{k_s}) \simeq M^\vee$ and a 2-divisible profinite group, hence we have a quasi-isomorphism $\mathrm{R}\Gamma(\pi_1^{\text{ét}}(A_{k_s}), \mathbb{Z}_2) \simeq \mathrm{R}\Gamma(M^\vee, \mathbb{Z}_2)$. We get a quasi-isomorphism in $D(\Gamma, \mathbb{Z}_2)$:

$$\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}_2) \simeq \mathrm{R}\Gamma(M^\vee, \mathbb{Z}_2).$$

Thus we can calculate $\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)$ using the standard bar complex for computing continuous cohomology of the group $M^\vee$:

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{d_0=0} \mathrm{Func}(M^\vee, \mathbb{Z}_2) \xrightarrow{d_1} \mathrm{Func}(M^\vee \times M^\vee, \mathbb{Z}_2) \xrightarrow{d_2} \dots,$$

where Func denotes the $\mathbb{Z}_2$-module of continuous functions. The differential d_1 sends a function $f\colon M^\vee \to \mathbb{Z}_2$ to the function of two arguments $(x, y) \mapsto f(x+y) - f(x) - f(y)$. The inclusion $Q^2(M) \subset \mathrm{Func}(M^\vee, \mathbb{Z}_2)$ gives rise to a commutative diagram

$$\begin{array}{ccc} Q^2(M) & \longrightarrow & M^{\otimes 2} \\ \downarrow & & \downarrow \\ \mathrm{Func}(M^\vee, \mathbb{Z}_2) & \xrightarrow{d_1} & \mathrm{Func}(M^\vee \times M^\vee, \mathbb{Z}_2) \end{array} \tag{10}$$

The right-hand vertical map is the inclusion of bilinear functions on $M^\vee \times M^\vee$ into all continuous functions. Commutativity of the diagram is immediate from the definitions of the maps. The submodule $M \subset Q^2(M)$ of linear functions maps isomorphically to $\mathrm{H}^1_{\text{cont}}(M^\vee, \mathbb{Z}_2)$. Moreover, the right vertical map lands in the kernel of the next differential d_2, and the induced map $M^{\otimes 2} \to \mathrm{H}^2_{\text{cont}}(M^\vee, \mathbb{Z}_2)$ coincides with the cup-product map $\mathrm{H}^1_{\text{cont}}(M^\vee, \mathbb{Z}_2)^{\otimes 2} \to \mathrm{H}^2_{\text{cont}}(M^\vee, \mathbb{Z}_2)$.

Exactness of (5) then implies that (10) gives a quasi-isomorphism in $D(\Gamma, \mathbb{Z}_2)$ between the two-term complex $Q^2(M) \to M^{\otimes 2}$ and the truncation $\tau^{[1,2]}\mathrm{R}\Gamma_{\text{cont}}(M^\vee, \mathbb{Z}_2) \simeq \tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)$, as desired. □

The description of the same differential with coefficients $\mathbb{Z}/2^m$ for $m \geq 2$ is a little more complicated. We work in the derived category $D(\Gamma, \mathbb{Z}_2)$ of $\mathbb{Z}_2$-modules with continuous Γ-action, introduced in notation section of the introduction.

When N is a free, finitely generated $\mathbb{Z}_2$-module with continuous action of Γ, we denote by Bock the morphism $N/2^m \to N[1]$ defined by the exact sequence

$$0 \to N \xrightarrow{2^m} N \to N/2^m \to 0.$$

For $i \geq 1$, by a slight abuse of notation, we also denote by Bock the composition $N/2^m \to N[1] \to (N/2^i)[1]$. This map is defined by the exact sequence

$$0 \to N/2^i \to N/2^{m+i} \to N/2^m \to 0,$$

which is the push-out of the previous exact sequence by the map $N \to N/2^i$.

Let us also introduce the following notation. For an $\mathbb{F}_2$-vector space V, equipped with an action of Γ continuous with respect to the discrete topology on V, consider the short exact sequence

$$0 \to V \xrightarrow{v \mapsto v \cdot v} S^2V \xrightarrow{v_1 \cdot v_2 \mapsto v_1 \wedge v_2} \Lambda^2 V \to 0. \tag{11}$$

Its extension class is a map $\Lambda^2 V \to V[1]$ in $D(\Gamma, \mathbb{F}_2)$ which we denote by $\alpha(V)$. Note that the extension (8) is obtained from (11) for $V = M/2$ by pulling back along $\Lambda^2 M \to \Lambda^2(M/2)$. Note that $S^2V = (V^{\otimes 2})_{S_2} \cong (V^{\otimes 2})_{S_2,\mathrm{sgn}}$. The following lemma shows that when the dimension of V is small, the connecting map $\alpha(V)$ defined by (11) is zero.

Lemma 1.7. (i) *If the rank of V is 2, then* (11) *is split.*

(ii) *If the rank of M is 3, then the connecting map $(\wedge^2 V)^\Gamma \to \mathrm{H}^1(\Gamma, V)$ defined by* (11) *is zero.*

Proof. (i) Let u, v, w be the three non-zero elements of V. Then the unique non-zero element in $\wedge^2 V$ lifts to $uv + vw + wu \in S^2V$. This lifting is $\mathrm{GL}(2, \mathbb{F}_2)$-equivariant.

(ii) Assume that there is a non-zero element $x \in (\wedge^2 V)^\Gamma$. We have a perfect bilinear pairing of $\mathbb{F}_2$-vector spaces $V \times \wedge^2 V \to \mathbb{F}_2$. The elements of V that pair trivially with x form an $\mathbb{F}_2$-subspace $N \subset V$ of dimension 2 which is stable under the action of Γ. Moreover, $x \in \wedge^2 N$. We have a commutative diagram of Γ-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & S^2N & \longrightarrow & \wedge^2 N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & S^2V & \longrightarrow & \wedge^2 V & \longrightarrow & 0 \end{array}$$

By part (i), the top sequence is split. Thus x lifts to an element of $(S^2V)^\Gamma$. ☐

Lemma 1.8. (i) *Suppose that V is a permutational Γ-module, that is, the action of Γ on V preserves an $\mathbb{F}_2$-basis of V. Then* (11) *is split.*

(ii) *Suppose that $k = \mathbb{R}$. Then* (11) *is split.*

Proof. (i) Let $e_1, \ldots, e_n$ be a Γ-stable basis of V. The union of sets $\{e_i^2\}$ and $\{e_i e_j | i < j\}$ is a Γ-stable basis of S^2V, which gives a Γ-equivariant splitting of (11).

(ii) By (i) it is enough to show that any representation of $\mathbb{Z}/2$ in V is permutational, but this is well-known. Indeed, endow V with the structure of an $\mathbb{F}_2[x]$-module by letting x act as the generator of $\mathbb{Z}/2$. By the classification of finitely generated torsion modules over a PID, the $\mathbb{F}_2[x]$-module V is a direct sum of cyclic submodules isomorphic to $\mathbb{F}_2 = \mathbb{F}_2[x]/(x-1)$ or $\mathbb{F}_2[x]/(x^2-1)$, both of which are permutational. ☐

We will now derive from Theorem 1.6 a description of the extension class of $\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}/2^m)$ in cohomology with coefficients modulo 2^m. It will rely on the following general computation:

Lemma 1.9. *Let p be a prime. Let X, Y be free, finitely generated $\mathbb{Z}_p$-modules equipped with a continuous action of Γ. Suppose we are given an extension*

$$0 \to X/p \to E \to Y/p \to 0$$

of $\mathbb{F}_p$-vector spaces equipped with an action of Γ. Let $\delta_E\colon Y/p \to X/p[1]$ be the corresponding map in $D(\Gamma, \mathbb{F}_p)$. Let $\widetilde{E}$ be the $\mathbb{Z}_p$-module with Γ-action obtained by pulling back $E \twoheadrightarrow Y/p$ along $Y \twoheadrightarrow Y/p$. The exact sequence

$$0 \to X \xrightarrow{p} X \xrightarrow{\psi} \widetilde{E} \to Y \to 0,$$

where ψ is the composition $X \to X/p \hookrightarrow \widetilde{E}$, defines a map $\beta_E : Y \to X[2]$ in $D(\Gamma, \mathbb{Z}_p)$.

Then the reduction of β_E modulo p^m is equal to the result of subtracting the composition (12) from (13):

$$Y/p^m \xrightarrow{\mathrm{Bock}_{Y/p^{m+1}}} Y/p[1] \xrightarrow{\delta_E[1]} X/p[2] \to X/p^m[2] \tag{12}$$

$$Y/p^m \to Y/p \xrightarrow{\delta_E} X/p[1] \xrightarrow{\mathrm{Bock}_{X/p^{m+1}}[1]} X/p^m[2] \tag{13}$$

Proof. The mod p^m reduction of the map $Y \to X[2]$ is the Yoneda class of the complex $(X \xrightarrow{\psi} \widetilde{E}) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$ concentrated in degrees -1 and 0. It can be computed as the totalisation of the bicomplex

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \widetilde{E} \\ {\scriptstyle p^m}\uparrow & & {\scriptstyle p^m}\uparrow \\ X & \xrightarrow{\psi} & \widetilde{E} \end{array} \tag{14}$$

The compositions of (12) and (13) are represented, respectively, by the following Yoneda extensions

$$0 \to X/p^m \xrightarrow{a} R_m \xrightarrow{b} Y/p^{m+1} \xrightarrow{c} Y/p^m \to 0 \tag{15}$$

$$0 \to X/p^m \xrightarrow{a'} X/p^{m+1} \xrightarrow{b'} T_m \xrightarrow{c'} Y/p^m \to 0, \tag{16}$$

where R_m (respectively, T_m) is the following pushout (respectively, pullback)

$$\begin{array}{ccc} X/p & \longrightarrow & E \\ \downarrow & & \downarrow \\ X/p^m & \longrightarrow & R_m \end{array} \qquad\qquad \begin{array}{ccc} T_m & \longrightarrow & Y/p^m \\ \downarrow & & \downarrow \\ E & \longrightarrow & Y/p \end{array}$$

and the map $b : R_m \to Y/p^{m+1}$ is the composition $R_m \to Y/p \hookrightarrow Y/p^{m+1}$, while $b' : X/p^{m+1} \to T_m$ is the composition $X/p^{m+1} \twoheadrightarrow X/p \to T_m$. The result of subtracting the map $Y/p^m \to X/p^m[2]$ corresponding to (15) from that corresponding to (16) is then represented by the extension

$$X/p^m \xrightarrow{(a,0)} \frac{R_m \oplus X/p^{m+1}}{(a(x), -a'(x))|x \in X/p^m} \xrightarrow{b \oplus b'} \ker(Y/p^{m+1} \oplus T_m \xrightarrow{c-c'} Y/p^m) \xrightarrow{(c,c')} Y/p^m \tag{17}$$

We can now write down a chain level map from the totalisation of (14) to the two-term complex from (17):

$$\begin{array}{ccccc} X & \xrightarrow{(\psi,-p^m)} & \widetilde{E} \oplus X & \xrightarrow{(p^m,\psi)} & \widetilde{E} \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ & & \frac{R_m \oplus X/p^{m+1}}{(a(x),-a'(x))|x\in X/p^m} & \xrightarrow{b\oplus b'} & \ker(Y/p^{m+1} \oplus T_m \xrightarrow{c-c'} Y/p^m) \end{array} \tag{18}$$

Let f_1 be the map induced by the direct sum of maps $\widetilde{E} \to E \to R_m$ and $X \to X/p^m \to R_m$, and f_2 is the sum of compositions $\widetilde{E} \to Y \to Y/p^{m+1}$ and $\widetilde{E} \twoheadrightarrow T_m$. The map f_2 indeed lands in the kernel of $c - c'$, the square in (18) commutes, and the composition $f_1 \circ (\psi, -p^m)$ is zero.

The complexes given by rows of (18) both have two non-zero cohomology groups given by X/p^m and Y/p^m, and the map of complexes induced by f_1, f_2 induces the identity map, giving the desired identification between the two classes in $\mathrm{Ext}^2(Y/p^m, X/p^m)$. $\square$

We now specialise the above computation to our case of interest. As before, for our semi-abelian variety A we denote the continuous Γ-module $\mathrm{H}^1_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)$ by M. Recall that we write $\alpha(M/2)\colon \wedge^2 (M/2) \to (M/2)[1]$ for the morphism in $D(\Gamma, \mathbb{F}_2)$ (as well as the corresponding morphism between the same objects in $D(\Gamma, \mathbb{Z}_2)$) defined by (11) with $V = M/2$.

The next statement is a more general variant of the particular case $p = 2$ of [P, Corollary 9.5 (2)]. We give an elementary proof of this result using Lemma 1.9.

Theorem 1.10. *Let k be a field of characteristic different from 2. Let A be a semiabelian variety over k. Consider the spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A_{k_s}, \mathbb{Z}/2^m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbb{Z}/2^m).$$

The differential $\delta_2^{p,2}\colon \mathrm{H}^p(k, \wedge^2(M/2^m)) \to \mathrm{H}^{p+2}(k, M/2^m)$ is given by

$$\alpha(M/2) \circ \mathrm{Bock}^{m,1}_{\Lambda^2 M} - \mathrm{Bock}^{1,m}_{M} \circ \alpha(M/2),$$

where $\mathrm{Bock}^{m,1}_{\Lambda^2 M} : \mathrm{H}^p(k, \Lambda^2(M/2^m)) \to \mathrm{H}^{p+1}(k, \Lambda^2(M/2))$ *and* $\mathrm{Bock}^{1,m}_{M} : \mathrm{H}^{p+1}(k, M/2) \to \mathrm{H}^{p+2}(k, M/2^m)$ *are Bockstein homomorphisms.*

Proof. By Theorem 1.6 the degree 2 extension class corresponding to $\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}_2)$ is equal to the composition

$$\Lambda^2 M \to \Lambda^2(M/2) \xrightarrow{\alpha(M/2)} M/2[1] \xrightarrow{\mathrm{Bock}_M} M[2]$$

To calculate the induced map on mod 2^m reductions, we apply Lemma 1.9 to the case $p = 2$ with $X = M$, $Y = \Lambda^2 M$, and the extension E given by

$$0 \to M/2 \to S^2(M/2) \to \Lambda^2(M/2) \to 0.$$

We obtain that the extension class of $\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}/2^m)$ is equal to the difference between the following compositions:

$$\wedge^2(M/2^m) \longrightarrow \wedge^2(M/2) \xrightarrow{\alpha(M/2)} (M/2)[1] \xrightarrow{\text{Bock}} (M/2^m)[2], \tag{19}$$

$$\wedge^2(M/2^m) \xrightarrow{\text{Bock}} \wedge^2(M/2)[1] \xrightarrow{\alpha(M/2)[1]} (M/2)[2] \longrightarrow (M/2^m)[2]. \tag{20}$$

The differential $\delta_2^{p,2}$ is equal to the map induced by this difference on the degree p derived functor of Γ-invariants, which gives the desired result. □

2 Jacobians

Let C be a smooth, projective, geometrically integral curve over a field k. Let $\mathbf{Pic}_{C/k}$ be the Picard scheme of C. Denote by $J := \mathbf{Pic}^0_{C/k}$ the Jacobian of C. The connected component $\mathbf{Pic}^1_{C/k} \subset \mathbf{Pic}_{C/k}$ parametrizing divisors of degree 1 is a torsor for J, usually called the Albanese torsor. There is a canonical map $C \to \mathbf{Pic}^1_{C/k}$. After a choice of a point $x_0 \in C(k_s)$, this map is identified with the usual Abel–Jacobi map $C_{k_s} \to J_{k_s}$ sending x to the class of the divisor $x - x_0$.

It is well known that $\mathbf{Pic}_{C/k}(k)$ is canonically isomorphic to $\mathrm{Pic}(C_{k_s})^\Gamma$, see, e.g., [CTS21, Corollary 2.5.9]. Thus the class of the torsor $\mathbf{Pic}^1_{C/k}(k)$ in $\mathrm{H}^1(k, J)$ is zero if and only if C has a k-rational divisor classes of degree 1.

Let n be an integer not divisible by $\mathrm{char}(k)$. As before, we denote by $D(\Gamma, \mathbb{Z}/n)$ the derived category of the abelian category of discrete $\mathbb{Z}/n$-modules equipped with a continuous action of $\Gamma = \mathrm{Gal}(k_s/k)$.

Suppose that A and B are abelian varieties over k such that there are quasi-isomorphisms

$$\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}/n) \simeq \bigoplus_{i\geq 0} \mathrm{H}^i_{\text{ét}}(A_{k_s}, \mathbb{Z}/n)[-i], \quad \mathrm{R}\Gamma_{\text{ét}}(B_{k_s}, \mathbb{Z}/n) \simeq \bigoplus_{j\geq 0} \mathrm{H}^j_{\text{ét}}(B_{k_s}, \mathbb{Z}/n)[-j]$$

in $D(\Gamma, \mathbb{Z}/n)$. The Künneth formula gives a quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n) \simeq \mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}/n) \otimes^L_{\mathbb{Z}/n} \mathrm{R}\Gamma_{\text{ét}}(B_{k_s}, \mathbb{Z}/n).$$

The groups $\mathrm{H}^i_{\text{ét}}(A_{k_s}, \mathbb{Z}/n)$ and $\mathrm{H}^j_{\text{ét}}(B_{k_s}, \mathbb{Z}/n)$ are free $\mathbb{Z}/n$-modules for all i and j, thus for each $m \geq 0$ we have isomorphisms

$$\mathrm{H}^m_{\text{ét}}(A_{k_s} \times B_{k_s}) \cong \bigoplus_{i+j=m} \mathrm{H}^i_{\text{ét}}(A_{k_s}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \mathrm{H}^j_{\text{ét}}(B_{k_s}, \mathbb{Z}/n).$$

We deduce a quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n) \simeq \bigoplus_{m\geq 0} \mathrm{H}^m_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n)[-m].$$

Conversely, given a quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n) \simeq \bigoplus_{m\geq 0} \mathrm{H}^m_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n)[-m],$$

we can produce a quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{ét}}(A_{k_s}, \mathbb{Z}/n) \xrightarrow{p_1^*} \mathrm{R}\Gamma_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n) \simeq \bigoplus_{m\geq 0} \mathrm{H}^m_{\text{ét}}(A_{k_s} \times B_{k_s}, \mathbb{Z}/n)[-m]$$
$$\xrightarrow{(\mathrm{id}_{A_{k_s}} \times e_B)^*} \bigoplus_{m\geq 0} \mathrm{H}^m_{\text{ét}}(A_{k_s}, \mathbb{Z}/n)[-m]. \quad (21)$$

Hence direct products of abelian varieties with decomposable etale cohomology complex, as well as direct factors of such varieties also have decomposable etale cohomology complexes. Therefore Theorem 3 of the introduction is a consequence of the following

Theorem 2.1. *Let C be a smooth, projective, geometrically integral curve over a field k. Let n be an integer not divisible by* $\mathrm{char}(k)$*. If C has a k-rational divisor class of degree 1, then there exists a quasi-isomorphism*

$$\mathrm{R}\Gamma_{\text{ét}}(J_{k_s}, \mathbb{Z}/n) \simeq \bigoplus_{i\geq 0} \mathrm{H}^i_{\text{ét}}(J_{k_s}, \mathbb{Z}/n)[-i] \quad (22)$$

in $D(\Gamma, \mathbb{Z}/n)$. In particular, the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(J_{k_s}, \mathbb{Z}/n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(J, \mathbb{Z}/n)$$

degenerates at the second page, and for all $r \geq 0$ there are isomorphisms

$$\mathrm{H}^r_{\text{ét}}(J, \mathbb{Z}/n) \cong \bigoplus_{i+j=r} \mathrm{H}^i(k, \mathrm{H}^j_{\text{ét}}(J_{k_s}, \mathbb{Z}/n)). \quad (23)$$

Proof. We will deduce this decomposition from the fact that J admits integral Künneth projectors defined over the field k, as proved in [Suh17, Theorem 1.4].

Composing the canonical map $C \to \mathbf{Pic}^1_{C/k}$ with an isomorphism of torsors $\mathbf{Pic}^1_{C/k} \cong J$ provided by the given degree 1 divisor class, we get a map $\alpha : C \to J$. Denote by $\alpha_n : \mathrm{Sym}^n C \to J$ the maps from the symmetric powers of C induced by α using the group structure on J. For each $1 \leq n \leq g$, denote by $w^{[n]} \in Z^n(J)$ the codimension n cycle in the Jacobian variety obtained by pushing forward the fundamental cycle $[\mathrm{Sym}^{n-g} C]$ along α_{n-g}. We also denote by $w^{[0]} \in Z^0(J)$ the fundamental cycle of J itself.

Following [Suh17, Theorem 4.2.3], for each $0 \leq i \leq 2g$ we define a codimension g cycle on $J \times J$ by the formula

$$\pi_i := (-1)^i \sum_{\substack{2a+b=2g-i \\ b+2c=i}} p_1^* w^{[a]} \cdot p_2^* w^{[c]} \cdot \sum_{e+d+f=b} (-1)^{d+f} p_1^* w^{[d]} \cdot \mu^* w^{[e]} \cdot p_2^* w^{[f]}$$

where $p_1, p_2 : J \times J \to J$ are the two projections, and $\mu : J \times J \to J$ is the multiplication map.

We will use the following key property of the cycles π_i, established in [Suh17]:

Proposition 2.2. *The endomorphism* $\mathrm{H}^j(J_{k_s}, \mathbb{Z}/n) \to \mathrm{H}^j(J_{k_s}, \mathbb{Z}/n)$ *of cohomology of* J_{k_s} *induced by the correspondence* π_i *is zero for* $j \neq i$*, and is the identity map for* $j = i$*.*

Proof. This is proved in [Suh17, Theorem 4.2.3] in the case $k_s = \mathbb{C}$. This formally implies the case of an arbitrary field k of characteristic zero, because étale cohomology is invariant under the base change along an extension of algebraically closed fields of characteristic zero.

If k is a field of positive characteristic p, then we can find a lift $\widetilde{C}$ of C_{k_s} to a smooth projective curve over the ring of p-typical Witt vectors $W(k_s)$, and the cycle $(\pi_i)_{k_s}$ can be extended to a cycle $\widetilde{\pi}_i$ on $\mathbf{Pic}^0_{\widetilde{C}/W(k_s)} \times \mathbf{Pic}^0_{\widetilde{C}/W(k_s)}$. Let K be the fraction field of $W(k_s)$. The base change isomorphisms $\mathrm{H}^j_{\text{ét}}(J_{k_s}, \mathbb{Z}/n) \simeq \mathrm{H}^j_{\text{ét}}(J(\widetilde{C}_{\overline{K}}), \mathbb{Z}/n)$ then intertwine the actions of the correspondences $(\pi_i)_{k_s}$ and $(\widetilde{\pi}_i)_{\overline{K}}$. This reduces the case of the characteristic p field k to that of the characteristic zero field $\overline{K}$, which we already handled. □

Let us now upgrade the action of π_i from individual cohomology groups to the complex $\mathrm{R}\Gamma_{\text{ét}}(J_{k_s}, \mathbb{Z}/n)$, as an object of $D(\Gamma, \mathbb{Z}/n)$.

Lemma 2.3. *For a smooth proper geometrically integral variety* Y *of dimension* d *over the field* k*, a cycle* $c \in Z^i(Y \times_k Y)$ *gives rise to a morphism in* $D(\Gamma, \mathbb{Z}/n)$

$$[c] : \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)(i-d)[2i-2d]$$

such that the induced maps on cohomology $\mathrm{H}^r_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \mathrm{H}^{r+2i-2d}_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n(i-d))$ *coincide with the usual action of* c *via correspondences on individual cohomology groups.*

Proof. The two projections $p_1, p_2 : Y \times Y \to Y$ define maps in $D(\Gamma, \mathbb{Z}/n)$:

$$p_1^* : \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \mathrm{R}\Gamma_{\text{ét}}((Y \times Y)_{k_s}, \mathbb{Z}/n)$$

and

$$p_{2*} : \mathrm{R}\Gamma_{\text{ét}}((Y \times Y)_{k_s}, \mathbb{Z}/n) \to \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)(-d)[-2d],$$

where $(-d)$ referes to the twist by the $(-d)$-th power of the cyclotomic character $\Gamma \to \mathrm{Aut}_{\mathbb{Z}/n}(\mu_n)$.

There are cycle classes $\mathrm{cl}(c) \in \mathrm{H}^{2i}_{\text{ét}}(Y \times Y, \mathbb{Z}/n(i))$ in absolute étale cohomology of $Y \times Y$, as defined in [SGA $4\frac{1}{2}$, Cycle, 2.2.10]. The absolute étale cohomology complex $\mathrm{R}\Gamma_{\text{ét}}(Y \times Y, \mathbb{Z}/n(i))$ can be identified with the continuous group cohomology complex $\mathrm{R}\Gamma_{\text{cont}}(\Gamma, \mathrm{R}\Gamma_{\text{ét}}((Y \times Y)_{k_s}, \mathbb{Z}/n(i)))$ with coefficients in geometric étale cohomology, hence $\mathrm{cl}(c)$ corresponds to a map $\mathrm{cl}(c) : \mathbb{Z}/n[-2i] \to \mathrm{R}\Gamma_{\text{ét}}((Y \times Y)_{k_s}, \mathbb{Z}/n(i))$ in the derived category $D(\Gamma, \mathbb{Z}/n)$.

We can now define the endomorphism of the complex $\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)$ induced by c as the composition

$$\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \xrightarrow{p_1^*} \mathrm{R}\Gamma_{\text{ét}}((Y\times Y)_{k_s}, \mathbb{Z}/n) \xrightarrow{\cup \mathrm{cl}(c)} \mathrm{R}\Gamma_{\text{ét}}((Y\times Y)_{k_s}, \mathbb{Z}/n)(i)[2i] \xrightarrow{p_{2*}} \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)(i-d)[2i-2d]$$

Here the middle map $\cup\mathrm{cl}(c)$ denotes the composition

$$\mathrm{R}\Gamma_{\text{ét}}((Y\times Y)_{k_s}, \mathbb{Z}/n) \xrightarrow{\mathrm{id}\otimes \mathrm{cl}(c)} \mathrm{R}\Gamma_{\text{ét}}((Y\times Y)_{k_s}, \mathbb{Z}/n)^{\otimes 2}(i)[2i] \to \mathrm{R}\Gamma_{\text{ét}}((Y\times Y)_{k_s}, \mathbb{Z}/n)(i)[2i]$$

where the second map is the cup-product on the level of cohomology complexes. □

We can now construct the desired quasi-isomorphism (22). It suffices to produce, for each $i \geq 0$, a map $\mathrm{R}\Gamma_{\text{ét}}(J_{k_s}, \mathbb{Z}/n) \to \mathrm{H}^i_{\text{ét}}(J_{k_s}, \mathbb{Z}/n)[-i]$ in $D(\Gamma, \mathbb{Z}/n)$ that induces the identity map on the i-th cohomology. By Lemma 2.3, the cycle π_i defines a map $[\pi_i] : \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)$ in $D(\Gamma, \mathbb{Z}/n)$ inducing 0 on H^j for $j \neq i$, and the identity map on H^i. The endomorphism $[\pi_i]$ fits into the following map of distinguished triangles

$$\begin{array}{ccccc}
\tau^{\leq i}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) & \longrightarrow & \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) & \longrightarrow & \tau^{\geq i+1}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \\
\downarrow{\scriptstyle \tau^{\leq i}[\pi_i]} & & \downarrow{\scriptstyle [\pi_i]} & & \downarrow{\scriptstyle \tau^{\geq i+1}[\pi_i]} \\
\tau^{\leq i}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) & \longrightarrow & \mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) & \longrightarrow & \tau^{\geq i+1}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)
\end{array}$$

The map $\tau^{\geq i+1}[\pi_i]$ induces the zero map on all cohomology groups, but a priori it may be a non-zero map in $D(\Gamma, \mathbb{Z}/n)$. Since the complex $\tau^{\geq i+1}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)$ has non-zero cohomology only in the range $[i+1, 2g]$, the $(2g-i)$-th power of the endomorphism $\tau^{\geq i+1}[\pi_i]$ is nonetheless equal to zero in $D(\Gamma, \mathbb{Z}/n)$, see Lemma 2.4 below.

Therefore $[\pi_i]^{2g-i}$ gives rise to a map $\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \tau^{\leq i}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)$ inducing the identity map on i-th cohomology. Composing it with the natural map

$$\tau^{\leq i}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \tau^{[i,i]}\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \simeq \mathrm{H}^i_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)[-i]$$

we obtain the desired map $\mathrm{R}\Gamma_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n) \to \mathrm{H}^i_{\text{ét}}(Y_{k_s}, \mathbb{Z}/n)[-i]$ in $D(\Gamma, \mathbb{Z}/n)$ that induces the identity map on H^i. Summing up these maps over all $i \in [0, 2g]$ we obtain the quasi-isomorphism (22).

Applying the derived functor of Γ-invariants to (22) gives the isomorphisms (23). □

Lemma 2.4. *Let A be an abelian category and let $D^b(A)$ be the bounded derived category of A. Let C be an object of $D^b(A)$ such that $\mathrm{H}^i(C) = 0$ for $i \notin [a, a+n]$, for some integers a and $n \geq 0$. Let $f\colon C \to C$ be an endomorphism in $D^b(A)$ such that $\mathrm{H}^i(f) = 0$ for every i. Then we have $f^{n+1} = 0$ in $\mathrm{Hom}_{D^b(A)}(C, C)$.*

Proof. Without loss of generality we can assume $a = 0$. Proceed by induction on n. If $n = 0$, then C is isomorphic in $D^b(A)$ to an object represented by the one-term complex $\mathrm{H}^0(C)$ in degree 0. Under this isomorphism, f corresponds to an endomorphism induced by $\mathrm{H}^0(f) \in \mathrm{Hom}_A(\mathrm{H}^0(C), \mathrm{H}^0(C))$, but $\mathrm{H}^0(f) = 0$.

Now suppose that f^n is zero on $\tau^{<n}C$. The composition $C \xrightarrow{f} C \to \mathrm{H}^n(C)[-n]$ is zero, because it is equal to the composition $C \to \mathrm{H}^n(C)[-n] \to \mathrm{H}^n(C)[-n]$, where the last map is $\mathrm{H}^n(f) = 0$. Given the distinguished triangle

$$\tau^{<n}C \to C \to \mathrm{H}^n(C)[-n],$$

this implies that $f\colon C \to C$ factors as $C \xrightarrow{g} \tau^{<n}C \to C$ for some map g. Then f^{n+1} is the composition $C \xrightarrow{g} \tau^{<n}C \to C \xrightarrow{f^n} C$. The truncation $\tau^{<n}C$ is functorial in C, so the composition of the last two maps is equal to $\tau^{<n}C \xrightarrow{f^n} \tau^{<n}C \to C$. By the induction assumption f^n is zero on $\tau^{<n}C$, so the iterate f^{n+1} is zero on C. □

3 An abelian surface over $\mathbb{Q}$ with $\delta_2^{0,2} \neq 0$

In this section we present an abelian variety A over $\mathbb{Q}$ such that the differential $\delta_2^{0,2}$ of the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A_{k_{\mathrm{s}}}, \mathbb{Z}_2(1))) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbb{Z}_2(1)), \tag{24}$$

and the same differential of the spectral sequence with coefficients $\mathbb{Z}/2$, are both non-zero.

For an abelian variety A over k we denote by $\beta_2 : \mathrm{H}^1(k, A) \to \mathrm{H}^2(k, A[2])$ the connecting homomorphism induced by the short exact sequence of Γ-modules

$$0 \to A[2](k_{\mathrm{s}}) \to A(k_{\mathrm{s}}) \xrightarrow{2} A(k_{\mathrm{s}}) \to 0.$$

We use the following crucial proposition.

Proposition 3.1. *Let A be an abelian variety over a field k of characteristic zero with a polarization*

$$\lambda \in \mathrm{NS}\,(A_{k_{\mathrm{s}}})^\Gamma = \mathrm{Hom}_k(A, A^\vee)^{\mathrm{sym}}.$$

Let $\mathrm{c}_1(\lambda) \in \mathrm{H}^2_{\text{ét}}(A_{k_{\mathrm{s}}}, \mathbb{Z}_2(1))^\Gamma$ be the first Chern class of λ. Let $c'_\lambda \in \mathrm{H}^1(k, A^\vee)$ be the image of λ under the connecting map of the exact sequence of Γ-modules

$$0 \longrightarrow A^\vee(k_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(A_{k_{\mathrm{s}}}) \longrightarrow \mathrm{NS}\,(A_{k_{\mathrm{s}}}) \longrightarrow 0. \tag{25}$$

The image of $\delta_2^{0,2}(\mathrm{c}_1(\lambda)) \in \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(A_{k_{\mathrm{s}}}, \mathbb{Z}_2(1)))$ in $\mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(A_{k_{\mathrm{s}}}, \mathbb{Z}/2))$ is equal to $\beta_2(c'_\lambda) \in \mathrm{H}^2(k, A^\vee[2])$. In particular, $\delta_2^{0,2}(\mathrm{c}_1(\lambda)) \in \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(A_{k_{\mathrm{s}}}, \mathbb{Z}_2(1)))$ is divisible by 2 if and only if $c'_\lambda \in \mathrm{H}^1(k, A^\vee)$ is.

Proof. The antipodal involution $[-1]\colon A \to A$ induces an action of $\mathbb{Z}/2$ on $\mathrm{Pic}(A_{k_s})$ which turns (25) into an exact sequence of abelian groups with an action of $\mathbb{Z}/2$. The induced action on $\mathrm{NS}\,(A_{k_s})$ is trivial. The involution $[-1]_A$ induces the involution $[-1]_{A^\vee}$ on $A^\vee$. Since $A^\vee(k_s)$ is 2-divisible, we obtain $\mathrm{H}^1(\mathbb{Z}/2, A^\vee(k_s)) = 0$. Thus the long exact sequence of cohomology for the group $\mathbb{Z}/2$ gives an exact sequence of Γ-modules

$$0 \longrightarrow A^\vee[2] \longrightarrow \mathrm{Pic}(A_{k_s})^{[-1]^*} \longrightarrow \mathrm{NS}\,(A_{k_s}) \longrightarrow 0. \tag{26}$$

Let $c_\lambda \in \mathrm{H}^1(k, A^\vee[2])$ be the image of λ under the connecting map of (26). It is clear that c'_λ is the image of c_λ in $\mathrm{H}^1(k, A^\vee)$.

Let $T_2(A^\vee)$ be the 2-adic Tate module, and let $e_2\colon A[2] \times A^\vee[2] \to \mathbb{Z}/2$ be the Weil pairing. The short exact sequences (26) and (11) are compatible, so that there is a commutative diagram (see diagram (16) in [PR11]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\vee[2] & \longrightarrow & \mathrm{Pic}(A_{k_s})^{[-1]^*} & \longrightarrow & \mathrm{NS}\,(A_{k_s}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^\vee[2] & \longrightarrow & S^2(A^\vee[2]) & \longrightarrow & \wedge^2(A^\vee[2]) & \longrightarrow & 0
\end{array}$$

Furthermore, the right-hand vertical map sends λ to the element of $\wedge^2(A^\vee[2])$ corresponding to

$$e_2(x, \lambda(y)) \in \mathrm{Hom}(\wedge^2 A[2], \mathbb{Z}/2).$$

This map factors as the first Chern class map c_1

$$\mathrm{NS}\,(A_{k_s}) \cong \mathrm{Hom}(A_{k_s}, A^\vee_{k_s})^{\mathrm{sym}} \xrightarrow{\mathrm{c}_1} \mathrm{Hom}(T_2(A), T_2(A^\vee))^{\mathrm{sym}} \cong \wedge^2 T_2(A^\vee)(-1) \cong \mathrm{H}^2_{\text{ét}}(A_{k_s}, \mathbb{Z}_2(1))$$

followed by reduction modulo 2 map $\mathrm{H}^2_{\text{ét}}(A_{k_s}, \mathbb{Z}_2(1)) \to \mathrm{H}^2_{\text{ét}}(A_{k_s}, \mathbb{Z}/2)$, see [OSZ, Lemma 2.6]. Theorem 1.6 gives

$$\delta_2^{0,2}(\mathrm{c}_1(\lambda)) = \mathrm{Bock}_{T_2(A^\vee)}(c_\lambda), \tag{27}$$

where $\mathrm{Bock}_{T_2(A^\vee)}$ is the connecting map attached to the exact sequence

$$0 \to T_2(A^\vee) \xrightarrow{[2]} T_2(A^\vee) \to A^\vee[2] \to 0.$$

This sequence fits into the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\vee[2] & \longrightarrow & A^\vee & \xrightarrow{[2]} & A^\vee & \longrightarrow & 0 \\
 & & {\scriptstyle \cong}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^\vee[2] & \longrightarrow & A^\vee[4] & \longrightarrow & A^\vee[2] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & T_2(A^\vee) & \xrightarrow{[2]} & T_2(A^\vee) & \longrightarrow & A^\vee[2] & \longrightarrow & 0
\end{array}$$

This implies a relation between connecting homomorphisms induced by the first and last rows: the image of $\mathrm{Bock}_{T_2(A^\vee)}(c_\lambda) \in \mathrm{H}^2(k, T_2A^\vee)$ in $\mathrm{H}^2(k, A^\vee[2])$ equals $\beta_2(c'_\lambda)$, where $\beta_2 : \mathrm{H}^1(k, A^\vee) \to \mathrm{H}^2(k, A^\vee[2])$ is the connecting homomorphism induced by the top exact row, as defined above. Combining this with (27) gives the desired equality. □

Proposition 3.1 demonstrates the importance of the class c'_λ. This class has an explicit interpretation when A is the Jacobian of a curve, which is particularly convenient when C is hyperelliptic.

Let C be a smooth, projective, geometrically integral curve of genus g over a field k of characteristic not equal to 2. Let $J := \mathbf{Pic}^0_{C/k}$ be the Jacobian of C with its canonical principal polarization λ. We have a canonical map $C \hookrightarrow \mathbf{Pic}^1_{C/k}$.

By [PR11, Theorem 3.9], c_λ is the class of the torsor of theta characteristics, which is the closed subvariety of $\mathbf{Pic}^{g-1}_{C/k}$ given by $2x = K_C$, where K_C is the canonical class. Thus c'_λ is the class of $\mathbf{Pic}^{g-1}_{C/k}$ in $\mathrm{H}^1(k, J)$. If C is a hyperelliptic curve of odd genus g or with a rational Weierstrass point, then $c_\lambda = 0$, see [PR11, Proposition 3.11]. If g is even, then we have an isomorphism $\mathbf{Pic}^{g-1}_{C/k} \cong \mathbf{Pic}^1_{C/k}$ given by subtracting $\frac{g-2}{2}H_C$, where H_C is the hyperelliptic divisor class. Thus c_λ is the class of the torsor for $J^\vee[2]$ in $\mathbf{Pic}^1_{C/k}$ given by $2x = H_C$, and $c'_\lambda = [\mathbf{Pic}^1_{C/k}]$.

Now let k be a number field. In this case we have $c_\lambda \in \mathrm{Sel}_2(J^\vee)$, see [PS99, Corollary 2]. Thus $c'_\lambda \in Ш(J^\vee)[2]$. By [PS99, Theorem 5] we have $\langle x, \lambda_*(x) + c'_\lambda \rangle = 0$ for any $x \in Ш(J)$, where

$$\langle x, y \rangle_{\mathrm{CT}} \colon Ш(J) \times Ш(J^\vee) \to \mathbb{Q}/\mathbb{Z}$$

is the Cassels–Tate pairing, see [Mil86, Ch. I, §6].

For a finite Γ-module M one defines $Ш^1(k, M)$ as the subgroup of $\mathrm{H}^1(k, M)$ consisting of the classes that go to zero under the restriction map to $\mathrm{H}^1(k_v, M)$, for all places v of k. It is clear that $Ш^1(k, J[2]) \subset \mathrm{Sel}_2(J)$.

Over a number field, we can test the divisibility of c'_λ in $\mathrm{H}^1(k, J^\vee)$ in terms of the Cassels–Tate pairing. Let us denote by i the natural map $\mathrm{H}^1(k, J[2]) \to \mathrm{H}^1(k, J)$.

Lemma 3.2. *The class c'_λ is divisible by 2 in $\mathrm{H}^1(k, J^\vee)$ if and only if $i(Ш^1(k, J[2]))$ is orthogonal to c'_λ with respect to the Cassels–Tate pairing.*

Proof. This is a particular case of [Cre13, Theorem 4], which is based on the non-degeneracy of the Poitou–Tate pairing [Mil86, Ch. I, §4]

$$Ш^1(k, J[2])) \times Ш^2(k, J^\vee[2]) \to \mathbb{Z}/2$$

and its compatibility with the Cassels–Tate pairing. □

Manin pointed out that the Cassels–Tate pairing can be interpreted as a particular case of what is now called the Brauer–Manin pairing. Let us recall this interpretation.

Let $\mathrm{Br}(C)$ be the Brauer group of C and let $\mathrm{Br}_0(C) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(C)]$. Since $\mathrm{Br}(C_{k_s}) = 0$, the Leray spectral sequence $\mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(C_{k_s}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(C, \mathbb{G}_m)$ gives an exact sequence

$$0 \to \mathrm{Br}_0(C) \to \mathrm{Br}(C) \to \mathrm{H}^1(k, \mathrm{Pic}(C_{k_s})) \to 0, \tag{28}$$

where we used that $\mathrm{H}^3(k, {k_s}^\times) = 0$ which holds since k is a number field. We have an exact sequence of Γ-modules

$$0 \to J(k_s) \to \mathrm{Pic}(C_{k_s}) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

It induces an isomorphism of $\mathrm{H}^1(k, \mathrm{Pic}(C_{k_s}))$ with the quotient of $\mathrm{H}^1(k, J)$ by the cyclic subgroup generated by $[\mathbf{Pic}^1_{C/k}]$. For an element $x \in \mathrm{H}^1(k, J)$ we call any $\mathcal{A} \in \mathrm{Br}(C)$ a Brauer class *associated* to x if the images of x and $\mathcal{A}$ in $\mathrm{H}^1(k, \mathrm{Pic}(C_{k_s}))$ are equal. Similar considerations show that $\mathcal{A}$ extends to a Brauer class on $\mathbf{Pic}^1_{C/k}$.

Lemma 3.3. *Let C be a hyperelliptic curve of even genus g over a number field k that is everywhere locally soluble. The class c'_λ is not divisible by 2 in $\mathrm{H}^1(k, J^\vee)$ if the class in $\mathrm{Br}(C)$ associated to some element of $i(\text{Ш}^1(k, J[2])) \subset \mathrm{H}^1(k, J)$ obstructs the Hasse principle on C.*

Proof. Since $c_\lambda \in \mathrm{Sel}_2(J^\vee)$, the k-variety $\mathbf{Pic}^1_{C/k}$ has points everywhere locally. For each place of k choose a local point $P_v \in C(k_v)$. By a theorem of Manin [Sk01, Theorem 6.2.3], we have

$$\sum_v \mathrm{inv}_v(\mathcal{A}(P_v)) = -\langle x, c'_\lambda \rangle_{\mathrm{CT}} \in \mathbb{Q}/\mathbb{Z},$$

for any $x \in \text{Ш}(J)$ and $\mathcal{A} \in \mathrm{Br}(C)$ associated to x, using that $c'_\lambda = [\mathbf{Pic}^1_{C/k}] \in \text{Ш}(J^\vee)$. Therefore, Lemma 3.2 implies our statement. □

Using an explicit example due to Creutz [Cre13, p. 941] and Creutz and Viray [CV15, Theorem 6.7] we get the following result.

Theorem 3.4. *Let J be the Jacobian of the hyperelliptic curve of genus 2 over $\mathbb{Q}$ given by*

$$y^2 = 3(x^2 + 1)(x^2 + 17)(x^2 - 17).$$

Then the differential $\delta_2^{0,2}$ of the spectral sequence (4), and the same differential of the analogous spectral sequence with coefficients in $\mathbb{Z}/2$, are both non-zero.

Proof. We sketch the computation of Creutz and Creutz–Viray for the convenience of the reader. Let C be the hyperelliptic curve over a field k given by $y^2 = cf(x)$ where $c \in k^\times$ and $f(x) \in k[x]$ is a separable monic polynomial of even degree. Let J be the Jacobian of C. Let $L = k[x]/(f(x))$ and let $\theta \in L$ be the image of x. The well-known identification of the group k-scheme $J[2]$ with the quotient of the kernel of the norm map $R_{L/k}(\mu_2) \to \mu_2$ by the image of μ_2 gives a natural inclusion

$$(L^\times/k^\times L^{\times 2})_1 \subset \mathrm{H}^1(k, J[2]),$$

where the subscript 1 denotes the subgroup of elements with norm $1 \in k^\times/k^{\times 2}$. To $l \in L^\times$ we associate the class of the quaternion algebra $(l, x-\theta) \in \mathrm{Br}(L(x))$, which only depends on the image of l in $L^\times/L^{\times 2}$. The corestriction

$$\mathrm{cores}_{L(x)/k(x)}\big((l, x-\theta)\big) \in \mathrm{Br}(k(\mathbb{P}^1_k))$$

is unramified away from the ramification locus of $C \to \mathbb{P}^1_k$ and the point at infinity. If the norm of l is a square in $k^\times$, then it is unramified at the infinity. It follows that the pullback of this element to $\mathrm{Br}(k(C))$ is unramified, and so belongs to the subgroup $\mathrm{Br}(C) \subset \mathrm{Br}(k(C))$. We denote the resulting element by $\mathcal{A}_l \in \mathrm{Br}(C)$. Finally, multiplying l by $s \in k^\times$ gives $\mathcal{A}_{ls} = \mathcal{A}_l + (s, c)$, so this does not change the image of $\mathcal{A}_l$ in $\mathrm{Br}(C)/\mathrm{Br}_0(C)$. An important property of this construction is that the map sending l to $\mathcal{A}_l$ coincides with the composition

$$\mathrm{H}^1(k, J[2]) \xrightarrow{i} \mathrm{H}^1(k, J) \longrightarrow \mathrm{Br}(C)/\mathrm{Br}_0(C),$$

where the second arrow is given by (28).

Now let C be the hyperelliptic curve of genus 2 over $\mathbb{Q}$ given by

$$y^2 = c(x^2+1)(x^2+17)(x^2-17),$$

where c is a positive integer. The 0-dimensional scheme $(x^2+1)(x^2+17)(x^2-17) = 0$ is everywhere locally soluble, hence C is everywhere locally soluble too.

We have $L = \mathbb{Q}(\sqrt{-1}) \oplus \mathbb{Q}(\sqrt{-17}) \oplus \mathbb{Q}(\sqrt{17})$. Take $l = (1, 1, -1) \in L$. We note that l gives rise to an everywhere locally trivial element of $(L^\times/\mathbb{Q}^\times L^{\times 2})_1$, that is, to an element of $\text{Ш}^1(\mathbb{Q}, J[2])$. We have $\mathcal{A}_l = (-1, x^2-17)$. A local computation [CV15, Lemma 6.8] shows that if the number of odd prime factors p of c such that neither 17 nor -1 is a square modulo p is odd, then $\sum_v \mathrm{inv}_v(\mathcal{A}_l(P_v)) = 1/2$. For example, one can take $c = 3$. Thus Lemma 3.3 can be applied, so that c'_λ is not divisible by 2 in $\mathrm{H}^1(k, J^\vee)$, hence $\delta_2^{0,2}$ is non-zero by Proposition 3.1. □

4 The Brauer group of a torus

Let T be a torus over a field k of characteristic exponent p. Let $\widehat{T} = \mathrm{Hom}_{k_s-\mathrm{gps}}(T_{k_s}, \mathbb{G}_m)$ be the Γ-module of characters of T. The Brauer group $\mathrm{Br}(T) = \mathrm{H}^2_{\text{ét}}(T, \mathbb{G}_m)$ is computed by the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(T_{k_s}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(T, \mathbb{G}_m). \tag{29}$$

We denote by $d_r^{p,q}$ the differential on the rth page emanating from the (p,q)-entry.

Since T_{k_s} is a dense open subscheme of $\mathbb{A}^n_{k_s}$, we have $\mathrm{H}^1(T_{k_s},\mathbb{G}_m)\cong \mathrm{Pic}(T_{k_s})=0$. Thus $d_2^{0,2}=0$, and the first interesting differential is $d_3^{0,2}\colon \mathrm{Br}(T_{k_s})^\Gamma\to \mathrm{H}^3(k,k_s[T]^\times)$.

The origin $e\in T(k)$ of the group law on T gives a section of the structure morphism $T\to \mathrm{Spec}(k)$, hence $\mathrm{H}^r(k,\mathbb{G}_m)\to \mathrm{H}^r_{\text{ét}}(T,\mathbb{G}_m)$ is injective for all $r\geq 0$. Likewise, the natural map $\mathrm{Br}(k)\to\mathrm{Br}(T)$ is injective. Let $\mathrm{Br}_e(T)=\mathrm{Ker}[\mathrm{Br}(T)\to \mathrm{Br}(k)]$ be the kernel of specialisation at e.

We have an isomorphism of Γ-modules $\mathrm{H}^0_{\text{ét}}(T_{k_s},\mathbb{G}_m)=k_s[T]^\times\cong k_s^\times\oplus\widehat{T}$. The spectral sequence (29) thus gives rise to an exact sequence

$$0\to \mathrm{H}^2(k,\widehat{T})\to \mathrm{Br}(T)_e\to \mathrm{Br}(T_{k_s})^\Gamma \xrightarrow{\bar{d}_3^{0,2}} \mathrm{H}^3(k,\widehat{T}). \tag{30}$$

To compute $\mathrm{Br}(T)$ one needs to describe the map $\bar{d}_3^{0,2}\colon \mathrm{Br}(T_{k_s})^\Gamma\to \mathrm{H}^3(k,\widehat{T})$, which is the composition of the differential $d_3^{0,2}$ with the map induced by the projection $k_s[T]^\times\to\widehat{T}$.

Let ℓ be a prime number not equal to p. For an abelian group A we write $A(p')$ for the subgroup of A consisting of the elements of finite order not divisible by p.

Since $\mathrm{Pic}(T_{k_s})=0$, the Kummer sequence gives rise to an isomorphism

$$\kappa\colon \mathrm{H}^2_{\text{ét}}(T_{k_s},\mu_{\ell^n})\xrightarrow{\sim}\mathrm{Br}(T_{k_s})[\ell^n].$$

Using the isomorphism of Γ-modules $\mathrm{H}^0_{\text{ét}}(T_{k_s},\mathbb{G}_m)\cong k_s^\times\oplus\widehat{T}$, we also deduce from the Kummer sequence a natural isomorphism

$$\widehat{T}/\ell^n\xrightarrow{\sim}\mathrm{H}^1_{\text{ét}}(T_{k_s},\mu_{\ell^n}).$$

We note that $\mathrm{H}^2_{\text{ét}}(T_{k_s},\mathbb{Z}/\ell^n)\cong \wedge^2\mathrm{H}^1_{\text{ét}}(T_{k_s},\mathbb{Z}/\ell^n)$ and thus multiplication by m map $[m]\colon T\to T$ acts on $\mathrm{H}^2_{\text{ét}}(T_{k_s},\mu_{\ell^n})$, and hence on $\mathrm{Br}(T_{k_s})(p')$, as m^2. On the other hand, $[m]$ acts on $\widehat{T}$ as m. Taking $m=-1$ we see that $2\bar{d}_3^{2,0}=0$, so that $\bar{d}_3^{2,0}$ sends the elements of $\mathrm{Br}(T_{k_s})(p')$ of odd order to zero. Then it follows from (30) that every element of odd order in $\mathrm{Br}(T_{k_s})(p')^\Gamma$ lifts to $\mathrm{Br}(T)$.

The question of an explicit description of the map $\bar{d}_3^{2,0}$ on the 2-primary torsion subgroup of $\mathrm{Br}(T_{k_s})^\Gamma$ was asked on top of [CTS21, p. 220]. We now give such a description.

Theorem 4.1. *Let T be a torus over a field k of characteristic different from 2. Let $\bar{d}_3^{0,2}$ be the composition*

$$\mathrm{Br}(T_{k_s})^\Gamma\xrightarrow{d_3^{0,2}}\mathrm{H}^3(k,k_s[T]^\times)\to\mathrm{H}^3(k,\widehat{T}),$$

where the last map is induced by the projection $k_s[T]^\times\cong k_s^\times\times\widehat{T}\to\widehat{T}$. The restriction of $\bar{d}_3^{0,2}$ to the 2^n-torsion subgroup is the composition

$$\mathrm{Br}(T_{k_s})[2^n]^\Gamma\underset{\simeq}{\xrightarrow{\kappa^{-1}}}\wedge^2(\widehat{T}/2^n)(-1)^\Gamma\xrightarrow{\mathrm{Bock}}\mathrm{H}^1(k,\wedge^2(\widehat{T}/2))\xrightarrow{\alpha[1]}\mathrm{H}^2(k,\widehat{T}/2)\xrightarrow{\mathrm{Bock}}\mathrm{H}^3(k,\widehat{T}).$$

Proof. We will show that the restriction of $d_3^{0,2}$ to the 2^n-torsion subgroup can be read off from the Hochschild–Serre spectral sequence for the cohomology of T with coefficients in μ_{2^n}. The natural map $\tau^{\leq 2}\mathrm{R}\Gamma_{\text{ét}}(T_{k_s}, \mu_{2^n}) \to \tau^{\leq 2}\mathrm{R}\Gamma_{\text{ét}}(T_{k_s}, \mathbb{G}_m)$ of truncations in degrees ≤ 2 induces a commutative diagram relating the extensions between H^2 and $\tau^{\leq 1}$ in these complexes:

$$\begin{array}{ccc} \mathrm{H}^2_{\text{ét}}(T_{k_s}, \mu_{2^n}) & \longrightarrow & \tau^{\leq 1}\mathrm{R}\Gamma(T_{k_s}, \mu_{2^n})[3] \\ \downarrow & & \downarrow \\ \mathrm{H}^2_{\text{ét}}(T_{k_s}, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(T_{k_s}, \mathbb{G}_m)[3] \end{array} \tag{31}$$

where we used that $\mathrm{H}^1_{\text{ét}}(T_{k_s}, \mathbb{G}_m) = \mathrm{Pic}(T_{k_s})$ vanishes. The differential $d_3^{0,2}$ is obtained by applying the functor $\mathrm{H}^0(k, -)$ to the bottom horizontal map in the diagram.

Galois invariants of the left vertical map is exactly the inclusion of the 2^n-torsion $\Lambda^2(\widehat{T}/2^n)(-1)^\Gamma \cong \mathrm{H}^2_{\text{ét}}(T_{k_s}, \mu_{2^n})$ into the Galois invariants in the Brauer group of T_{k_s}, so we are looking to compute the result of applying $\mathrm{H}^0(k, -)$ to the counter-clockwise composition in (31), composed with the projection $\mathrm{H}^3(k, \mathrm{H}^0_{\text{ét}}(T_{k_s}, \mathbb{G}_m)) \to \mathrm{H}^3(k, \widehat{T})$.

From the Kummer sequence, we see that $\tau^{\leq 1}\mathrm{R}\Gamma_{\text{ét}}(T_{k_s}, \mu_{2^n})$ can be represented by the two-term complex $\mathrm{H}^0_{\text{ét}}(T_{k_s}, \mathbb{G}_m) \xrightarrow{2^n} \mathrm{H}^0_{\text{ét}}(T_{k_s}, \mathbb{G}_m)$ with the right vertical arrow in (31) given by shift by [3] of the projection onto the 0th term of this complex. This map followed by the projection to $\widehat{T}$ sends $\mathrm{H}^0_{\text{ét}}(T_{k_s}, \mu_{2^n})$ to zero, so the composition factors through $\tau^{[1,1]}\mathrm{R}\Gamma_{\text{ét}}(T_{k_s}, \mu_{2^n})$. Applying $\mathrm{H}^0(k, -)$ to this composition, we obtain a map that factors as follows:

$$\mathrm{H}^0(k, \tau^{\leq 1}\mathrm{R}\Gamma_{\text{ét}}(T_{k_s}, \mu_{2^n})[3]) \to \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(T_{k_s}, \mu_{2^n})) \cong \mathrm{H}^2(k, \widehat{T}/2^n) \xrightarrow{-\mathrm{Bock}_{\widehat{T}}} \mathrm{H}^3(k, \widehat{T}).$$

Therefore the clock-wise composition in (31) evaluated on $\mathrm{H}^0(k-,)$, composed with the projection onto $\mathrm{H}^3(k, \widehat{T})$, is given by

$$\mathrm{H}^2_{\text{ét}}(T_{k_s}, \mu_{2^n})^\Gamma \xrightarrow{\delta_2^{0,2}} \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(T_{k_s}, \mu_{2^n})) \xrightarrow{-\mathrm{Bock}_{\widehat{T}}} \mathrm{H}^3(k, \widehat{T})$$

where $\delta_2^{0,2}$ is a differential of the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(T_{k_s}, \mu_{2^n})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(T, \mu_{2^n}).$$

By Theorem 1.10, $\delta_2^{0,2}$ is obtained by applying $- \otimes^L_{\mathbb{Z}_2} \mu_{2^n}$ to the terms of the 2-extension given by the difference of the maps (19) and (20) with $M = \mathrm{H}^1_{\text{ét}}(T_{k_s}, \mathbb{Z}_2)$. The composition of two Bockstein maps is zero, so only (20) contributes to $\bar{d}_3^{0,2}$, thus proving that $\bar{d}_3^{0,2}$ is the composition of four maps in the theorem. □

Remark 4.2. It would be interesting to construct a torus with a non-zero map $\bar{d}_3^{0,2}$, or prove that none exist.

When the first version of this paper was completed, the authors became aware of the following result of Julian Demeio [Dem, Theorem 1.1]. Recall that a torus is called *quasi-trivial* if $\widehat{T}$ is a permutational Γ-module, that is, $\widehat{T}$ has a Γ-stable $\mathbb{Z}$-basis.

Corollary 4.3. *Let k be a field of characteristic zero. If T is a quasi-trivial torus, or if k is a local or global field, then the natural map $\mathrm{Br}(T) \to \mathrm{Br}(T_{k_\mathrm{s}})^\Gamma$ is surjective.*

Proof. Let us show that $\bar{d}_3^{0,2} = 0$. The case of quasi-trivial torus is immediate from Theorem 4.1 and Lemma 1.8 (i). If k is a p-adic field, then $\mathrm{H}^3(k, \widehat{T}) = 0$ since k has strict cohomological dimension 2 [Har20, Theorem 10.6], so there is nothing to prove. The case $k_v = \mathbb{R}$ follows from Theorem 4.1 and Lemma 1.8 (ii). If k is a number field, we have an isomorphism $\mathrm{H}^3(k, \widehat{T}) \xrightarrow{\sim} \prod \mathrm{H}^3(k_v, \widehat{T})$, where the product is over the real places of k, see [Har20, Exercise 18.1], so this case follows from the case of local fields. □

5 Torsors

Let X be a k-torsor for a semiabelian variety A. In this section we address the problem of computing the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n)$, where n is not divisible by $\mathrm{char}(k)$.

Since A_{k_s} is connected, translations by elements of $A(k_\mathrm{s})$ act trivially on the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n)$. Therefore, any choice $X_{k_\mathrm{s}} \simeq A_{k_\mathrm{s}}$ of a trivialization of the torsor X over k_s gives rise to the same Γ-equivariant isomorphism

$$\mathrm{H}^i_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n) \cong \mathrm{H}^i_{\text{ét}}(A_{k_\mathrm{s}}, \mathbb{Z}/n), \quad i \geq 0.$$

But the complexes $\mathrm{R}\Gamma_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n)$ and $\mathrm{R}\Gamma_{\text{ét}}(A_{k_\mathrm{s}}, \mathbb{Z}/n)$ need not be isomorphic as objects of $D(\Gamma, \mathbb{Z}/n)$: the truncation of the latter in degrees $[0,1]$ is the direct sum of its cohomology groups, but the truncation $\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n)$ need not be split as we demonstrate in this section.

The Hochschild–Serre spectral sequence for X has the form

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A_{k_\mathrm{s}}, \mathbb{Z}/n)) \cong \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Z}/n).$$

The question we address is the explicit form of the differentials

$$\delta_2^{p,1} \colon \mathrm{H}^p(k, \mathrm{H}^1_{\text{ét}}(A_{k_\mathrm{s}}, \mathbb{Z}/n)) \to \mathrm{H}^{p+2}(k, \mathbb{Z}/n),$$

where $p \geq 0$. Each of these differentials is induced by the map in the derived category $\mathbb{Z}/n$-modules with a continuous discrete action of Γ

$$\delta_X : \mathrm{H}^1_{\text{ét}}(A_{k_\mathrm{s}}, \mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n) \to \mathbb{Z}/n[2]$$

arising from the complex $\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_\mathrm{s}}, \mathbb{Z}/n) \in D(\Gamma, \mathbb{Z}/n)$.

Recall from notation section of the introduction that $\mathcal{A}(G,\mathbb{Z}/n)$ is the abelian category of discrete $\mathbb{Z}/n$-modules equipped with a continuous action of G. Writing $M = \mathrm{H}^1_{\text{ét}}(A_{k_s},\mathbb{Z}/n)$, we can think of δ_X as an element of $\mathrm{Ext}^2_{\mathcal{A}(G,\mathbb{Z}/n)}(M,\mathbb{Z}/n)$. Since $\mathrm{Hom}_{\mathcal{A}(G,\mathbb{Z}/n)}(M,\mathbb{Z}/n) = \mathrm{Hom}_{\mathbb{Z}/n}(M,\mathbb{Z}/n)^{\Gamma}$, we have a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{Ext}^q_{\mathbb{Z}/n}(M,\mathbb{Z}/n)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}(G,\mathbb{Z}/n)}(M,\mathbb{Z}/n). \tag{32}$$

In our case, M is a free, hence projective $\mathbb{Z}/n$-module, thus $\mathrm{Ext}^q_{\mathbb{Z}/n}(M,\mathbb{Z}/n) = 0$ for $q > 0$. Now (32) gives a natural isomorphism

$$\mathrm{Ext}^2_{\mathcal{A}(G,\mathbb{Z}/n)}(M,\mathbb{Z}/n) \cong \mathrm{H}^2(k, \mathrm{Hom}_{\mathbb{Z}/n}(M,\mathbb{Z}/n)) \cong \mathrm{H}^2(k, A[n]), \tag{33}$$

where the second isomorphism is the $\mathbb{Z}/n$-linear dual of a natural isomorphism of Γ-modules $\mathrm{Hom}_{\mathbb{Z}/n}(A[n],\mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(A_{k_s},\mathbb{Z}/n)$ induced by the map (9). Using isomorphisms (33), we identify δ_X with an element of $\mathrm{H}^2(k, A[n])$.

We will express the class δ_X in terms of the class of the torsor X. The exact sequence of Γ-modules

$$0 \to A[n] \to A \to A \to 0 \tag{34}$$

gives rise to the homomorphism of Galois cohomology groups

$$\beta_n \colon \mathrm{H}^1(k, A) \to \mathrm{H}^2(k, A[n]). \tag{35}$$

Theorem 5.1. *Let k be a field and let n be a positive integer not divisible by* $\mathrm{char}\, k$. *Let X be a k-torsor for a semiabelian variety A over k with class $[X] \in \mathrm{H}^1(k, A)$. The class $\delta_X \in \mathrm{H}^2(k, A[n])$ is equal to the image of $[X]$ under the map β_n. In particular, the differentials $\delta_2^{p,1}$ are given by cupping with the class*

$$\beta_n([X]) \in \mathrm{H}^2(k, A[n]) \cong \mathrm{Ext}^2_{\mathcal{A}(G,\mathbb{Z}/n)}(\mathrm{H}^1(A_{k_s},\mathbb{Z}/n),\mathbb{Z}/n).$$

Proof. Our goal is to compute the morphism

$$\delta_X : \mathrm{Hom}_{\mathbb{Z}/n}(A[n],\mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(X_{k_s},\mathbb{Z}/n) \to \mathrm{H}^0_{\text{ét}}(X_{k_s},\mathbb{Z}/n)[2] \cong \mathbb{Z}/n[2]$$

in $D(\Gamma,\mathbb{Z}/n)$ corresponding to the complex $\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s},\mathbb{Z}/n)$. The $\mathbb{Z}/n$-linear dual of the map δ_X shifted by [2] is a map $\delta_X^{\vee}[2] : \mathbb{Z}/n \to (A[n])[2]$ which is the following composition

$$\mathbb{Z}/n \xrightarrow{\Delta} \mathrm{End}_{\mathbb{Z}/n}(A[n]) = \mathrm{Hom}_{\mathbb{Z}/n}(A[n],\mathbb{Z}/n) \otimes^L_{\mathbb{Z}/n} A[n] \xrightarrow{\delta_X \otimes \mathrm{id}} \mathbb{Z}/n[2] \otimes^L_{\mathbb{Z}/n} A[n],$$

where Δ sends $1 \in \mathbb{Z}/n$ to $\mathrm{id}_{A[n]} \in \mathrm{End}_{\mathbb{Z}/n}(A[n])$. After applying the derived functor of the functor of Γ-invariants, we see that $\mathrm{id}_{A[n]} \in \mathrm{End}_{\mathbb{Z}/n}(A[n])^{\Gamma}$ goes to $\delta_X \in \mathrm{H}^2(k, A[n])$.

The map $\delta_X \otimes \mathrm{id}_{A[n]}$ can be identified with the connecting map of the complex $\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s},\mathbb{Z}/n) \otimes^L_{\mathbb{Z}/n} A[n] \simeq \tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, A[n])$. Thus the differential

$$d \colon \mathrm{H}^1_{\text{ét}}(X_{k_s}, A[n])^{\Gamma} \to \mathrm{H}^2_{\text{ét}}(k, A[n])$$

of the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X_{k_s}, A[n])) \to \mathrm{H}^{p+q}_{\text{ét}}(X, A[n])$$

is induced by $\delta_X \otimes \mathrm{id}_{A[n]}$. There are natural isomorphisms of Γ-modules

$$\mathrm{H}^1_{\text{ét}}(X_{k_s}, A[n]) \cong \mathrm{H}^1_{\text{ét}}(A_{k_s}, A[n]) \cong \mathrm{H}^1_{\text{ét}}(A_{k_s}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} A[n] \cong \mathrm{End}_{\mathbb{Z}/n}(A[n]).$$

In the notation of [Sk01, Proposition 3.2.2], under these isomorphisms, $\mathrm{id}_{A[n]} \in \mathrm{End}_{\mathbb{Z}/n}(A[n])$ corresponds to the class $\tau \in \mathrm{H}^1_{\text{ét}}(X_{k_s}, A[n])$ of the X_{k_s}-torsor for $A[n]$ given by (34) after an identification of X_{k_s} with A_{k_s}. (Note that τ does not depend on the choice of a k_s-point of X, thus $\tau \in \mathrm{H}^1_{\text{ét}}(X_{k_s}, A[n])^\Gamma$.) This gives $d(\tau) = \delta_X$.

On the other hand, by [Sk01, Proposition 3.2.2], which is a restatement of [Gir71, Proposition V.3.2.9], we have $d(\tau) = \beta_n([X])$. □

Remark 5.2. Using [Sk01, Lemma 2.4.5] as in [HS09, Proposition 2.2], one sees easily that $\beta_n([X]) \in \mathrm{H}^2(k, A[n])$ can also be interpreted as the class of the group extension

$$0 \to A[n] \to G_{X,n} \to \Gamma \to 0$$

obtained from the fundamental exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{k_s}) \to \pi_1^{\text{ét}}(X) \to \Gamma \to 1$$

by pushing out along the surjection $\pi_1^{\text{ét}}(X_{k_s}) \cong \pi_1^{\text{ét}}(A_{k_s}) \to A[n]$.

Among the k-torsors X for a semiabelian variety A whose class in $\mathrm{H}^1(k, A)$ is not divisible by n, we have the following well-known examples in dimension 1.

Example 5.3. (i) Let k be a field of characteristic not equal to 2. Let X be the affine curve $x^2 - ay^2 = b$, where $a, b \in k^\times$. This is a torsor for the norm 1 torus

$$S = R^1_{k(\sqrt{a})/k}(\mathbb{G}_{m,k(\sqrt{a})}) := \mathrm{Ker}[R_{k(\sqrt{a})/k}(\mathbb{G}_{m,k(\sqrt{a})}) \to \mathbb{G}_{m,k}],$$

where the arrow is given by the norm $\mathrm{N}\colon k(\sqrt{a}) \to k$. Hilbert's Theorem 90 gives an isomorphism $\mathrm{H}^1(k, S) \cong k^\times/\mathrm{N}(k(\sqrt{a})^\times)$. This group is annihilated by 2, so if $[X] \neq 0$ then $[X]$ is not divisible by 2. The class $[X] \in \mathrm{H}^1(k, S)$ is zero if and only if the projective conic $x^2 - ay^2 = bz^2$ has a k-point, that is, if and only if the symbol $(a, b) \in \mathrm{H}^2(k, \mathbb{Z}/2)$ is zero. In fact, we have an isomorphism of group k-schemes $S[2] \cong \mathbb{Z}/2$, and $\delta_X = \beta_2([X]) = (a, b) \in \mathrm{H}^2(k, \mathbb{Z}/2)$, as follows, for example, from [CTS21, Proposition 7.1.11].

(ii) Let $k = \mathbb{R}$ and let X be a smooth projective curve of genus 1 over $\mathbb{R}$ such that $X(\mathbb{R}) = \emptyset$. Let E be the Jacobian of X. The group $\mathrm{H}^1(\mathbb{R}, E)$ is annihilated by 2, and since $[X] \neq 0$, we see that $\delta_X = \beta_2([X]) \neq 0$.

Remark 5.4. (i) More generally, if X is a geometrically connected scheme over k, then the map $\mathbb{Z}/n \cong \mathrm{H}^0_{\text{ét}}(X_{k_s}, \mathbb{Z}/n) \to \mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}/n)$ admits a splitting in $D(\Gamma, \mathbb{Z}/n)$ if X has a 0-cycle of degree coprime to n. Indeed, for a 0-cycle $\sum_i a_i[Z_i]$ with each $f_i : Z_i \hookrightarrow X$ isomorphic to $\mathrm{Spec}(L_i)$, where L_i is a finite extension of k, consider the induced maps

$$r_i : \mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}/n) \xrightarrow{f_i^*} \mathrm{R}\Gamma(\mathrm{Spec}(L_i \otimes_k k_s), \mathbb{Z}/n) \cong \bigoplus_{L_{i,s} \hookrightarrow k_s} \mathbb{Z}/n \xrightarrow{\Sigma} \mathbb{Z}/n.$$

Here the direct sum is taken over all embeddings of the maximal separable subextension $L_{i,s}$ of L_i into k_s. The map Σ is Γ-equivariant because the Galois group permutes the summands in the direct sum. The map on H^0 induced by r_i is then the multiplication by the degree $[L_{i,s} : k]$, so the sum $\sum a_i \cdot [L_i : L_{i,s}] \cdot r_i$ is a map $\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}/n) \to \mathbb{Z}/n$ inducing multiplication by the degree of the cycle $\sum_i a_i[Z_i]$.

(ii) In particular, if X is a geometrically connected smooth proper variety over k, then the map $\mathbb{Z}/n \to \mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}/n)$ admits a splitting if n is coprime to the Euler characteristic $\chi(X_{k_s})$ (defined, e.g., as $\sum_i (-1)^i \dim_{\mathbb{Q}_\ell} \mathrm{H}^i_{\text{ét}}(X_{k_s}, \mathbb{Q}_\ell)$ for any $\ell \neq \mathrm{char}\, k$). Indeed, the top Chern class $c_{\dim X}(T_X) \in \mathrm{CH}^{\dim X}(X)$ of the tangent bundle is a 0-cycle of degree $\chi(X)$, by the Hirzebruch–Riemann–Roch formula. By (i), such a cycle gives a map $\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mathbb{Z}/n) \to \mathbb{Z}/n$ which induces $\chi(X) \cdot \mathrm{id}$ on H^0.

6 Curves

Let C be a smooth, projective, geometrically integral curve over a field k. Let $J := \mathbf{Pic}^0_{C/k}$ be the Jacobian of C and let $X := \mathbf{Pic}^1_{C/k}$ be the Albanese torsor of C. The canonical map $C \to X$ gives isomorphisms of Γ-modules

$$\mathrm{H}^1_{\text{ét}}(C_{k_s}, \mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(X_{k_s}, \mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(J_{k_s}, \mathbb{Z}/n).$$

We use this identification in the following proposition.

Proposition 6.1. *Let n be an integer not divisible by* $\mathrm{char}\, k$.

(a) *For all $i \in \mathbb{Z}$ and all $p \geq 0$ the differential $\delta^{p,1}_{2,C}$ of the spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(C_{k_s}, \mu_n^{\otimes i})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(C, \mu_n^{\otimes i}),$$

is equal to $\delta^{p,1}_{2,X}$, and is induced by the class $\beta_n([X]) \in \mathrm{H}^2(k, J[n])$, as in Theorem 5.1;

(b) *the differential $\delta^{0,2}_{2,C}$ of the spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(C_{k_s}, \mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(C, \mu_n) \qquad (36)$$

sends the generator of $\mathrm{H}^2(C_{k_s}, \mu_n)^\Gamma \cong \mathrm{H}^2(C_{k_s}, \mu_n) \cong \mathbb{Z}/n$ *to the image of* $[X]$ *under the map*

$$\mathrm{H}^1(k, J) \xrightarrow{\beta_n} \mathrm{H}^2(k, J[n]) \cong \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(J^\vee_{k_s}, \mu_n)) \cong \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(J_{k_s}, \mu_n)).$$

Here the last isomorphism is induced by the principal polarization of J. Consequently, for all $p \geq 0$ the differential $\delta_{2,C}^{p,2}$ is given by cupping with this class in $\mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(J_{k_s}, \mu_n))$.

Proof. The canonical map $C \to X$ induces a map $\mathrm{R}\Gamma_{\text{ét}}(X_{k_s}, \mu_n^{\otimes i}) \to \mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mu_n^{\otimes i})$ in $D(\Gamma, \mathbb{Z}/n)$, which induces isomorphisms of cohomology groups in degrees 0 and 1. This gives (a).

Poincaré duality states that in $D(\Gamma, \mathbb{Z}/n)$ the objects $\mathrm{Hom}_{\mathbb{Z}/n}(\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{Z}/n), \mu_n)$ and $\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{Z}/n)[2]$ are canonically isomorphic. Since each cohomology group of $\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module, this implies a duality on truncations: the objects $\mathrm{Hom}_{\mathbb{Z}/n}(\tau^{[0,1]}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{Z}/n), \mu_n)$ and $\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{Z}/n)[2]$ are canonically isomorphic. Therefore the differential $\delta_{2,C}^{0,2}$ of the Leray spectral sequence with coefficients $\mathbb{Z}/n$ is given by the class of a 2-extension in $\mathrm{Ext}^2_k(\mathbb{Z}/n, \mathrm{H}^1_{\text{ét}}(J^\vee_{k_s}, \mu_n))$ that is dual to the extension defining $\delta_{2,C}^{0,1}$. It follows that $\delta_{2,C}^{0,2}$ sends the generator of $\mathrm{H}^2_{\text{ét}}(C_{k_s}, \mu_n)^\Gamma \cong \mathbb{Z}/n$ to the image of $[X]$ in $\mathrm{H}^2(k, J[n]) \cong \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(J^\vee_{k_s}, \mu_n))$. □

Example 6.2. Let C be a smooth, projective, geometrically integral curve over $\mathbb{R}$ such that $C(\mathbb{R}) = \emptyset$. If the genus of C is odd, then the Albanese torsor is non-trivial, that is, $[X] \neq 0$ in $\mathrm{H}^1(\mathbb{R}, J)$, see [GH81, Proposition 3.3 (2)]. Since $\mathrm{H}^1(\mathbb{R}, J)$ is annihilated by 2, we have $\beta_2([X]) \neq 0$, thus for $n = 2$ the differential $\delta_{2,C}^{0,2}$ is non-zero. See the next example and Example 6.7 below for the case of even genus.

Example 6.3. The differentials on the 3rd page of the Hochschild–Serre spectral sequence of a smooth projective curve can be non-zero. Indeed, consider the spectral sequence (36) for $n = 2$, where C is the conic $ax^2 + by^2 = z^2$ with $a, b \in k^\times$, $\mathrm{char}\, k \neq 2$. Then $\mathrm{H}^1_{\text{ét}}(C_{k_s}, \mathbb{Z}/2) = 0$, so all differentials on the 2nd page are zero. On the 3rd page we have a differential

$$\delta_{3,C}^{0,2} \colon \mathrm{H}^2_{\text{ét}}(C_{k_s}, \mathbb{Z}/2)^\Gamma \to \mathrm{H}^3(k, \mathbb{Z}/2).$$

Suslin's lemma [Sus82, Lemma 1] states that the generator of $\mathrm{H}^2_{\text{ét}}(C_{k_s}, \mathbb{Z}/2)^\Gamma \cong \mathbb{Z}/2$ goes to the symbol $(a, b, -1)$. In particular, if $k = \mathbb{R}$ and $a = b = -1$, then $\delta_{3,C}^{0,2} \neq 0$.

Example 6.4. More generally, let X be a smooth, projective, geometrically integral variety over $\mathbb{R}$ such that $X(\mathbb{R}) = \emptyset$. Then the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{R}, \mathrm{H}^q_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Z}/2)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Z}/2) \tag{37}$$

necessarily has non-zero differentials by the following argument that we learned from Vadim Vologodsky. The Artin comparison quasi-isomorphism $\mathrm{R}\Gamma_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Z}/2) \simeq \mathrm{R}\Gamma_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}/2)$ can be made $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant with the Galois group acting on the left on the scheme $X_{\mathbb{C}}$ by functoriality, and on the right on the topological space $X(\mathbb{C})$ via the continuous automorphisms of $\mathbb{C}$. Hence $\mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{Z}/2)$ can be identified with $\mathrm{R}\Gamma(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{R}\Gamma_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}/2))$ which is the equivariant cohomology of the topological space $X(\mathbb{C})$ with respect to the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Since

$X(\mathbb{R})$ is empty, the Galois action on $X(\mathbb{C})$ is free, and hence this equivariant cohomology coincides with the cohomology of the quotient space $X(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ which is necessarily concentrated in finitely many degrees. But if the spectral sequence (37) had no non-zero differentials, the cohomology of the complex $\mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{Z}/2)$ would be non-zero in infinitely many degrees.

In fact, one can describe the differentials $\delta_3^{p,2}$ in the spectral sequence (36) in general, assuming that all differentials on the 2nd page vanish. It follows from Proposition 6.1 that the term $E_2^{0,2} = \mathrm{H}^2_{\text{ét}}(C_{k_s}, \mu_n)^\Gamma \simeq \mathbb{Z}/n$ remains intact on the 3rd page if and only if the class of Albanese torsor $[\mathbf{Pic}^1_{C/k}] \in \mathrm{H}^1(k, J)$ is divisible by n. Let us assume that this is the case and choose a k-torsor for J, which we denote $\mathbf{Pic}^{1/n}_{C/k}$, together with an isomorphism $[n]_*\mathbf{Pic}^{1/n}_{C/k} \simeq \mathbf{Pic}^1_{C/k}$. Equivalently, we have a pushout diagram of extensions of group schemes

$$\begin{array}{ccccc} J & \longrightarrow & A & \xrightarrow{\pi} & \mathbb{Z} \\ \| & & {\scriptstyle g}\uparrow & & {\scriptstyle [n]}\uparrow \\ J & \longrightarrow & \mathbf{Pic}_{C/k} & \longrightarrow & \mathbb{Z} \end{array} \tag{38}$$

with $\mathbf{Pic}^{1/n}_{C/k}$ isomorphic to the fibre of π over $1 \in \mathbb{Z}$.

The Picard stack of C is a $\mathbb{G}_m$-gerbe over the Picard scheme $\mathbf{Pic}_{C/k}$ and it has the structure of a group stack compatible with the group structure on $\mathbf{Pic}_{C/k}$. In particular, it defines a degree 2 extension of Galois modules

$$\delta : \mathbf{Pic}_{C/k}(k_s) \to k_s^\times[2].$$

By definition, the map g in the diagram (38) defines a short exact sequence of Galois modules

$$0 \to \mathbf{Pic}_C(k_s) \xrightarrow{g} A(k_s) \to \mathbb{Z}/n \to 0 \tag{39}$$

and we define a degree 3 extension as the composition

$$\mathbb{Z}/n \to \mathbf{Pic}_C(k_s)[1] \xrightarrow{\delta[1]} k_s^\times[3] \tag{40}$$

with the first map being the connecting morphism in $D(\Gamma, \mathbb{Z})$ induced by (39).

Recall that for an object $M \in D(\Gamma, \mathbb{Z})$ there are natural isomorphisms for all i:

$$\mathrm{H}^i(k, M \otimes^L_{\mathbb{Z}} \mathbb{Z}/n) \cong \mathrm{Ext}^{i+1}_\Gamma(\mathbb{Z}/n, M) \tag{41}$$

Indeed, the right-hand side can be calculated as $\mathrm{H}^{i+1}(k, \mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}/n, M))$ and $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}/n, M) \in D(\Gamma, \mathbb{Z})$ is quasi-isomorphic to $M \otimes^L_{\mathbb{Z}} \mathbb{Z}/n[-1]$.

For a Galois module M, which is n-divisible as an abelian group, there is a natural isomorphism $\mathrm{Ext}^i_\Gamma(\mathbb{Z}/n, M) \cong \mathrm{Ext}^i_\Gamma(\mathbb{Z}, M[n]) = \mathrm{H}^i(k, M[n])$. Let $c_{1/n} \in \mathrm{H}^3(k, \mu_n)$ be the image of the composition of maps in (40) under this isomorphism for $M = k_s^\times$. As a consequence of the following proposition, the class $c_{1/n}$ does not depend on the choice of the torsor $\mathbf{Pic}^{1/n}_{C/k}$.

Proposition 6.5. *Let $\mathbf{Pic}^{1/n}_{C/k}$ be a k-torsor for J such that $[n]_*[\mathbf{Pic}^{1/n}_{C/k}] = [\mathbf{Pic}^1_{C/k}]$ in $\mathrm{H}^1(k, J)$. Then all differentials on the second page of the Hochschild–Serre spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(C_{k_s}, \mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(C, \mu_n)$$

are zero. The only non-zero differentials on the 3rd page

$$E_3^{p,2} = \mathrm{H}^p(k, \mathrm{H}^2_{\text{ét}}(C_{k_s}, \mu_n)) = \mathrm{H}^p(k, \mathbb{Z}/n) \to$$
$$\to E_3^{p+3,0} = \mathrm{H}^{p+3}(k, \mathrm{H}^0_{\text{ét}}(C_{k_s}, \mu_n)) = \mathrm{H}^{p+3}(k, \mu_n)$$

are induced by the class $c_{1/n}$.

Proof. Under the assumption that $[\mathbf{Pic}^1_{C/k}]$ is divisible by n, all differentials on the second page vanish by Proposition 6.1. To access the differentials on the next page we will relate the extensions between the cohomology modules of the complex $\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mu_n)$ to those of $\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{G}_m)$.

The complex $\tau^{\leq 2}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{G}_m)$ has non-zero cohomology modules only in degrees 0 and 1, isomorphic to $k_s^\times$ and $\mathbf{Pic}_{C/k}(k_s)$, respectively, cf. [CTS21, Theorem 5.6.1 (iv)]. The degree 2 extension between them is exactly the class δ. According to the Kummer sequence, applying the functor $- \otimes^L_{\mathbb{Z}} \mathbb{Z}/n : D(\Gamma, \mathbb{Z}) \to D(\Gamma, \mathbb{Z}/n)$ to the object $\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{G}_m)$ gives $\mathrm{R}\Gamma(C_{k_s}, \mu_n)[1]$, and the extension

$$\mathrm{H}^0_{\text{ét}}(C_{k_s}, \mathbb{G}_m) \to \tau^{\leq 2}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{G}_m) \to \mathrm{H}^1_{\text{ét}}(C_{k_s}, \mathbb{G}_m)[-1]$$

is carried to

$$\mathrm{H}^0_{\text{ét}}(C_{k_s}, \mu_n)[1] \to \mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mu_n)[1] \to (\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{G}_m))[1].$$

Hence the extension

$$\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mu_n)[2] \to \mu_n[3] \tag{42}$$

arising from the complex $\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mu_n)$ is the result of applying the functor $-\otimes^L_{\mathbb{Z}}\mathbb{Z}/n$ to the map $\delta : \mathbf{Pic}_{C/k}(k_s) \to k_s^\times[2]$.

Under the assumption that $[\mathbf{Pic}^1_{C/k}]$ is divisible by n, the natural map

$$\tau^{[1,2]}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mu_n)[2] \simeq \mathbf{Pic}_{C/k}(k_s) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n \to \mathbb{Z}/n \tag{43}$$

has a section in $D(\Gamma, \mathbb{Z}/n)$ and the task of computing the 3rd differential is equivalent to computing the composition of this section with (42).

Recall from (41) that for any Galois module M there is a natural isomorphism $\mathrm{Ext}^i_\Gamma(\mathbb{Z}/n, M[1]) \cong \mathrm{H}^i(k, M \otimes^L_{\mathbb{Z}} \mathbb{Z}/n)$. The morphism $h : \mathbb{Z}/n \to \mathbf{Pic}_{C/k}(k_s)[1]$ corresponding to the extension $0 \to \mathbf{Pic}_{C/k}(k_s) \to A(k_s) \to \mathbb{Z}/n \to 0$ is carried to a class in $\mathrm{H}^0(k, \mathbf{Pic}_{C/k}(k_s) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n)$ which defines a splitting of (43) because the composition

$$\mathbb{Z}/n \xrightarrow{h} \mathbf{Pic}_{C/k}(k_s)[1] \xrightarrow{\deg[1]} \mathbb{Z}[1]$$

is the Bockstein map.

The composition of this splitting with (42) is then the map $\mathbb{Z}/n \to \mu_n[3]$ whose class in $\mathrm{H}^3(k, \mu_n)$ corresponds to the composition $\mathbb{Z}/n \xrightarrow{h} \mathbf{Pic}_{C/k}(k_{\mathrm{s}})[1] \xrightarrow{\delta[1]} k_{\mathrm{s}}^{\times}[3]$, and we arrive at the definition of the class $c_{1/n}$, as desired. □.

If the torsion order of the Albanese torsor $[\mathbf{Pic}^1_{C/k}] \in \mathrm{H}^1(k, J)$ is coprime to n then we can make the above formula for the 3rd differential more explicit. The map δ arising from the Picard stack of C induces a map $\mathbf{Pic}_{C/k}(k) \to \mathrm{Br}(k)$ sending a rational point to the obstruction to lifting it to an actual line bundle on C. We denote by $\mathrm{Bock}_{\mathbb{G}_m} : \mathrm{Br}(k) = \mathrm{H}^2(k, \mathbb{G}_m) \to \mathrm{H}^3(k, \mu_n)$ the connecting homomorphism of the Kummer sequence.

Corollary 6.6. *Suppose that* $[\mathbf{Pic}^1_{C/k}] \in \mathrm{H}^1(k, J)$ *is annihilated by an integer* m *coprime to* n*. Then all differentials on the second page of the Hochschild–Serre spectral sequence*

$$\mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(C_{k_{\mathrm{s}}}, \mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(C, \mu_n)$$

are zero. Let $x \in \mathbf{Pic}^d_{C/k}(k)$*, where* $d \equiv 1 \bmod n$*. The only non-zero differentials*

$$E_3^{p,2} = \mathrm{H}^p(k, \mathbb{Z}/n) \to \mathrm{H}^{p+3}(k, \mu_n) = E_3^{p+3,0}$$

on the 3*rd page are given by cupping with* $\mathrm{Bock}_{\mathbb{G}_m}(\delta(x)) \in \mathrm{H}^3(k, \mu_n)$*.*

Proof. Since n is invertible modulo m, the class $[\mathbf{Pic}^1_{C/k}]$ is divisible by n, so we are in the setup of Proposition 6.5. Specifically, choose m' to be any integer such that $m' \cdot n \equiv 1 \bmod m$ and let $\mathbf{Pic}^{1/n}_{C/k} := [m']_* \mathbf{Pic}^1_{C/k}$. The point x defines a map of Galois modules $f_x : \mathbb{Z} \to \mathbf{Pic}_{C/k}(k_{\mathrm{s}})$ sending $1 \in \mathbb{Z}$ to x. The composition

$$\mathbb{Z} \xrightarrow{f_x} \mathbf{Pic}_{C/k}(k_{\mathrm{s}}) \xrightarrow{\delta} k_{\mathrm{s}}^{\times}[2] \xrightarrow{\mathrm{Bock}_{\mathbb{G}_m}} \mu_n[3]$$

corresponds under the isomorphism $\mathrm{H}^3(k, \mu_n) \simeq \mathrm{Ext}^3_{\Gamma}(\mathbb{Z}/n, k_{\mathrm{s}}^{\times})$ to the composition

$$\mathbb{Z}/n \to \mathbb{Z}[1] \xrightarrow{f_x[1]} \mathbf{Pic}_{C/k}(k_{\mathrm{s}})[1] \xrightarrow{\delta[1]} k_{\mathrm{s}}^{\times}[3]. \tag{44}$$

The composition of the first two maps in (44) equals

$$\mathbb{Z}/n \xrightarrow{f_x \otimes^L_{\mathbb{Z}} \mathbb{Z}/n} \mathbf{Pic}_C(k_{\mathrm{s}}) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n \xrightarrow{\mathrm{id} \otimes \mathrm{Bock}} \mathbf{Pic}_C(k_{\mathrm{s}})[1].$$

Since $f_x \otimes^L_{\mathbb{Z}} \mathbb{Z}/n$ is a section of the natural map $\mathbf{Pic}_C(k_{\mathrm{s}}) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n \to \mathbb{Z}/n$, the above composition (44) is the map inducing the differentials on the 3rd page of our spectral sequence, as in the proof of Proposition 6.5. □.

Example 6.7. Let C be a smooth, projective, geometrically integral curve over $\mathbb{R}$ such that $C(\mathbb{R}) = \emptyset$. The spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{R}, \mathrm{H}^q_{\text{ét}}(C_{\mathbb{C}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(C, \mathbb{G}_m)$$

gives rise to the short exact sequence

$$0 \to \mathrm{Pic}(C) \to \mathrm{Pic}(C_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} = \mathbf{Pic}_{C/\mathbb{R}}(\mathbb{R}) \xrightarrow{\delta} \mathrm{Br}(\mathbb{R}) \to 0,$$

as follows from [GH81, Proposition 2.2 (2)]. If the genus of C is even, then the Albanese torsor is trivial, that is, $[\mathbf{Pic}^1_{C/\mathbb{R}}] = 0 \in \mathrm{H}^1(\mathbb{R}, J)$, see [GH81, Proposition 3.3 (1)]. Moreover, by the second statement of [GH81, Proposition 2.2 (2)], for any $x \in \mathbf{Pic}^1_{C/\mathbb{R}}(\mathbb{R})$ we have $\delta(x) \neq 0$ in $\mathrm{Br}(\mathbb{R})$. By Corollary 6.6 the 3rd page differentials $\mathrm{H}^p(\mathbb{R}, \mathbb{Z}/2) \to \mathrm{H}^{p+3}(\mathbb{R}, \mathbb{Z}/2)$ are cup-products with the generator $(-1,-1,-1)$ of $\mathrm{H}^3(\mathbb{R}, \mathbb{Z}/2)$, so they are isomorphisms $\mathbb{Z}/2 \xrightarrow{\sim} \mathbb{Z}/2$ for all $p \geq 0$.

Remark 6.8. One can also give an explicit degree 3 Yoneda model for the extension $\mathbb{Z}/n \to \mu_n[3]$ discussed above, at least after applying the forgetful functor $D(\Gamma, \mathbb{Z}/n) \to D(\Gamma, \mathbb{Z})$.

Let $\mathrm{Div}_{C_{k_s}}$ be the group of divisors on C_{k_s}, that is, the free abelian group with the set of closed points of C_{k_s} as the basis. The exact sequence

$$0 \to k_s^\times \to k_s(C)^\times \to \mathrm{Div}_{C_{k_s}} \to \mathbf{Pic}_{C/k}(k_s) \to 0 \tag{45}$$

represents the extension class of $\tau^{\leq 1}\mathrm{R}\Gamma_{\text{ét}}(C_{k_s}, \mathbb{G}_m)$, see [CTS21, Proposition 5.4.5, Remark 5.4.6]. From diagram (38) we see that multiplication by n on $\mathbf{Pic}_{C/k}$ factors as

$$\mathbf{Pic}_{C/k} \hookrightarrow A \xrightarrow{f} \mathbf{Pic}_{C/k}$$

for some map f. Let $\widetilde{\mathrm{Div}}$ be the Γ-module defined as the pushout

$$\begin{array}{ccc} \widetilde{\mathrm{Div}} & \longrightarrow & \mathrm{Div}_{C_{k_s}} \\ \downarrow & & \downarrow \\ A(k_s) & \xrightarrow{f} & \mathbf{Pic}_{C/k}(k_s) \end{array}$$

We will view elements of $\widetilde{\mathrm{Div}}$ as pairs $(\alpha, D) \in A(k_s) \oplus \mathrm{Div}_{C_{k_s}}$ satisfying $f(\alpha) = [D]$. We can then form the complex

$$0 \to \mu_n \to k_s(C)^\times \xrightarrow{a} k_s(C)^\times \oplus \mathrm{Div}_{C_{k_s}} \xrightarrow{b} \widetilde{\mathrm{Div}} \xrightarrow{c} \mathbb{Z}/n \to 0. \tag{46}$$

Here the map a sends a rational function φ to the pair $(\varphi^n, \mathrm{div}(\varphi)) \in k_s(C)^\times \oplus \mathrm{Div}_{C_{k_s}}$. The map b on $k_s(C)^\times$ is induced from the map $-\mathrm{div} : k_s(C)^\times \to \mathrm{Div}_{C_{k_s}}$, and on $\mathrm{Div}_{C_{k_s}}$ the map b sends a divisor D to the element of $\widetilde{\mathrm{Div}}$ given by $(g([D]), n \cdot D)$. Finally, c is the composition $\widetilde{\mathrm{Div}} \to A(k_s) \to \operatorname{coker} g \simeq \mathbb{Z}/n$.

The sequence (46) is exact and hence defines a map $\mathbb{Z}/n \to \mu_n[3]$ in $D(\Gamma, \mathbb{Z})$. Unwinding definitions one checks that it is equal to the image of $c_{1/n} \in \mathrm{H}^3(k, \mu_n)$ under the map $\mathrm{H}^3(k, \mu_n) \to \mathrm{Ext}^3_{\Gamma,\mathbb{Z}}(\mathbb{Z}/n, \mu_n)$.

We finally explain why Corollary 6.6 implies Suslin's lemma [Sus82, Lemma 1] discussed in Example 6.3.

Lemma 6.9. *Suppose that* $\operatorname{char} k \neq 2$. *Consider the extension of* Γ*-modules*

$$0 \to \mathbb{Z}/2 \to \mu_4 \to \mathbb{Z}/2 \to 0. \tag{47}$$

For even n *the connecting map* $\mathrm{H}^n(k, \mathbb{Z}/2) \to \mathrm{H}^{n+1}(k, \mathbb{Z}/2)$ *of* (47) *is the cup-product with the class of* -1 *in* $\mathrm{H}^1(k, \mathbb{Z}/2) = k^\times/k^{\times 2}$.

Proof. The spectral sequence $\mathrm{H}^p(k, \mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)) \Rightarrow \mathrm{Ext}^{p+q}_{\Gamma,\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$ gives an exact sequence

$$0 \to \mathrm{H}^1(k, \mathbb{Z}/2) \to \mathrm{Ext}^1_{\Gamma,\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2).$$

As pointed out in [Sus82, the proof of Lemma 4], the difference between the image of $[-1] \in \mathrm{H}^1(k, \mathbb{Z}/2)$ in $\mathrm{Ext}^1_{\Gamma,\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$ and the class of (47) is the class of the extension where $\mathbb{Z}/4$ is equipped with the trivial Galois action

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0. \tag{48}$$

By the Milnor–Bloch–Kato conjecture, the natural map

$$\mathrm{H}^n(k, \mu_4^{\otimes n}) \to \mathrm{H}^n(k, \mu_2^{\otimes n}) = \mathrm{H}^n(k, \mathbb{Z}/2)$$

is surjective. For even n we have $\mu_4^{\otimes n} \cong \mathbb{Z}/4$, so the connecting map of (48) is zero on $\mathrm{H}^n(k, \mathbb{Z}/2)$. ☐

If C is a conic, then δ sends $1 \in \mathbb{Z} \cong \mathrm{Pic}(C_{k_s})^\Gamma \cong \mathbf{Pic}_{C/k}(k)$ to the class of the associated quaternion algebra in $\mathrm{Br}(k)$ (see [CTS21, Proposition 7.1.3]), that is, to the image of the symbol (a, b) under the natural map $\mathrm{H}^2(k, \mathbb{Z}/2) \to \mathrm{H}^2(k, \mathbb{G}_m)$. By Lemma 6.9 the image of (a, b) under the Bockstein map $\mathrm{H}^2(k, \mathbb{G}_m) \to \mathrm{H}^3(k, \mathbb{Z}/2)$ is $(a, b, -1)$. We of course do not need to invoke the Milnor–Bloch–Kato conjecture for this particular computation, because the Kummer classes of a and b in $\mathrm{H}^1(k, \mathbb{Z}/2)$ lift to $\mathrm{H}^1(k, \mu_4)$, so $(a, b) \in \mathrm{H}^2(k, \mathbb{Z}/2)$ lifts to $\mathrm{H}^2(k, \mu_4^{\otimes 2}) \cong \mathrm{H}^2(k, \mathbb{Z}/4)$.

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