NON-ABELIAN HODGE THEORY IN POSITIVE CHARACTERISTIC, AFTER OGUS-VOLOGODSKY

ALEXANDER PETROV

These are notes for two expository talks on non-abelian Hodge theory in positive characteristic at the workshop ‘Complex and p-Adic Simpson Correspondence’ at the University of Maryland in November 2025. Responsbility for all the mathematical errors lies solely with me, and I welcome all comments and corrections. We attempted to summarize the main ideas of the proofs of the results mentioned, and refer to [OV07], [Kat70], [BMR08], [BB07] for the detailed proofs.

1. Notation

Let $f : X \to S$ be a smooth morphism of schemes. We first introduce the two main categories of coefficients that we will be studying:

$$\mathcal{D}\text{-mod}(X/S) := \left\{ \begin{array}{c} \text{pairs } (M, \nabla) \text{ where } M \text{ is a quasi-coherent sheaf on } X, \\ \text{and } \nabla : M \to M \otimes \Omega^1_{X/S} \text{ is an } f^{-1}(\mathcal{O}_S)\text{-linear map} \\ \text{satisfying Leibniz rule with vanishing curvature: } \nabla^2 = 0 \end{array} \right\} \tag{1.1}$$

The data of a flat connection on a given M is equivalent to the action of the quasi-coherent sheaf of $f^{-1}(\mathcal{O}_S)$-algebras $\mathcal{D}_{X/S}$ on M, such that $\mathcal{O}_X \subset \mathcal{D}_{X/S}$ acts by the given $\mathcal{O}_X$-module structure on M. Here $\mathcal{D}_{X/S}$ is the sheaf of so-called ‘crystalline’ differential operators, generated over $\mathcal{O}_X$ by symbols ∂_v for each vector field $v \in T_{X/S}$, subject to the relations $[\partial_v, f] = \partial_v(f), [\partial_v, \partial_w] = \partial_{[v,w]}, \partial_{f\cdot v} = f \cdot \partial_v$ for local sections $v, w \in T_{X/S}, f \in \mathcal{O}_X$.

The second category is the category of Higgs sheaves:

$$\text{Hig}(X/S) := \left\{ \begin{array}{c} \text{pairs } (E, \theta) \text{ with } E \text{ a quasi-coherent sheaf on } X \text{ and } \theta : E \to E \otimes \Omega^1_{X/S} \\ \text{and a } \mathcal{O}_X\text{-linear map such that } \theta \wedge \theta = 0 \end{array} \right\} \tag{1.2}$$

The data of a map θ is equivalent to the action of the tensor algebra $\bigoplus\limits_{i\geq 0} T^{\otimes i}_{X/S}$ on E, and the condition $\theta \wedge \theta = 0$ is equivalent to the factorization of this action through the commutative sheaf of algebras $\operatorname{Sym} T_{X/S}$.

2. p-curvature

To begin with, we specialize to the case $S = \operatorname{Spec} k$ where k be a perfect field of characteristic $p > 0$, e.g. $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$. We will later mention how the theory generalized over general characteristic p bases.

Version of December 15th, 2025.
The author was supported by the Clay Research Fellowship. I am grateful to Arthur Ogus and Vadim Vologodsky for numerous useful comments about these notes.

The key to defining the complex Riemann-Hilbert equivalence between flat vector bundles and local systems is that all vector bundles with a flat connection are locally trivial in the analytic topology on the underlying variety. Let us recall why they are at least trivial in the formal neighborhood of every point:

Lemma 2.1. *Let X be a smooth variety over a field F of characteristic zero. For a quasi-coherent sheaf with a flat connection $(M,\nabla)\in\mathcal{D}\text{-mod}(X)$, for every point $x\in X(F)$ there is a natural isomorphism of $\widehat{\mathcal{O}}_{X,x}$-modules with a connection*

$$M|_{\widehat{\mathcal{O}}_{X,x}}\simeq M_x\widehat{\otimes}_F\widehat{\mathcal{O}}_{X,x}. \tag{2.1}$$

Here $M|_{\widehat{\mathcal{O}}_{X,x}}$ is defined as $\lim\limits_{n} M\otimes_{\mathcal{O}_X}\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ and similarly the completed tensor product on the RHS denotes the limit $\lim\limits_{n} M_x\otimes_F\mathcal{O}_{X,x}/\mathfrak{m}_x^n$. Choosing local parameters of X at x we can identify $\widehat{\mathcal{O}}_{X,x}$ with $F[[t_1,\dots,t_d]]$ with $d=\dim X$.

Proof. The data of a map from the RHS to LHS is equivalent to the data of a map of F-vector spaces $\alpha: M_x\to M|^{\nabla}_{\widehat{\mathcal{O}}_{X,x}}$ from the stalk of M to the space of flat sections on the formal neighborhood. Given an element $s\in M_x$ we choose an arbitrary $\tilde{s}\in M|_{\widehat{\mathcal{O}}_{X,x}}$ with value s at x, and define

$$\alpha(s)=\sum_{i_1,\dots,i_d\geq 0}(-1)^{i_1+\dots+i_d}\frac{t_1^{i_1}\dots t_d^{i_d}}{i_1!\dots i_d!}\nabla^{\circ i_1}_{\partial_{t_1}}\circ\dots\circ\nabla^{\circ i_d}_{\partial_{t_d}}(\tilde{s}). \tag{2.2}$$

One checks that the RHS indeed does not depend on the choice of $\tilde{s}$, and is annihilated by the connection (it therefore does not depend on the choice of coordinates on X as well). ☐

The above proof breaks down in characteristic p because the terms in the formula (2.2) do not make sense if at least one i_j is $\geq p$. The p-curvature of a module with connection $(M,\nabla)\in\mathcal{D}\text{-mod}(X/k)$ records the values of the p-th iterates of the connection $\nabla^{\circ p}_{\partial_{t_j}}$ which are exactly the lowest-degree terms in (2.2) that do not make sense in characteristic p. To state the definition, we first introduce two constructions specific to algebraic geometry in positive characteristics.

For an $\mathbb{F}_p$-scheme Y we denote by $F_Y: Y\to Y$ the absolute Frobenius endomorphism given by the identity map on the underlying topological spaces, and by $\mathcal{O}_Y\ni a\mapsto a^p$ on the structure sheaves. For a morphism of $\mathbb{F}_p$-schemes $X\to S$ we let $X^{(1)}$ be the S-scheme obtained as the fiber product $X\times_{S,F_S}S$, and we let $F_{X/S}:=F_X\times\mathrm{id}_S: X\to X^{(1)}$ be the relative Frobenius morphism, we will often drop the subscript of $F_{X/S}$ in what follows.

For a smooth morphism $f: X\to S$ of $\mathbb{F}_p$-schemes the sheaf of vector fields $T_{X/S}$ is defined as the sheaf of $f^{-1}(\mathcal{O}_S)$-linear derivations of $\mathcal{O}_X$. For a vector field $v\in T_{X/S}(U)$ on some open $U\subset X$ the p-th power $\partial_v^{\circ p}$ of the corresponding derivation $\partial_v:\mathcal{O}_U\to\mathcal{O}_U$ again happens to be the derivation, and we denote by $v^{[p]}$ the corresponding vector field, so that $\partial_v^{\circ p}=\partial_{v^{[p]}}$.

Definition 2.2. The p-curvature of a module with a flat connection $(M,\nabla)\in\mathcal{D}\text{-mod}(X/k)$ is the $\mathcal{O}_X$-linear map

$$\psi: M\to M\otimes F^*\Omega^1_{X^{(1)}/k} \tag{2.3}$$

uniquely characterized by the property that for every local section $v\in T_{X/k}\simeq\mathrm{Der}_k(\mathcal{O}_X,\mathcal{O}_X)$ the induced endomorphism $\psi_v: M\to M$ equals $\nabla_v^{\circ p}-\nabla_{v^{[p]}}$

Here ψ_v denotes the composition of ψ with the map $F^*\Omega^1_{X^{(1)}/k} \to \mathcal{O}_X$ induced by the section $v \otimes 1 \in T_{X/k} \otimes_{\mathcal{O}_X, F_X} \mathcal{O}_X = F^* T_{X^{(1)}/k}$. It is a remarkable fact that each ψ_v is an $\mathcal{O}_X$-linear map, and that they organize into an $\mathcal{O}_X$-linear map ψ (the latter crucially uses flatness of the connection ∇).

Example 2.3. For $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$ the p-curvature along the vector field ∂_t is simply $\nabla^{\circ p}_{\partial_t} : M \to M$, because the derivation $\partial_t^{\circ p}$ of $k[t]$ vanishes.

It turns out that non-vanishing of the p-curvature is the only obstruction to M being locally spanned by flat sections, as the following theorem shows:

Theorem 2.4 (Cartier descent, [Kat70, 5.1])**.** *For a smooth k-scheme X there is a natural equivalence of categories*

$$\{(M, \nabla) \in \mathcal{D}\text{-mod}(X/k) \text{ with } \psi = 0\} \simeq \operatorname{QCoh}(X^{(1)}) \tag{2.4}$$

via the functors $(M, \nabla) \mapsto M^{\nabla}$ and $(F^*E, \nabla^{\text{can}}) \leftarrow E$.

Here, for a quasi-coherent sheaf $E \in \operatorname{QCoh}(X^{(1)})$ the canonical connection $\nabla^{\text{can}} : F^*E \to F^*E \otimes \Omega^1_{X^{(1)}}$ is defined by $\nabla(e \otimes f) = (e \otimes 1) \otimes df \in F^*E \otimes \Omega^1_{X^{(1)}}$ for a local section $e \otimes f \in E \otimes_{\mathcal{O}_{X^1}, F^*} \mathcal{O}_X = F^*E$. This is a well-defined map because the derivative d is zero on the image of F^*. For the same reason, $M^{\nabla} \subset M$ has a natural structure of an $\mathcal{O}_{X^{(1)}}$ module, because for a flat section $m \in M$ the section $F^*(f)m$ is likewise flat for every $f \in \mathcal{O}_{X^{(1)}}$.

Proof. The main part of the proof is to show that the natural map $F^*(M^{\nabla=0}) \to M$ is an isomorphism for every $(M, \nabla) \in \mathcal{D}\text{-mod}(X)$ with vanishing p-curvature. Let us just indicate that the proof analogous to Lemma 2.1 shows that M Zariski-locally has many flat sections. Indeed, for every local section $m \in M$ the formula (obtained by summing the same expression as in (2.2) over the range of indices where denominators are non-zero modulo p)

$$\alpha(m) = \sum_{0 \le i_1, \ldots, i_d < p} (-1)^{i_1 + \ldots + i_d} \frac{t_1^{i_1} \ldots t_d^{i_d}}{i_1! \ldots i_d!} \nabla^{\circ i_1}_{\partial_{t_1}} \circ \ldots \circ \nabla^{\circ i_d}_{\partial_{t_d}}(\tilde{s}). \tag{2.5}$$

defines a map $\alpha : M \to M^{\nabla}$. □

In contrast with the characteristic zero situation, we see that there are many non-trivial sheaves with a flat connection that are spanned by flat sections Zariski-locally, and complementarily a flat section is *not* determined by its value at one point.

Additionally to being $\mathcal{O}_X$-linear, p-curvature satisfies the following two crucial properties:

Lemma 2.5 ([Kat70, 5.2])**.** *For every $(M, \nabla) \in \mathcal{D}\text{-mod}(X)$ we have*

(1) $\psi : M \to M \otimes F^*\Omega^1_{X^{(1)}}$ is a map of sheaves with a flat connection, if we endow the target with the tensor product connection $\nabla \otimes \nabla_{\text{can}} := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^{\text{can}}$.

(2) $\psi \wedge \psi = 0$, that is p-curvature induces an action of the sheaf of commutative algebras $\operatorname{Sym} F^*T_{X^{(1)}}$ on the $\mathcal{D}$-module M (the action is compatible with the $\mathcal{D}$-module structure by the previous point).

In other words, every vector field on X provides a canonical endomorphism of every $\mathcal{D}$-module on X – this phenomenon will allow us to localize the category of $\mathcal{D}_X$-modules over the cotangent bundle, and it plays a key role in the construction of the non-abelian Hodge correspondence.

Next, we define subcategories of $\mathcal{D}$-modules and Higgs sheaves cut out by a nilpotence condition:

(2.6)

$$\mathrm{Hig}^{\leq n} := \{E \in \mathrm{Hig}(X) \text{ such that the action of } \operatorname{Sym} T_X \text{ factors through } \operatorname{Sym}^{\leq n} T_X := \bigoplus_{i=0}^{n} \operatorname{Sym}^i T_{X/S}\}$$

Equivalently, for every $n+1$ local sections $v_0, \ldots, v_n \in T_X$ the composition $\theta_{v_0} \circ \ldots \circ \theta_{v_n}$ is zero on M.

$$\mathcal{D}\text{-mod}^{\leq n} := \left\{ \begin{array}{c} M \in \mathcal{D}\text{-mod}(X) \text{ such that the action of } p\text{-curvature} \\ \text{factors through } \operatorname{Sym}^{\leq n} F^*T_{X^{(1)}} := \bigoplus_{i=0}^{n} \operatorname{Sym}^i F^*T_{X^{(1)}} \end{array} \right\} \tag{2.7}$$

3. Main results and corollaries

We can now state the first version of the main theorem:

Theorem 3.1 ([OV07, Theorem 2.8]). *If $\widetilde{X}^{(1)}$ is flat scheme over $W_2(k)$ with $\widetilde{X}^{(1)} \times_{W_2(k)} k \simeq X^{(1)}$, there is a natural equivalence of k-linear categories*

$$C_{\widetilde{X}^{(1)}} : \mathcal{D}\text{-mod}^{\leq p-1}(X) \simeq \mathrm{Hig}^{\leq p-1}(X^{(1)}). \tag{3.1}$$

Moreover, this equivalence, that we will refer to as 'Cartier transform', satisfies the following properties:

1. On connections with vanishing p-curvature this equivalence coincides with the Cartier functor: $C_{\widetilde{X}^{(1)}}(F^*E, \nabla_{\mathrm{can}}) \simeq (E, 0)$. In particular, it does not depend on the choice of the lift $\widetilde{X}^{(1)}$ on that subcategory.
2. ([OV07, Theorem 3.8]) The equivalence respect the cohomology in the following sense: for $(M, \nabla) \in \mathcal{D}\text{-mod}^{\leq k}$ for some $k \leq p-1$ there is an isomorphism
$$H^n_{\mathrm{dR}}(X, M) \simeq H^n_{\mathrm{Dol}}(X, C_{\widetilde{X}^{(1)}}(M, \nabla))$$
for all $n \leq p-1-k$. In particular, applied to $(\mathcal{O}_X, d) \in \mathcal{D}\text{-mod}^{\leq 0}(X)$ this recovers Deligne-Illusie decomposition [DI87, Théorème 2.1]
$$H^n_{\mathrm{dR}}(X) \simeq \bigoplus_{i+j=n} H^i(X^{(1)}, \Omega^j_{X^{(1)}}) \tag{3.2}$$
for $n < p$.
3. ([OV07, Theorem 2.8(3)]) Generalizing (1), the image of the map $\psi : M \to M \otimes F^*\Omega^1_{X^{(1)}}$ under the functor $C_{\widetilde{X}^{(1)}}$ is the Higgs field $E \to E \otimes \Omega^1_{X^{(1)}}$ on $(E, \theta) = C_{\widetilde{X}^{(1)}}(M, \nabla)$.
4. The subcategory $\mathcal{D}\text{-mod}^{\leq p-1}(X) \subset \mathcal{D}\text{-mod}(X)$ (and, analogously, for Higgs sheaves) is not preserved by the monoidal structure given by the tensor product of the underlying quasi-coherent sheaves. [OV07, Theorem 2.8] in fact proves an equivalence of larger categories that are in fact monoidal. In particular, (3.1) is compatible with tensor products in the sense that for any $M \in \mathcal{D}\text{-mod}^{\leq k}(X), N \in \mathcal{D}\text{-mod}^{\leq p-1-k}(X)$ we do have a natural isomorphism $C_{\widetilde{X}^{(1)}}(M \otimes N) \simeq C_{\widetilde{X}^{(1)}}(M) \otimes C_{\widetilde{X}^{(1)}}(N)$.

An object $(M,\nabla)\in\mathcal{D}\text{-mod}(X)$ is nilpotent of order $\leq n$ if and only if it admits a filtration $0=\mathrm{Fil}_{-1}\subset\mathrm{Fil}_0\subset\dots\mathrm{Fil}_n=M$ by subsheaves preserved by the connection such that the graded pieces $\mathrm{gr}_i=\mathrm{Fil}_i/\mathrm{Fil}_{i-1}$ have vanishing p-curvature. In contrast with the situation in characteristic zero, there are many more interesting reducible $\mathcal{D}$-modules:

Example 3.2. Let $f:Y\to X$ be an arbitrary smooth morphism to a smooth k-scheme X. The n-th relative de Rham cohomology

$$\mathcal{H}^n_{\mathrm{dR}}(Y/X):=R^nf_*(\mathcal{O}_Y\xrightarrow{d}\Omega^1_{Y/X}\xrightarrow{d}\dots) \tag{3.3}$$

is a quasi-coherent sheaf on X with a flat Gauss-Manin connection. The canonical filtration on the relative de Rham complex induces a filtration on $\mathcal{H}^n_{\mathrm{dR}}(Y/X)$, and by the Cartier isomorphism (Lemma 4.9) this filtration happens to be preserved by the Gauss-Manin connection, and the associated graded pieces of the filtration are subquotients of $F_X^*R^{n-i}f_*\Omega^i_{Y^{(1)}/X^{(1)}}$ with its canonical connection. In particular, the p-curvature of the Gauss-Manin connection is nilpotent of nilpotence length $\leq n+1$, that is $\mathcal{H}^n_{\mathrm{dR}}(Y/X)\in\mathcal{D}\text{-mod}^{\leq n}(X)$, cf. [Kat70, Theorem 5.10].

Moreover, if $f^{(1)}$ admits a lift $\widetilde{Y}^{(1)}\to\widetilde{X}^{(1)}$ to a morphism of smooth schemes over $W_2(k)$, then for $n<p$ the Cartier transform of $\mathcal{H}^n_{\mathrm{dR}}(Y/X)$ recovers the Higgs bundle given by the associated graded of the Hodge filtration [OV07, Theorem 3.8]:

$$C_{\widetilde{X}^{(1)}}(\mathcal{H}^n_{\mathrm{dR}}(Y/X),\nabla^{\mathrm{GM}})\simeq(\bigoplus_i R^{n-i}f_*\Omega^i_{Y^{(1)}/X^{(1)}},\mathrm{gr}\,\nabla) \tag{3.4}$$

where $\mathrm{gr}\,\nabla: R^{n-i}f_*\Omega^i_{Y^{(1)}/X^{(1)}}\to R^{n-i-1}f_*\Omega^{i+1}_{Y^{(1)}/X^{(1)}}\otimes\Omega^1_{X^{(1)}}$ is the $\mathcal{O}_{X^{(1)}}$-linear map induced by the Griffiths transversality of the connection. More explicitly, $\mathrm{gr}\,\nabla$ is given by cupping with the Kodaira-Spencer class $\mathrm{ks}_{f^{(1)}}\in H^0(X,\Omega^1_X\otimes R^1f_*T_{Y/X})^{(1)}$ of the family $f^{(1)}:Y^{(1)}\to X^{(1)}$. Note though that this compatibility of $C_{\widetilde{X}^{(1)}}$ with de Rham and Higgs cohomology of Y over X is *not* an application of the Cartier transform on Y relative to X, the latter would require liftability of the Frobenius twisted family $Y\times_{X,F_X}X\to X$ and would yield a stronger statement that $\mathcal{H}^n_{\mathrm{dR}}(Y/X)$ decomposes as a direct sum of Hodge cohomology sheaves.

Since $C_{\widetilde{X}^{(1)}}$ is an equivalence of categories, (3.4) implies that $\mathcal{H}^n_{\mathrm{dR}}(Y/X)$ with its Gauss-Manin connection is recoverable just from the associated graded of the Hodge filtration with the induced Higgs field. Over $\mathbb{C}$ the analogous fact follows from Simpson's correspondence[1].

In the proof of the complex Simpson correspondence one first establishes an equivalence between subcategories of semi-simple objects (i.e. stable Higgs bundles with vanishing Chern classes and irreducible local systems), and then deduces an equivalence on all object after the fact, by comparing the differential graded algebras $\mathrm{REnd}(E)$ of derived endomorphisms of the objects on both sides.

In contrast, all irreducible $\mathcal{D}$-modules with nilpotent p-curvature have vanishing p-curvature, so by analogous logic Theorem 3.1 would follow from Theorem 2.4 together with an equivalence

$$\mathrm{RHom}_{\mathcal{D}\text{-mod}}(F^*E,F^*E)\simeq\mathrm{RHom}_{\mathrm{Hig}}((E,0),(E,0)) \tag{3.5}$$

[1]Notably, according to Simpson [Sim91, p. 330], recovering the variation of Hodge structures from the associated infinitesimal variation of Hodge structures was one of the questions he considered early on when developing his theory.

of associative dg algebras for every $E \in \mathrm{QCoh}(X^{(1)})$. However, contrary to the formality theorems in Hodge theory over $\mathbb{C}$, these dg algebras are generally not equivalent as we will see[2] in Remark 6.3. Instead, Ogus-Vologodsky's result can be interpreted as an equivalence between certain modifications of these algebras that are enough to control all the objects in $\mathcal{D}\text{-mod}^{\leq p-1}(X)$ and $\mathrm{Hig}^{\leq p-1}(X^{(1)})$.

Remark 3.3. Theorem 3.1 continues to hold over an arbitrary base $\mathbb{F}_p$-scheme S in place of $\operatorname{Spec} k$. Namely, if S has a flat lift $\widetilde{S}$ over $\mathbb{Z}/p^2$ then a lift of $X^{(1)}$ over $\widetilde{S}$ (which no longer induces a lift of X itself) gives rise to an equivalence $\mathcal{D}\text{-mod}^{\leq p-1}(X/S) \simeq \mathrm{Hig}(X^{(1)}/S)$, see [OV07, Theorem 2.8]. Crucially, the choice of a lift of S is not necessary: one can instead require the existence of a lift of X over the second Witt vector scheme $W_2(S)$. In the presence of a flat lift $\widetilde{S}$ smooth lifts of $X^{(1)}$ over $\widetilde{S}$ are in a natural equivalence with smooth lifts of X over $W_2(S)$. This point of view on the non-abelian Hodge theory over an arbitrary base S, among with other extensions of it will appear in a forthcoming work of Terentiuk [Ter25].

Another key piece of structure in non-abelian Hodge theory is the notion of a (polarized) complex variation of Hodge structures. In positive characteristic, an analog of this notion is provided by:

Definition 3.4 ([OV07, Definition 4.6])**.** Let X be a smooth scheme over k equipped with a flat lift $\widetilde{X}^{(1)}$ of $X^{(1)}$ over $W_2(k)$. A Fontaine-Laffaile-Faltings module on X is an object $M \in \mathcal{D}\text{-mod}^{\leq p-1}(X)$ equipped with

(1) a filtration $F^pM = 0 \subset \ldots \subset F^{i+1}M \subset F^iM \subset \ldots \subset F^0M = M$ by quasicoherent subsheaves indexed by integers $i \in [0, p-1]$ such that $\nabla(F^iM) \subset F^{i-1}M \otimes \Omega^1_{X/k}$.
(2) An isomorphism of Higgs sheaves $C_{\widetilde{X}^{(1)}}(M, \nabla) \simeq (\bigoplus_i F^iM/F^{i+1}M, \mathrm{gr}\,\nabla)^{(1)}$ on $X^{(1)}$.

The main example of this structure arises from cohomology of a family of varieties over X, as discussed in Example 3.2: for a smooth liftable morphism $f : Y \to X$ relative de Rham cohomology sheaf $M = \mathcal{H}^n_{\mathrm{dR}}(Y/X)$ with $n < p$ has a Fontaine-Laffaile-Faltings structure given by the Hodge filtration.

Let us illustrate this notion by deducing semi-stability of relative de Rham cohomology bundles of smooth proper liftable families. The Hodge-theoretic proof of the analogous fact over $\mathbb{C}$ crucially uses the polarization on a VHS, while this proof is powered the fact that a FLF-module admits two filtrations, with graded pieces of one being the Frobenius pullbacks of the other.

Proposition 3.5 ([OV07, 4.17,4.18,4.19])**.** *Let X be a smooth projective variety X over k with a lift $\widetilde{X}^{(1)}$ of $X^{(1)}$ over k. If $(M, \nabla) \in \mathcal{D}\text{-mod}^{\leq p-1}(X)$ is a coherent sheaf admitting a structure of a FLF-module, then M is locally free, each graded piece $F^iM/F^{i+1}M$ is locally free, Chern classes of M in $\mathrm{CH}^i(X)_{\mathbb{Q}}$ are zero, and (M, ∇) is curve-semistable as a vector bundle with a flat connection.*

---

[2]More precisely, that remark only shows that they are not equivalent as DG algebras in quasi-coherent sheaves on $X^{(1)}$, that is (3.5) cannot exist for every E and be functorial in E. There are also more complicated examples as in [Pet25] showing that even the underlying complexes of these algebras need not be isomorphic already for $E = \mathcal{O}_{X^{(1)}}$.

Proof. We will discuss the proof of semi-stability and the vanishing of Chern classes. The local freeness is proven in [DI87, Corollaire 4.1.4] for M arising from the de Rham cohomology of a family, and the proof in general relies on the same idea.

By (1), for every $(N,\nabla)\in\mathcal{D}\text{-mod}^{\leq p-1}(X)$ with N coherent as an $\mathcal{O}_X$-module, the classes of N and $F^*C_{\widetilde{X}^{(1)}}(N,\nabla)$ in the Grothendieck group $G_0(X)=K_0(X)$ are equal: they both have filtrations induced by the filtration on N by the order of nilpotence of p-curvature which have isomorphic graded pieces.

On the other hand, if (M,∇) has a FLF module structure, (2) implies that classes of $C_{\widetilde{X}^{(1)}}(M,\nabla)$ and $M^{(1)}$ in $K_0(X^{(1)})$ are equal. Therefore, the class of $[M]$ equals $[F^*_{\text{abs}}M]=[F^*M^{(1)}]$. Hence, $c_i(M)$ equals $F^*_{\text{abs}}c_i(M)$ in $\mathrm{CH}^i(X)$, but $F^*_{\text{abs}}c_i(M)$ equals $p^ic_i(M)$ as one can see, by applying the splitting principle, from the fact that $F^*_{\text{abs}}L\simeq L^{\otimes p}$ for every line bundle L. Hence $c_i(M)\in\mathrm{CH}^i(M)$ is a (p^i-1)-torsion element.

To prove semi-stability on every curve in X, by definition we may assume that X is a curve, and our goal is to prove that every subbundle $(M',\nabla')\subset(M,\nabla)$ has non-positive degree. The following argument was suggested by Daniel Litt.

We equip the subsheaf $M'\subset M$ with the induced filtration $F^iM'=M'\cap F^iM:=\ker(M'\to M\to M/F^iM)$. This filtration satisfies Griffiths transversality, and we consider the induced map of associated graded Higgs sheaves:

$$(E',\theta'):=(\bigoplus_i F^iM'/F^{i+1}M',\mathrm{gr}\,\nabla')\to(\bigoplus_i F^iM/F^{i+1}M,\mathrm{gr}\,\nabla) \tag{3.6}$$

of Higgs sheaves. Note that this map is injective: by definition, the kernel of the map $F^iM'\to F^iM/F^{i+1}M\subset M/F^{i+1}M$ is precisely the subsheaf $F^{i+1}M'$. Applying the functor $C^{-1}_{\widetilde{X}^{(1)}}$ (composed with Frobenius twist on the base field) we then get an injection of $\mathcal{D}$-modules on X:

$$C^{-1}_{\widetilde{X}^{(1)}}(E'^{(1)},\theta'^{(1)})\hookrightarrow C^{-1}_{\widetilde{X}^{(1)}}(\mathrm{gr}_F\,M^{(1)},\mathrm{gr}\,\nabla^{(1)})\simeq(M,\nabla) \tag{3.7}$$

Note that E' is a locally free sheaf, because it is a subsheaf of the locally free sheaf $\mathrm{gr}_F\,M$ on a smooth curve, and $\deg C^{-1}_{\widetilde{X}^{(1)}}(E'^{(1)},\theta'^{(1)})=p\cdot\deg E'$. On the other hand, $\deg E'$ equals $\deg M'$ because E' is the associated graded of a filtration on the vector bundle M'.

Summarizing, we have described a procedure that starts with a subbundle $(M',\nabla')\subset(M,\nabla)$ and produces another subbundle $(M'',\nabla''):=C^{-1}_{\widetilde{X}^{(1)}}(E'^{(1)},\theta'^{(1)})\subset(M,\nabla)$ (that is also preserved by the connection) with $\deg M''=p\cdot\deg M'$. If $\deg M'$ was positive, iterating this procedure we would obtain that M has subbundles of arbitrarily large degree, which is absurd. ☐

Before turning to discussing the proof of Theorem 3.1, let us state what non-abelian Hodge theory in positive characteristic can say about $\mathcal{D}$-modules with p-curvature of nilpotence order $\geq p$ or even general $\mathcal{D}$-modules with non-nilpotent p-curvature.

Let X be a smooth scheme over k equipped with a flat lift $\widetilde{X}^{(1)}$ over $W_2(k)$ of $X^{(1)}$. To make the statements more tangible, we will restrict to discussing vector bundles of a fixed rank $r\geq 0$, though all of the theory goes through for arbitrary objects of the derived category of $\mathcal{D}$-modules and Higgs sheaves.

Denote by $M_{\text{dR}}(X,r)$ the moduli stack of vector bundles with a flat connection of rank r on X, and similarly let $M_{\text{Hig}}(X^{(1)},r)$ be the moduli stack of Higgs bundles of rank r

on $X^{(1)}$. Formally, these are stacks[3] over k defined by

$$M_{\mathrm{dR}}(X,r)(S)=\{(M,\nabla)\in\mathcal{D}\text{-mod}(X\times S/S)\text{ with }M\text{ locally free of rank }r\} \tag{3.8}$$
$$M_{\mathrm{Hig}}(X^{(1)},r)(S)=\{(E,\theta)\in\mathrm{Hig}(X\times S/S)\text{ with }E\text{ locally free of rank }r\}$$

For $d\geq 0$ we denote by $M^{\leq d}_{\mathrm{dR}}\subset M_{\mathrm{dR}}$ (resp. $M^{\leq d}_{\mathrm{Hig}}\subset M_{\mathrm{Hig}}$) the closed substacks parametrizing objects for which p-curvature (resp. the Higgs field) is nilpotent of order d.

Note that for each k-scheme S the lift $\widetilde{X}^{(1)}$ naturally gives rise to the lift $\widetilde{X}^{(1)}\times_{W_2(k)} W_2(S)$ of $X\times S$ over $W_2(S)$. In particular, the relative version of Theorem 3.1 mentioned in Remark 3.3 gives rise to an equivalence of stacks

$$M^{\leq p-1}_{\mathrm{dR}}(X,r)\simeq M^{\leq p-1}_{\mathrm{Hig}}(X^{(1)},r) \tag{3.9}$$

On the moduli stack of Higgs bundles $M_{\mathrm{Hig}}(X,r)$ there is an action of $\mathbb{G}_m$ given by $t\in\mathbb{G}_m(S)$ sending (E,θ) to $(E,t\cdot\theta)$. By transport of structure across (3.9), for every choice of a lift $\widetilde{X}^{(1)}$ we get an action of $\mathbb{G}_m$ on $M^{\leq p-1}_{\mathrm{dR}}(X,r)$.

Theorem 3.6 (Travkin [Tra16, 5.5], Ogus-Vologodsky, forthcoming)**.** *For a chosen lift $\widetilde{X}^{(1)}$ of $X^{(1)}$ the aforementioned action of $\mathbb{G}_m$ on $M^{\leq p-1}_{\mathrm{dR}}(X,r)$ extends to an action of $\mathbb{G}^{\sharp}_m$ on $M^{\mathrm{nilp}}_{\mathrm{dR}}(X,r):=\bigcup_d M^{\leq d}_{\mathrm{dR}}(X,r)$. Moreover, this action extends to an action of $\mathbb{G}^{\widehat{\sharp}}_m$ on all of $M_{\mathrm{dR}}(X,r)$.*

Here $\mathbb{G}^{\sharp}_m$ is the divided power envelope of $1\in\mathbb{G}^{\sharp}_m$, that is the affine group scheme $\operatorname{Spec}k[x^{\pm1},\frac{(x-1)^n}{n!}|n\in\mathbb{N}]$, and $\mathbb{G}^{\widehat{\sharp}}_m$ is a pd-completed divided power envelope, that is a formal group scheme with the underlying ind-scheme given by $\operatorname*{colim}_N \operatorname{Spec}k[x^{\pm1},\frac{(x-1)^n}{n!}|n\in\mathbb{N}]/(\frac{(x-1)^n}{n!}|n\geq N)$. The are natural maps $\mathbb{G}^{\widehat{\sharp}}_m\to\mathbb{G}^{\sharp}_m\to\mathbb{G}_m$, with respect to which the three actions in the theorem are compatible.

Remark 3.7. The data of an action of $\mathbb{G}^{\widehat{\sharp}}_m$ on an arbitrary stack over k is equivalent to the data of a vector field on it. Here by a vector field on a generally non-smooth scheme or stack Y we mean a section of the map π from the tangent object $TY:=\mathrm{Map}(\operatorname{Spec}k[\varepsilon]/\varepsilon^2,Y)\xrightarrow{\pi}Y$ down to Y. The element $1+\varepsilon\in\mathbb{G}_m(k[\varepsilon]/\varepsilon^2)$ extends to a point of $\mathbb{G}^{\widehat{\sharp}}_m$ with coefficients in dual numbers, by declaring that all divided powers $\frac{\varepsilon^n}{n!}$ of ε are zero for $n\geq 2$. Pushing forward this vector field along the action map $\mathbb{G}^{\widehat{\sharp}}_m\times Y\to Y$ defines a vector field on Y, and one can check that every vector field arises this way from a unique action of $\mathbb{G}^{\widehat{\sharp}}_m$. This amounts to showing that the Cartier dual of $\mathbb{G}^{\widehat{\sharp}}_m$ is isomorphic to the group scheme $\mathbb{G}_a$.

That is, the final assertion of Theorem 3.6 is equivalent to producing a natural vector field on $M_{\mathrm{dR}}(X,r)$. We will be able to describe this vector field explicitly as a byproduct of the proof of Theorem 3.1. Note that the situation is formally analogous to Simpson correspondence over $\mathbb{C}$: for a smooth projective variety X over $\mathbb{C}$ there is a natural real-analytic action of $\mathbb{C}^{\times}$ on the $\mathbb{C}$-points of (the coarse moduli space of) $M_{\mathrm{dR}}(X,r)$ which does not extend to an algebraic action of $\mathbb{G}_{m,\mathbb{C}}$, but does give rise to a complex-valued real-analytic vector field on the manifold of $\mathbb{C}$-points of this moduli space.

[3]By a 'stack' we simply mean a functor from k-schemes to 1-groupoids satisfying fpqc descent

4. Azumaya property of differential operators

We now turn to the proof of Theorem 3.1. The first key piece of structure is the Azumaya property of the ring of differential operators in positive characteristic, discovered by Bezrukavnikov-Mirković-Rumynin [BMR08] and Berthelot.

For simplicity of notation, we work in the case of a smooth scheme X over a perfect field k of characteristic p. As we discussed, every vector field on $T_{X^{(1)}}$ defines an endomorphism of every $\mathcal{D}$-module on X, and we will now see that this is explained by the fact that every such vector field give rise to an element in the center of $\mathcal{D}_X$.

The sheaf of differential operators $\mathcal{D}_{X/k}$ has a natural structure of a $\mathcal{D}$-module via left multiplication, and its p-curvature defines a map of quasi-coherent sheaves of algebras

$$\psi : F^* \operatorname{Sym} T_{X^{(1)}} \to \mathcal{E}nd_{\mathcal{O}_X} \mathcal{D}_X \tag{4.1}$$

Explicitly, for a vector field $v \in T_{X^{(1)}}$ the map ψ sends $F^*(v)$ to the operator of left multiplication by $\partial_v^p - \partial_{v^{[p]}} \in \mathcal{D}_X$. Moreover, by Lemma 2.5 the map ψ is compatible with connections if we equip $F^* \operatorname{Sym} T_{X^{(1)}}$ with the canonical connection. In particular, sections of $\operatorname{Sym} T_{X^{(1)}} = (F^* \operatorname{Sym} T_{X^{(1)}})^{\nabla=0}$ get mapped to endomorphism of $\mathcal{D}_X$ as a $\mathcal{D}_X$-module, which are all given by *right* multiplication by a section of $\mathcal{D}_X$. This forces $\partial_v^p - \partial_{v^{[p]}}$ to be a section of the center of $\mathcal{D}_X$. Summarizing, by adjunction we get a map

$$v \mapsto \partial_v^p - \partial_{v^{[p]}} : \operatorname{Sym} T_{X^{(1)}} \to Z(F_* \mathcal{D}_X) \tag{4.2}$$

A local calculation (that will also follow from the Azumaya propety of $F_*\mathcal{D}_X$ over $\operatorname{Sym} T_{X^{(1)}}$, proven in Proposition 4.5 below) shows:

Lemma 4.1. *The map (4.2) is an isomorphism onto the center of $F_*\mathcal{D}_X$.*

This existence of the map (4.2) allows us to introduce the following crucial point of view on $\mathcal{D}$-modules in positive characteristic: they can be localized across the cotangent bundle $T^*X^{(1)}$ of $X^{(1)}$.

By the above calculation of the center, $F_*\mathcal{D}_X$ is not only an $\mathcal{O}_{X^{(1)}}$-algebra (that is, $\mathcal{O}_{X^{(1)}} \subset F_*\mathcal{D}_X$ lies in its center), but has a structure of a sheaf of $\operatorname{Sym} T_{X^{(1)}}$-algebras. In particular, $F_*\mathcal{D}_X$ can be viewed as a sheaf of algebras on $T^*X^{(1)} = \operatorname{Spec}_{X^{(1)}} \operatorname{Sym} T_{X^{(1)}}$, that is $F_*\mathcal{D}_X$ has the form $\pi_*\mathcal{A}$ for some sheaf of algebras $\mathcal{A}$ on $T^*X^{(1)} \xrightarrow{\pi} X$. To lighten the notation, we denote $\mathcal{A}$ by $F_*\mathcal{D}_X$ as well.

Example 4.2. For $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$ differential operators form the Weyl algebra $\mathcal{D}_X = k\langle t, \partial_t\rangle$ freely generated by symbols t, ∂_t subject to the condition $[\partial_t, t] = 1$. The above description of its center is explicitly given by $\operatorname{Sym} T_{X^{(1)}} = k[t^p, \partial_t^p] \subset F_*\mathcal{D}_X = k\langle t, \partial_t\rangle$.

Lemma 4.3. *There is an equivalence $M \mapsto F_*M : \mathcal{D}\text{-mod}(X) \simeq \operatorname{Mod}_{T^*X^{(1)}}(F_*\mathcal{D}_X)$ between $\mathcal{D}$-modules on X and quasi-coherent sheaves on $T^*X^{(1)}$ equipped with an action of the sheaf of algebras $F_*\mathcal{D}_X$.*

Proof. Since $F : X \to X^{(1)}$ is an affine morphism, there is an equivalence between $\mathcal{D}\text{-mod}(X)$ and the category of quasi-coherent sheaves on $X^{(1)}$ equipped with a compatible action of the sheaf of rings $F_*\mathcal{D}_X$, which is in turn equivalent to $\operatorname{Mod}_{T^*X^{(1)}}(F_*\mathcal{D}_X)$ because $\pi : T^*X^{(1)} \to X^{(1)}$ is an affine morphism as well. $\square$

Example 4.4. Under this equivalence, $\mathcal{D}\text{-mod}^{\leq n}(X)$ is identified with $\operatorname{Mod}_{X_n^{(1)}}(F_*\mathcal{D}_X|_{X_n^{(1)}})$ where $X_n^{(1)} := \operatorname{Spec}_{X^{(1)}} \operatorname{Sym}^{\leq n} T_{X^{(1)}} \hookrightarrow T^*X^{(1)}$ is the n-th order nilpotent neighborhood of the zero section in the cotangent bundle.

Similarly, there is a tautological description of the category $\mathrm{Hig}(X^{(1)}) = \mathrm{Mod}_{\mathcal{O}_{X^{(1)}}}(\mathrm{Sym}\, T_{X^{(1)}})$ as $\mathrm{Mod}_{T^*X^{(1)}}(\mathcal{O}_{T^*X^{(1)}}) = \mathrm{QCoh}(T^*X^{(1)})$ because $T^*X^{(1)}$ is the relative spectrum of the sheaf of commutative algebras $\mathrm{Sym}\, T_{X^{(1)}}$ on $X^{(1)}$.

In this language, the task of relating $\mathcal{D}\text{-mod}(X)$ with $\mathrm{Hig}(X^{(1)})$ is equivalent to relating categories of modules in sheaves on $T^*X^{(1)}$ over two different sheaves of algebras: $F_*\mathcal{D}_X$ and the structure sheaf $\mathcal{O}_{T^*X^{(1)}}$ itself. With that goal in mind, we turn to the key special property of $F_*\mathcal{D}_X$:

Proposition 4.5 ([BMR08, §2])**.** *$F_*\mathcal{D}_X$ is an Azumaya algebra on $T^*X^{(1)}$ of rank $p^{2\dim X}$, that is étale (or fppf-) locally it is isomorphic to the matrix algebra $\mathrm{Mat}_r(\mathcal{O}_{T^*X^{(1)}})$ for $r = p^{\dim X}$.*

Proof. We will explicitly produce a finite flat cover $h : Y \to T^*X^{(1)}$ such that $h^*F_*\mathcal{D}_X$ is isomorphic to $\mathrm{End}_{\mathcal{O}_Y}(V)$ for a rank $p^{\dim X}$ vector bundle V on Y, this will imply the Azumaya property of $F_*\mathcal{D}_X$. Let Y be the fiber product $X \times_{F, X^{(1)}} T^*X^{(1)} \xrightarrow{h := p_2} T^*X^{(1)}$, it is a degree $p^{\dim X} = \deg F$ finite flat cover of $T^*X^{(1)}$.

Since h is affine, proving that $h^*(F_*\mathcal{D}_X)$ is a matrix algebra is equivalent to finding an $h_*\mathcal{O}_Y \otimes_{\mathcal{O}_{T^*X^{(1)}}} F_*\mathcal{D}_X$-module E on $T^*X^{(1)}$ such that the action map

$$h_*\mathcal{O}_Y \otimes_{\mathcal{O}_{T^*X^{(1)}}} F_*\mathcal{D}_X \to \mathrm{End}_{h_*\mathcal{O}_Y}(E) \tag{4.3}$$

is an isomorphism. By definition, we have $h_*\mathcal{O}_Y = F_*\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathrm{Sym}\, T_{X^{(1)}}$. We take E to be $F_*\mathcal{D}_X$ with the action of a section $f \otimes \partial \in F_*\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} F_*\mathcal{D}_X = h_*\mathcal{O}_Y \otimes_{\mathcal{O}_{T^*X^{(1)}}} F_*\mathcal{D}_X$ given by left multiplication by $f \cdot \partial \in F_*\mathcal{D}_X$. This is indeed well-defined because $\mathcal{O}_{T^*X^{(1)}}$ is central in $F_*\mathcal{D}_X$. The construction of E and the action map (4.3) is compatible with étale pullback on X, so to check that this map is an isomorphism we may assume that X is an affine space $\mathbb{A}^d_k$.

For simplicity of notation, we make explicit here only the case $d = 1$. In this case the algebra $h_*\mathcal{O}_Y \otimes_{\mathcal{O}_{T^*X^{(1)}}} F_*\mathcal{D}_X$ on $T^*X^{(1)}$ is given by $k[t, \partial_t^p] \otimes_{k[t^p, \partial_t^p]} k\langle t, \partial_t\rangle$, and the action map

$$\alpha : k[t, \partial_t^p] \otimes_{k[t^p, \partial_t^p]} k\langle t, \partial_t\rangle \to \mathrm{End}_{k[t, \partial_t^p]}\, k\langle t, \partial_t\rangle \tag{4.4}$$

is a map between $k[t, \partial_t^p]$-algebras that are free of rank p^2 as modules. Note that elements $1 \otimes t$ and $1 \otimes \partial_t$ under α get mapped to matrices whose commutator is Id. Since ranks of the source and target of α are equal, it suffices to check that α is an isomorphism after base changing to every closed point of $\mathrm{Spec}\, k[t, \partial_t^p]$. This assertion follows from the following linear-algebraic fact: for any field k of characteristic p any pair of matrices $A, B \in \mathrm{Mat}_p(k)$ with $[A, B] = \mathrm{Id}$ generates $\mathrm{Mat}_p(k)$ as an algebra[4]. □

If $F_*\mathcal{D}_X$ was globally isomorphic to an algebra of the form $\mathrm{End}_{T^*X^{(1)}}(V)$ for a vector bundle V on $T^*X^{(1)}$, this would immediately yield an equivalence $\mathcal{D}\text{-mod}(X) \simeq \mathrm{Hig}(X^{(1)})$. However, $F_*\mathcal{D}_X$ is a non-split Azumaya algebra for every positive-dimensional X:

Lemma 4.6 ([BMR08, 2.2.3])**.** *If $\dim X > 0$ then the Azumaya algebra $F_*\mathcal{D}_X$ is not split.*

[4]Proof: we may assume that k is algebraically closed. By Burnside's theorem it suffices to prove that there is no non-zero proper subspace $V \subset k^{\oplus p}$ preserved by both A and B. Indeed, $\mathrm{tr}([A|_V, B|_V])$ would be equal to $\mathrm{tr}\,\mathrm{Id}_V = \dim V \in k$ but the trace of a commutator has to be zero, so $\dim V$ would have to be divisible by p.

Proof. We may assume that X is affine. If $F_*\mathcal{D}_X$ was split $\Gamma(T^*X^{(1)}, F_*\mathcal{D}_X) = \Gamma(X, \mathcal{D}_X)$ would have zero-divisors, because a matrix algebra of rank $p^{\dim X} > 1$ does so. But $\mathcal{D}_X$ admits an order filtration with the associated graded isomorphic to $\operatorname{Sym} T_X$ which does not have zero divisors. □

However, $F_*\mathcal{D}_X$ may be split on various loci on $T^*X^{(1)}$ which would translates into the following weaker version of non-abelian Hodge correspondence:

Lemma 4.7. *If $f : Z \to T^*X^{(1)}$ is any scheme over $T^*X^{(1)}$ then $f^*(F_*\mathcal{D}_X)$ is a split Azumaya algebra on Z if and only if there exists a $f^*(F_*\mathcal{D}_X)$-module V such that V a locally free of rank $p^{2\dim X}$ over the subalgebra $f^* \operatorname{Sym} T_{X^{(1)}} \subset f^*(F_*\mathcal{D}_X)$. In this case, there is an equivalence $\operatorname{Mod}_Z(f^*(F_*\mathcal{D}_X)) \simeq \operatorname{QCoh}(Z)$ given by $M \mapsto \operatorname{Hom}_{f^*(F_*\mathcal{D}_X)}(V, M)$.*

Proof. By definition, an Azumaya algebra $\mathcal{A}$ on Z is split if and only if there exists a vector bundle V on Z with an isomorphism $\mathcal{A} \simeq \operatorname{End}_Z(V)$.

Suppose that we are given an $\mathcal{A} := f^*(F_*\mathcal{D}_X)$-module V on Z such that V is locally free of rank $p^{2\dim X}$ over $f^* \operatorname{Sym} T_{X^{(1)}}$. We have the action map $\mathcal{A} \to \operatorname{End}_Z(V)$ that we will check to be an isomorphism. As this is a map of locally free $\mathcal{O}_Z$-modules, it suffices to check that it becomes an isomorphism over every field-valued point of Z. That now follows from the fact that for a field K any module over the matrix algebra $\operatorname{Mat}_r(K)$ which is r-dimensional as a K-vector space is isomorphic to the standard module $K^{\oplus r}$. □

Example 4.8. Azumaya algebra $F_*\mathcal{D}_X$ is naturally split over the zero section $X^{(1)} \hookrightarrow T^*X^{(1)}$. The splitting module is given by $F_*\mathcal{O}_X$ with the $F_*\mathcal{D}_X$-module structure coming from the trivial flat connection $(\mathcal{O}_X, d)$, where the condition of Lemma 4.7 is satisfied because $F_*\mathcal{O}_X$ is locally free of rank $p^{\dim X}$ over $X^{(1)}$, as a local calculation shows. The resulting equivalence $\mathcal{D}\text{-mod}^{\leq 0}(X) = \operatorname{Mod}_{X^{(1)}}(F_*\mathcal{D}_X|_{X^{(1)}}) \simeq \operatorname{QCoh}(X^{(1)})$ then recovers Cartier descent from Theorem 2.4.

The Azumaya property of $F_*\mathcal{D}_X$ has immediate consequences for the derived category of $\mathcal{D}_X$-modules, in particular for de Rham cohomology of X:

Lemma 4.9 (Cartier isomorphism)**.** *For each i the i-th cohomology sheaf of the $\mathcal{O}_{X^{(1)}}$-linear de Rham complex $F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_X \xrightarrow{d} \dots$ is isomorphic to $\Omega^i_{X^{(1)}}$.*

Proof. Recall that (in any characteristic) the de Rham complex computes derived endomorphisms of the object $\mathcal{O}_X \in \mathcal{D}\text{-mod}(X)$. To see this, note that we have the Koszul-type resolution of $\mathcal{O}_X$ by locally free $\mathcal{D}_X$-modules:
(4.5)
$$\dots \to \mathcal{D}_X \otimes \Lambda^2 T_X \xrightarrow{\partial\otimes v_1\wedge v_2 \mapsto \partial\cdot\partial_{v_1}\otimes v_2 - \partial\cdot\partial_{v_2}\otimes\partial_{v_1} - \partial\otimes[v_1,v_2]} \mathcal{D}_X \otimes_{\mathcal{O}_X} T_X \xrightarrow{\partial\otimes v\mapsto \partial\cdot\partial_v} \mathcal{D}_X$$
where the $\mathcal{D}$-module structure on each term is via left multiplication on the first tensor factor. Applying the functor $\operatorname{Hom}_{\mathcal{D}_X}(-, \mathcal{O}_X)$ to this resolution we obtain a quasi-isomorphism $\operatorname{RHom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots$. Keeping track of the $\mathcal{O}_{X^{(1)}}$-linear structure we similarly get a quasi-isomorphism

$$F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_X \xrightarrow{d} \dots \simeq \operatorname{RHom}_{F_*\mathcal{D}_X}(F_*\mathcal{O}_X, F_*\mathcal{O}_X). \tag{4.6}$$

of complexes of $\mathcal{O}_{X^{(1)}}$-modules.

We will compare the complex $\mathcal{C}_1 := \mathcal{RH}om_{F_*\mathcal{D}_X}(F_*\mathcal{O}_X, F_*\mathcal{O}_X)$ with

$$\mathcal{C}_2 := \mathcal{RH}om_{\mathcal{O}_{T^*X^{(1)}}}(i_*\mathcal{O}_{X^{(1)}}, i_*\mathcal{O}_{X^{(1)}}) \simeq \bigoplus_{i\geq 0} \Omega^i_{X^{(1)}}[-i]$$

They are in general not quasi-isomorphic, but our goal is to check that their cohomology sheaves are naturally isomorphic.

We call a flat map $f: U \to T^*X^{(1)}$ *tiny* if it satisfies the following two properties: first, there exists an isomorphism $\alpha: f^*(F_*\mathcal{D}_X) \simeq \mathrm{Mat}_r(\mathcal{O}_U)$, and moreover the image of the $f^*(F_*\mathcal{D}_X)$-module $f^*(F_*\mathcal{O}_X)$ under the equivalence $\mathrm{Mod}_U(\mathrm{Mat}_r(\mathcal{O}_U)) \simeq \mathrm{Mod}_U(\mathcal{O}_U) = \mathrm{QCoh}(U)$ is isomorphic to $f^*(i_*\mathcal{O}_{X^{(1)}})$.

Note that for any flat morphism $V \to T^*X^{(1)}$ there exists a further flat cover $U \to V$ which is tiny: the existence of α amounts to the fact that $f^*(F_*\mathcal{D}_X)$ is a split Azumaya algebra, and moreover the splitting module is free over $\mathcal{O}_U$. A priori, $f^*(F_*\mathcal{O}_X) \in \mathrm{Mod}_U(\mathrm{Mat}_r(\mathcal{O}_U))$ gets carried under the equivalence with $\mathrm{QCoh}(U)$ to a module of the form i_*L for a line bundle L on $U_0 := X^{(1)} \times_{0,T^*X^{(1)}} U$, because $f^*(F_*\mathcal{O}_X)$ is locally free of rank r over $f^*(i_*\mathcal{O}_{X^{(1)})}) = i_*\mathcal{O}_{U_0}$, and this Morita equivalence divides ranks by r. Replacing U by an appropriate Zariski cover will thus make the second condition hold as well.

For any tiny flat map $U \to T^*X^{(1)}$ there is a natural quasi-isomorphism $f^*\mathcal{C}_1 \simeq f^*\mathcal{C}_2$ and we will check that its effect on individual cohomology sheaves is independent of the choice of α, and of the trivialization of L. Indeed, we have a chain of quasi-isomorphism

$$
\begin{aligned}
(4.7)\quad f^*(\mathcal{C}_1) = \mathcal{REnd}_{f^*(F_*\mathcal{D}_X)}(f^*(F_*O_X)) \simeq \mathcal{REnd}_{\mathrm{Mat}_r(\mathcal{O}_U)}(f^*(F_*O_X)) \simeq \\
\simeq \mathcal{REnd}_{\mathcal{O}_U}(i_*L) \simeq \mathcal{REnd}_{\mathcal{O}_U}(i_*\mathcal{O}_{U_0}) = f^*(\mathcal{C}_2)
\end{aligned}
$$

where the second one is induced by α, the second-to-last one is given by the trivialization of L, and the other ones are completely canonical. Changing the trivialization of L by an invertible function on U_0 does not change the last quasi-isomorphism if that function can be lifted to an invertible function on the formal neighborhood of $U_0 \hookrightarrow U$, and that's always possible if U is affine. Similarly, by Skolem-Noether theorem locally on U every automorphism of $\mathrm{Mat}_r(\mathcal{O}_U)$ is given by conjugation by an element $g \in GL_r(\mathcal{O}_U)$, and in this case the effect on (4.7) of changing α by $\mathrm{ad}_g \circ g$ is equivalent to rescaling the trivialization of L by $\det(g)$, which again does not change the quasi-isomorphism (4.7).

That is, for every tiny flat map and any choice of α and the trivialization of L we constructed a quasi-isomorphism $f^*(\mathcal{C}_1) \simeq f^*(\mathcal{C}_2)$ whose effect on cohomology sheaves depends only on f, because that can be checked after replacing U by a further Zariski cover, and as we discussed any two choices give equal elements in $\mathrm{Hom}_{D(U)}(f^*(\mathcal{C}_1), f^*(\mathcal{C}_2))$ after replacing U by an appropriate open cover. By flat descent, we obtain isomorphism $\mathcal{H}^i(\mathcal{C}_1) \simeq \mathcal{H}^i(\mathcal{C}_2)$ on $T^*X^{(1)}$ (and both of these sheaves are supported on the zero section $X^{(1)} \hookrightarrow T^*X^{(1)}$), as desired. □

Remark 4.10. There is a version of Cartier isomorphism for de Rham cohomology with coefficients in an arbitrary object $E \in \mathcal{D}\text{-mod}(X)$, discovered by Ogus [Ogu04]. Consider the "Dolbeault" complex constructed from p-curvature of E:

$$
E \xrightarrow{\psi} E \otimes F^*\Omega^1_{X^{(1)}/k} \xrightarrow{\psi} \dots \tag{4.8}
$$

By Lemma 2.5 this a complex of $\mathcal{D}$-modules, so its cohomology sheaves $\mathcal{H}^i_\psi(E)$ inherit a flat connection, which necessarily has vanishing p-curvature. Then there is a natural isomorphism ([Ogu04, Theorem 1.2.1])

$$
\mathcal{H}^i(F_*(E \otimes \Omega^\bullet_X)) \simeq \mathcal{H}^i_\psi(E)^{\nabla=0}
$$

of the cohomology sheaves of the de Rham complex and flat sections in $\mathcal{H}^i_\psi(E)$. This can also be deduced from the Azumaya property of $F_*\mathcal{D}_X$ as in the above proof for $E = \mathcal{O}_X$, cf. [OV07, Theorem 2.26].

5. Brauer class of $F_*\mathcal{D}_X$ via Milne sequence

For $f : Z \to T^*X^{(1)}$ as above with Z smooth over k, it turns out that we can describe the Brauer class of the Azumaya algebra $f^*(F_*\mathcal{D}_X)$ on Z in terms intrinsic to Z, with the map f contributing only through the pullback $f^*\lambda$ of the canonical 1-form $\lambda \in H^0(T^*X^{(1)}, \Omega^1)$. In the following statement by the 'groupoid of splittings' of an Azumaya algebra $\mathcal{A}$ on a scheme Z we mean the category of pairs (V, α) where V is a locally free sheaf on Z, and $\alpha : \mathcal{A} \simeq \mathrm{End}_Z(V)$ is an isomorphism of algebras.

Lemma 5.1 ([BB07, Proposition 3.11])**.** *If Z is a smooth k-scheme with a map $f : Z \to T^*X$ the groupoid of splittings of $f^{(1)*}(F_*\mathcal{D}_X)$ on $Z^{(1)}$ is equivalent to the groupoid of line bundles (L, ∇) with a flat connection on Z whose p-curvature $\psi \in \mathrm{Hom}_\nabla(L, L \otimes F^*_{\mathrm{abs}}\Omega^1_{Z/k}) = H^0(Z, \Omega^1_Z)$ equals $f^*(\lambda)$.*

Proof. Consider the Azumaya algebra $F_*\mathcal{D}_Z$ on $T^*Z^{(1)}$ associated to Z itself. Differential 1-form $f^{(1)*}(\lambda)$ defines a section $s : Z^{(1)} \to T^*Z^{(1)}$ of the cotangent bundle, and we will first identify the groupoid of line bundles with p-curvature $f^*(\lambda)$ with the groupoid of splittings of the restriction $s^*(F_*\mathcal{D}_Z)$ of the Azumaya algebra to this section. This is a special case of Lemma 4.7. An $s^*(F_*D_Z)$-module is tantamount to a F_*D_Z-module F_*M on $T^*Z^{(1)}$ in which $T_{Z^{(1)}} \subset \mathrm{Sym}\, T_{Z^{(1)}} = Z(F_*\mathcal{D}_Z)$ acts via multiplication by the section $T_{Z^{(1)}} \xrightarrow{f^*(\lambda)} \mathcal{O}_{Z^{(1)}}$. The condition that the module is locally free of rank $p^{\dim Z}$ over $\mathcal{O}_{Z^{(1)}}$ is equivalent to F_*M being locally free of this rank over $F_*\mathcal{O}_Z$, which is to say that M is a line bundle.

Now we will relate splittings of $s^*(F_*\mathcal{D}_Z)$ to those of $f^*(F_*\mathcal{D}_X)$. This follows from the following functoriality property of differential operators, cf. [BB07, Proposition 3.7]: the pullbacks of $F_*\mathcal{D}_X$ and $F_*\mathcal{D}_Z$ along the respective maps

$$T^*X^{(1)} \xleftarrow{\ p_1\ } T^*X^{(1)} \times_{X^{(1)}} Z^{(1)} \xrightarrow{\ d(\pi\circ f)\ } T^*Z^{(1)} \tag{5.1}$$

are Morita-equivalent. Here df refers to the differential of the composition $Z^{(1)} \xrightarrow{f} T^*X^{(1)} \xrightarrow{\pi} X^{(1)}$. The map $f : Z^{(1)} \to T^*X^{(1)}$ then defines a section $s' : Z^{(1)} \to T^*X^{(1)} \times_{X^{(1)}} Z^{(1)}$ such that $p_1 \circ s' = f, df \circ s' = s$ so pulling back the Morita equivalence of $p_1^*(F_*\mathcal{D}_X)$ and $df^*(F_*\mathcal{D}_Z)$ along s' gives an equivalence between $f^*(F_*\mathcal{D}_X)$ and $s^*(F_*\mathcal{D}_Z)$, finishing the proof. □

We can recast the above description of the gerbe of splittings (that is, the data of groupoids of splitting of $F_*\mathcal{D}_X$ over each $Z \to T^*X^{(1)}$ as a functor in Z) of $F_*\mathcal{D}_X$ as a formula for the Brauer class of $F_*\mathcal{D}_X$ using the so-called Milne exact sequence. For a smooth scheme Y over k, consider the vector bundle Ω^1_Y as a sheaf on the small étale site, and likewise consider its subsheaf $\Omega^{1,\mathrm{cl}}_Y$ of closed 1-forms. By definition of étaleness, for every étale morphism $U \to Y$ the restriction of Ω^1_Y to the Zariski site of U is Ω^1_U.

Lemma 5.2. *(1) Cartier operator $C : \Omega^{1,\mathrm{cl}}_Y \to \Omega^1_Y$ induces an exact sequences of sheaves of abelian groups on Y_{et}:*

$$0 \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \xrightarrow{\mathrm{dlog}} \Omega^{1,\mathrm{cl}} \xrightarrow{1-C} \Omega^1_Y \to 0. \tag{5.2}$$

(2) For $Y = T^*X^{(1)}$ the image of the canonical 1-form $\lambda \in H^0(Y, \Omega^1_Y)$ under the connecting homomorphism $H^0(Y, \Omega^1_Y) \to H^2(Y, \mathbb{G}_m) = \mathrm{Br}(Y)$ induced by the degree 2 extension (5.2) equals to the class $[F_*\mathcal{D}_X] \in \mathrm{Br}(Y)$ of the Azumaya algebra $F_*\mathcal{D}_X$.

Proof. We refer to [OV07, Proposition 4.2] for the proof. The main observations going into it is that, in any characteristic, $\mathrm{R}\Gamma(T, \mathbb{G}_m \xrightarrow{\mathrm{dlog}} \Omega^{1,\mathrm{cl}}_T)[1]$ is the Picard groupoid of line bundles with a flat connection on a smooth scheme T, and that the p-curvature of the connection $(\mathcal{O}_T, d+\omega)$ defined by a closed 1-form $\omega \in H^0(T, \Omega^{1,\mathrm{cl}}_T)$ is precisely $\omega - C(\omega) \in H^0(T, \Omega^1_T)$. □

We now turn to the proof of Theorem 3.1 which, under the equivalence of Lemma 4.3 amounts to splitting the Azumaya algebra $F_*\mathcal{D}_X$ over the $(p-1)$-st neighborhood $X^{(1)}_{p-1} \subset T^*X^{(1)}$ of the zero section. By Lemma 4.7 doing so is equivalent to constructing a $\mathcal{D}_X$-module $V \in \mathcal{D}\text{-mod}^{\leq p-1}(X)$ such that the action of p-curvature makes F_*V into a locally free $\mathrm{Sym}^{\leq p-1} T_{X^{(1)}}$-module of rank $p^{\dim X}$.

6. Torsor of Frobenius splittings

The desired splitting module for $F_*\mathcal{D}_X|_{X^{(1)}_{p-1}}$ will be construced out of the torsor of Frobenius lifts onto $\widetilde{X}^{(1)}$. Given the lift $\widetilde{X}^{(1)}$ of $X^{(1)}$, let $\widetilde{X} := \widetilde{X}^{(1)} \times_{W_2(k), F_k^{-1}} W_2(k)$ be the corresponding lift of X that can be defined thanks to the fact that k is perfect. Consider the problem of lifting the Frobenius morphism $F : X \to X^{(1)}$ to a map $\widetilde{F} : \widetilde{X} \to \widetilde{X}^{(1)}$. This is always possible Zariski locally on X, but local lifts of F are not unique, and there is a naturally defined torsor on X for the vector bundle $F^*T_{X^{(1)}}$ whose splittings are in bijection with lifts of F. The data of a torsor for $F^*T_{X^{(1)}}$ is equivalent (by dualizing) to an extension of vector bundles

$$0 \to \mathcal{O}_X \to V_{\widetilde{X}^{(1)}} \to F^*\Omega^1_{X^{(1)}} \to 0. \tag{6.1}$$

Key observation of [OV07] is that $V_{\widetilde{X}^{(1)}}$ can be equipped with a natural flat connection:

Lemma 6.1 ([OV07, Theorem 1.1, Proposition 1.5])**.** *(1) There is a natural flat connection $\nabla : V_{\widetilde{X}^{(1)}} \to V_{\widetilde{X}^{(1)}} \otimes \Omega^1_X$ upgrading (6.1) to a short exact sequence of vector bundles with a flat connection*

$$0 \to (\mathcal{O}_X, d) \to (V_{\widetilde{X}^{(1)}}, \nabla) \to (F^*\Omega^1_{X^{(1)}}, \nabla^{\mathrm{can}}) \to 0 \tag{6.2}$$

(2) Since the p-curvature of 1st and 3rd terms of (6.2) is zero, p-curvature of ∇ defines a flat map $F^*\Omega^1_{X^{(1)}} \to \mathcal{O}_X \otimes F^*\Omega^1_{X^{(1)}} = F^*\Omega^1_{X^{(1)}}$. This map is equal to the identity.

Proof. We will outline the construction of the connection, and refer to [OV07] for the computation of its p-curvature. It will be slightly simpler to describe the connection on the dual bundle $V^\vee_{\widetilde{X}^{(1)}}$ which, by definition, fits into a short exact sequence

$$F^*T_{X^{(1)}} \to V^\vee_{\widetilde{X}^{(1)}} \xrightarrow{b} \mathcal{O}_X. \tag{6.3}$$

For any open $U \subset X$, the data of a section $s \in V^{\vee}_{\widetilde{X}^{(1)}}(U)$ that maps under b to $1 \in \mathcal{O}_X(U)$ is equivalent to a choice of a Frobenius lift $\widetilde{F}_U : \widetilde{U} \to \widetilde{U}^{(1)}$ onto the unique lift $\widetilde{U} \subset \widetilde{X}$ of the open $U \subset X$ inside $\widetilde{X}$.

On such s we define the value of the connection as follows. Note, first of all, that $1 \in \mathcal{O}_X(U)$ is a flat connection, so $\nabla(s)$ should be a section of $F^*T_{X^{(1)}} \otimes \Omega^1_X \subset V^{\vee}_{\widetilde{X}^{(1)}} \otimes \Omega^1_X$. The data of such a section over U is equivalent to the data of a map $\Omega^1_{U^{(1)}} \to F_*\Omega^1_U$, and we define this map to be the inverse of the Cartier operator induced by the lift $\widetilde{F}$, given by the formula

$$\omega \mapsto \frac{\widetilde{F}^*(\widetilde{\omega})}{p} : \Omega^1_{U^{(1)}} \to F_*\Omega^1_U, \tag{6.4}$$

cf. [DI87, Théorème 2.1(b)]. One then checks that the map $b^{-1}(1) \to (F^*T_{X^{(1)}} \otimes \Omega^1_X)(U)$ thus constructed extends uniquely to a flat connection on $V^{\vee}_{\widetilde{X}^{(1)}}$. □

Proof of Theorem 3.1. As discussed above, all we need to do is to produce a $\mathcal{D}_X$-module $(V, \nabla) \in \mathcal{D}\text{-mod}^{\leq p-1}(X)$ such that the action of the p-curvature makes F_*V into a locally free $\operatorname{Sym}^{\leq p-1} T_{X^{(1)}}$-module of rank $p^{\dim X}$. The desired equivalence $\mathcal{D}\text{-mod}^{\leq p-1}(X) \simeq \operatorname{Hig}^{\leq p-1}(X^{(1)})$ will then be given by

$$\mathcal{D}\text{-mod}^{\leq p-1}(X) \ni M \mapsto \mathcal{H}om_{F_*\mathcal{D}_X}(F_*V, F_*M) \in \operatorname{QCoh}(T^*X^{(1)}) = \operatorname{Hig}(X^{(1)}). \tag{6.5}$$

For any $k \in \mathbb{N}$ consider the vector bundle $V_k := \operatorname{Sym}^k V_{\widetilde{X}^{(1)}}$ with a flat connection, where $(V_{\widetilde{X}^{(1)}}, \nabla)$ is the vector bundle with a flat connection provided by Lemma 6.1. Sequence (6.2) can be viewed as a 2-step filtration on $V_{\widetilde{X}^{(1)}}$ with graded pieces having vanishing p-curvature. It induces a $(k+1)$-step filtration on V_k showing that it is an object of $\mathcal{D}\text{-mod}^{\leq k}(X)$. We claim that for $k < p$ the p-curvature action of $\operatorname{Sym}^{\leq k} T_{X^{(1)}}$ makes F_*V_k into a locally free module of rank $p^{\dim X}$.

Since F_*V_k is coherent over $\operatorname{Sym}^{\leq k} T_{X^{(1)}}$ it suffices to check local freeness after base change to every closed point $x \in |X^{(1)}|$. If $t_1, \dots, t_d$ are local coordinates at x, by Lemma 6.1(2) the module $F_*V_1|_x$ over $\operatorname{Sym} T_{X^{(1)}}|_x = k[\partial_{t_1}, \dots, \partial_{t_d}]$ can be identified with

$$(F_*\mathcal{O}_X)_x \otimes_k \{\text{degree } \leq 1 \text{ polynomials in } t_1, \dots, t_d\}$$

with the module structure given simply by the derivation on the second factor. This module is clearly free of rank $\dim(F_*\mathcal{O}_X)_x = p^{\dim X}$ over $\operatorname{Sym}^{\leq 1} T_{X^{(1)}}|_x = k[\partial_{t_1}, \dots, \partial_{t_d}]/(\partial_{t_1}, \dots, \partial_{t_d})^2$. Similarly, passing to n-th symmetric power we get

$$(F_*V_n)_x = \operatorname{Sym}^n_{(F_*\mathcal{O}_X)_x}(F_*V_1)_x = (F_*\mathcal{O}_X)_x \otimes_k \{\text{degree } \leq n \text{ polynomials in } t_1, \dots, t_d\} \tag{6.6}$$

As a module over $\operatorname{Sym}^{\leq n} T_{X^{(1)}}|_x = k[\partial_{t_1}, \dots, \partial_{t_d}]/(\partial_{t_1}, \dots, \partial_{t_d})^n$ then $(F_*V_n)_x$ is isomorphic to the sum of $p^{\dim X}$ copies of the divided power module $k[\frac{\partial^i_{t_1}}{i!}, \dots, \frac{\partial^i_{t_d}}{i!} | i \in \mathbb{N}]/(\partial_{t_1}, \dots, \partial_{t_d})^{[n]}$. Therefore, for $n < p$, in particular for $n = p$ this is simply a free rank $p^{\dim X}$ module over $\operatorname{Sym}^{\leq n} T_{X^{(1)}}|_x$ and V_{p-1} satisfies the condition for being a splitting module for $F_*\mathcal{D}_X|_{X_{p-1}}$. □

Remark 6.2. Moreover, one can show that the obstruction to splitting $F_*\mathcal{D}_X$ over X_1 is equal to the obstruction to lifting $X_1^{(1)}$ over $W_2(k)$. Note that both lie in the group $H^2(X^{(1)}, T_{X^{(1)}})$: the pushforward (as a sheaf on the fppf site) of $\mathbb{G}_{m,1}$ along the retraction

$X_1 \to X$ is $T_{X^{(1)}} \times \mathbb{G}_{m,X^{(1)}}$, hence $\mathrm{Br}(X_1)$ is isomorphic $H^2(X^{(1)}, T_{X^{(1)}}) \oplus \mathrm{Br}(X)$ and the class of $F_*\mathcal{D}_X|_{X_1}$ lies in the first factor because $F_*\mathcal{D}_X$ splits over the zero section.

This is closely related to Deligne-Illusie's equivalence [DI87, Proposition 3.3] between lifts of $X^{(1)}$ and decompositions of the truncated de Rham complex $\tau^{\leq 1}F_*\Omega^\bullet_X$. Splitting $F_*\mathcal{D}_X|_{X_1^{(1)}}$ is equivalent to finding a $\mathcal{D}_X$-module with 2-step nilpotent p-curvature that is locally free of rank $p^{\dim X}$ over $\mathrm{Sym}^{\leq 1} T_{X^{(1)}}$. Any such $\mathcal{D}_X$-module (W, ∇) fits into an extension

$$(\mathcal{O}_X, d) \to (W, \nabla) \to (F^*\Omega^1_{X^{(1)}}, \nabla^{\mathrm{can}}) \tag{6.7}$$

with p-curvature described as in Lemma 6.1(2). The isomorphism class of any extension of the form (6.7) is given by a class in $\mathrm{Ext}^1_{\mathcal{D}_X}((F^*\Omega^1_{X^{(1)}}, \nabla^{\mathrm{can}}), (\mathcal{O}_X, d)) = H^1_{\mathrm{dR}}(X, (F^*T_{X^{(1)}}, \nabla^{\mathrm{can}}))$. De Rham cohomology of the canonical connection $(F^*T_{X^{(1)}}, \nabla^{\mathrm{can}})$ is computed by the tensor product of the Frobenius-linearized de Rham complex

$$H^1_{\mathrm{dR}}(X, (F^*T_{X^{(1)}}, \nabla^{\mathrm{can}})) = \mathbb{H}^1(X^{(1)}, T_{X^{(1)}} \otimes_{\mathcal{O}_{X^{(1)}}} (F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_X \xrightarrow{d} \ldots)) \tag{6.8}$$

The spectral sequence associated to the canonical filtration on this de Rham complex, combined with the Cartier isomorphism, gives an exact sequence

$$0 \to H^1(X^{(1)}, T_{X^{(1)}}) \to H^1_{\mathrm{dR}}(X, (F^*T_{X^{(1)}}, \nabla^{\mathrm{can}})) \to H^0(X^{(1)}, T_{X^{(1)}} \otimes \Omega^1_{X^{(1)}}) \xrightarrow{\delta} H^2(X^{(1)}, T_{X^{(1)}}) \tag{6.9}$$

The condition that the p-curvature of (W, ∇) satisfies the condition of Lemma 6.1(2) is equivalent to the fact that $[W] \in H^1_{\mathrm{dR}}(X, (F^*T_{X^{(1)}}, \nabla^{\mathrm{can}}))$ maps to the counit of duality $\mathrm{id} \in H^0(X^{(1)}, T_{X^{(1)}} \otimes \Omega^1_{X^{(1)}})$. At the same time, δ sends id to the obstruction to decomposing the complex $\tau^{\leq 1}F_*\Omega^\bullet_X$, so Lemma 6.1 recovers Deligne-Illusie's decomposition arising from a lift of $X^{(1)}$.

Remark 6.3. The proof of Theorem 3.1 given above does not go through already for the p-th order neighborhood of $X^{(1)} \hookrightarrow T^*X^{(1)}$, and there are examples [OV07, 4.5] of liftable smooth proper surfaces over k for which $F_*\mathcal{D}_X$ does not split over $X_p^{(1)}$. Equivalently, the associative dg algebra $\mathcal{RH}om_{F_*\mathcal{D}_X}(F_*\mathcal{O}_X, F_*\mathcal{O}_X)$ in $\mathrm{QCoh}(X^{(1)})$ is not quasi-isomorphic to a dg algebra with vanishing differential.

7. Further results

Having introduced the object $V_{\widetilde{X}^{(1)}}$ we can also define the natural vector field on the moduli stack $M_{\mathrm{dR}}(X, r)$ of arbitrary vector bundles with a flat connection on X.

Proof of Theorem 3.6. We will only define the action of $\widehat{\mathbb{G}_m^\sharp}$ on $M_{\mathrm{dR}}(X, r)$. As discussed in Remark 3.7 this is equivalent to defining a vector field on $M_{\mathrm{dR}}(X, r)$, which, by definition, amounts to specifying, for every S-point $f : S \to M_{\mathrm{dR}}(X, r)$ an $S \times_k k[\varepsilon]/\varepsilon^2$-point of $M_{\mathrm{dR}}(X, r)$ lifting f. For simplicity of notation we describe this construction in the case $S = \mathrm{Spec}\, k$. We refer to [Tra16] for more on this construction.

Given a vector bundle with a flat connection (E,∇) consider the tensor product $E\otimes V_{\widetilde{X}^{(1)}}$ and form the pullback along (the negative of) the p-curvature map ψ_∇:

$$\begin{array}{ccccc} E & \longrightarrow & E\otimes V_{\widetilde{X}^{(1)}} & \longrightarrow & V\otimes F^*\Omega^1_{\widetilde{X}^{(1)}} \\ \| & & \uparrow & & \uparrow{\scriptstyle -\psi_\nabla} \\ E & \longrightarrow & \widetilde{E} & \longrightarrow & E \end{array} \tag{7.1}$$

We obtain a rank $2r$ vector bundle $\widetilde{E}$ on X that we make into a locally free sheaf with a flat connection relative to $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ on $X\times k[\varepsilon]/\varepsilon^2$ by letting $\varepsilon:\widetilde{E}\to\widetilde{E}$ be the composition $\widetilde{E}\twoheadrightarrow E\hookrightarrow\widetilde{E}$. The association $E\mapsto\widetilde{E}$ defines the desired vector field[5]. □

As we discussed, for any liftable X the Azumaya algebra $F_*\mathcal{D}_X$ splits on the $(p-1)$st infinitesimal neighborhood $X^{(1)}_{p-1}\hookrightarrow T^*X^{(1)}$ of the zero section, but need not split on larger nilpotent neighborhoods. Let us finish with a splitting result of a different flavor:

Lemma 7.1. *Let X be a smooth proper variety over an algebraically closed field k. If $X^{(1)}$ admits a lift over $W_2(k)$, then $F_*\mathcal{D}_X|_Z$ splits for every closed subscheme $Z\to T^*X^{(1)}$ that is finite étale over $X^{(1)}$.*

Proof. By functoriality of the Milne sequence the Brauer class $[(F_*\mathcal{D}_X)|_Z]\in\operatorname{Br}(Z)$ equals to the image of the restriction $\lambda|_Z\in H^0(Z,\Omega^1_Z)$ under the connecting homomorphism induced by the Milne sequence on Z_{et}:

$$0\to\mathbb{G}_m\xrightarrow{p}\mathbb{G}_m\xrightarrow{\text{dlog}}\Omega^{1,\text{cl}}_Z\xrightarrow{1-C}\Omega^1_Z\to 0 \tag{7.2}$$

This connecting homomorphism is zero if the map $H^0(Z,\Omega^{1,\text{cl}}_Z)\xrightarrow{1-C}H^0(Z,\Omega^1_Z)$ is surjective, and we will now establish this surjectivity. In other words, we will find a closed 1-form ω such that the p-curvature of $(\mathcal{O}_Z,d+\omega)$ is $\lambda|_Z$, implying that $F_*\mathcal{D}_X|_Z$ splits by Lemma 5.1.

Since Z is étale over $X^{(1)}$, it also admits a lift over $W_2(k)$ because of the nil-invariance of the étale site. As Z is smooth and proper, its Hodge-to-de Rham spectral sequence has no non-zero differentials coming out of terms $H^i(Z,\Omega^j_Z)$ wit $i+j<p$, in particular all global 1-forms on Z are closed. Hence the Cartier operator $C:H^0(Z,\Omega^{1,\text{cl}}_Z)\to H^0(Z,\Omega^1_Z)=H^0(Z,\Omega^{1,\text{cl}}_Z)$ can be viewed as an $\mathbb{F}_p$-linear endomorphism of the finite-dimensional k-vector space which is moreover inverse-Frobenius k-linear, that is $C(\lambda^p\omega)=\lambda\omega$ for any $\lambda\in k$. Composing C and Id with any Frobenius-linear isomorphism $\varphi:H^0(Z,\Omega^{1,\text{cl}}_Z)\simeq H^0(Z,\Omega^{1,\text{cl}}_Z)$ we get endomorphisms $\varphi,\varphi\circ C$ and $\varphi-\varphi\circ C$ is surjective by Lemma 7.2 below. □

Lemma 7.2. *Let V be a finite-dimensional vector space over an algebraically closed field k of characteristic p. Suppose that $F:V\to V$ is a Frobenius-linear automorphism (that is, F is additive and $F(\lambda v)=\lambda^pF(v)$ for $\lambda\in k$) and $L:V\to V$ is any linear endomorphism. Then the $\mathbb{F}_p$-linear map $F-L:V\to V$ is surjective.*

[5]Formally, the only content of the final statement of Theorem 3.6 is the compatibility of the $\widehat{\mathbb{G}^\sharp_m}$-action arising from this vector field and the $\mathbb{G}_m$-action arising from the equivalence in Theorem 3.1. I do not know whether the vector field defined above is literally compatible with Ogus-Vologodsky's equivalence, or if it is off by a sign. In the latter case the definition of the vector field should be modified by multiplying it by (-1).

Proof. The following argument was explained to me by Arthur Ogus. Choosing a basis in V, and composing F and L with an appropriate k-linear automorphism we may assume that F is given by $F(x_1,\dots,x_d)=(x_1^p,\dots,x_d^p)$ for $(x_1,\dots,x_d)\in k^d=V$, and L is given by some matrix $(a_{ij})_{1\leq i,j,\leq d}$.

Observe that the map between affine spaces $\pi:\mathbb{A}^d_k=\operatorname{Spec} k[x_1,\dots,x_d]\to\mathbb{A}^d_k=\operatorname{Spec} k[y_1,\dots,y_d]$ given by

$$(x_1,\dots,x_d)\mapsto(y_i=x_i^p-\sum_j a_{ij}x_j)_{i=1,\dots,d}$$

is finite: $k[x_1,\dots,x_d]$ is generated by monomials $x_1^{i_1}\cdot\ldots\cdot x_d^{i_d}$ with $0\leq i_j\leq p-1$ as a module over $k[y_1,\dots,y_d]$. Hence the image of π is closed, and generically on its image π is flat of relative dimension zero. Hence the image must be of dimension d, that is π is surjective. This morphism induces the map $F-L$ on k-points, hence $F-L:V\to V$ is surjective. $\square$

Remarkably, at the moment I am not aware of a single example of a smooth spectral subvariety on which the algebra $F_*\mathcal{D}_X$ does not split:

Question 7.3. (1) Does there exist a smooth variety X over k and a smooth subvariety $Z\subset T^*X^{(1)}$ finite over $X^{(1)}$ such that $F_*\mathcal{D}_X|_Z$ does not split?
(2) If we assume that X lifts to $W_2(k)$, does there exist a subscheme $Z\subset T^*X^{(1)}$ finite flat of degree $<p$ over $X^{(1)}$ such that $F_*\mathcal{D}_X|_Z$ does not split?

References

[BB07] Alexander Braverman and Roman Bezrukavnikov. Geometric Langlands correspondence for $\mathcal{D}$-modules in prime characteristic: the GL(n) case. *Pure Appl. Math. Q.*, 3(1):153–179, 2007.

[BMR08] Roman Bezrukavnikov, Ivan Mirković, and Dmitriy Rumynin. Localization of modules for a semisimple Lie algebra in prime characteristic. *Ann. of Math. (2)*, 167(3):945–991, 2008. With an appendix by Bezrukavnikov and Simon Riche.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[Kat70] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.

[Ogu04] Arthur Ogus. Higgs cohomology, p-curvature, and the Cartier isomorphism. *Compos. Math.*, 140(1):145–164, 2004.

[OV07] A. Ogus and V. Vologodsky. Nonabelian Hodge theory in characteristic p. *Publ. Math. Inst. Hautes Études Sci.*, (106):1–138, 2007.

[Pet25] Alexander Petrov. Non-decomposability of the de Rham complex and non-semisimplicity of the Sen operator. `https://arxiv.org/abs/2302.11389`, 2025.

[Sim91] Carlos T. Simpson. The ubiquity of variations of Hodge structure. In *Complex geometry and Lie theory (Sundance, UT, 1989)*, volume 53 of *Proc. Sympos. Pure Math.*, pages 329–348. Amer. Math. Soc., Providence, RI, 1991.

[Ter25] Gleb Terentiuk. Ogus-Vologodsky equivalence via stacks. in preparation, 2025.

[Tra16] Roman Travkin. Quantum geometric Langlands correspondence in positive characteristic: the GL_N case. *Duke Math. J.*, 165(7):1283–1361, 2016.