THE GAUSS-MANIN CONNECTION ON THE PERIODIC CYCLIC HOMOLOGY

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To Sasha Beilinson on his 60th birthday, with admiration

Abstract. Let R be the algebra of functions on a smooth affine irreducible curve S over a field k and let $A_\bullet$ be smooth and proper DG algebra over R. The relative periodic cyclic homology $HP_*(A_\bullet)$ of $A_\bullet$ over R is equipped with the Hodge filtration $F^\cdot$ and the Gauss-Manin connection ∇ ([Ge], [K1]) satisfying the Griffiths transversality condition. When k is a perfect field of odd characteristic p, we prove that if the relative Hochschild homology $HH_m(A_\bullet, A_\bullet)$ vanishes in degrees $|m| \geq p-2$ then a lifting $\tilde{R}$ of R over $W_2(k)$ and a lifting of $A_\bullet$ over $\tilde{R}$ determine the structure of a relative Fontaine-Laffaille module ([Fa], §2 (c), [OV] §4.6) on $HP_*(A_\bullet)$. That is, the inverse Cartier transform of the Higgs R-module $(Gr^F HP_*(A_\bullet), Gr^F \nabla)$ is canonically isomorphic to $(HP_*(A_\bullet), \nabla)$. This is noncommutative counterpart of Faltings' result ([Fa], Th. 6.2) for the de Rham cohomology of a smooth proper scheme over R. Our result amplifies the noncommutative Deligne-Illusie decomposition proven by Kaledin ([K4], Th. 5.1). As a corollary, we show that the p-curvature of the Gauss-Manin connection on $HP_*(A_\bullet)$ is nilpotent and, moreover, it can be expressed in terms of the Kodaira-Spencer class $\kappa \in HH^2(A, A) \otimes_R \Omega^1_R$ (a similar result for the p-curvature of the Gauss-Manin connection on the de Rham cohomology is proven by Katz in [Katz2]). As an application of the nilpotency of the p-curvature we prove, using a result from [Katz1]), a version of "the local monodromy theorem" of Griffiths-Landman-Grothendieck for the periodic cyclic homology: if $k = \mathbb{C}$, $\overline{S}$ is a smooth compactification of S, then, for any smooth and proper DG algebra $A_\bullet$ over R, the Gauss-Manin connection on the relative periodic cyclic homology $HP_*(A_\bullet)$ has regular singularities, and its monodromy around every point at $\overline{S} - S$ is quasi-unipotent.

1. Introduction

It is expected that the periodic cyclic homology of a DG algebra over $\mathbb{C}$ (and, more generally, the periodic cyclic homology of a DG category) carries a lot of additional structure similar to the mixed Hodge structure on the de Rham cohomology of algebraic varieties. Whereas a construction of such a structure seems to be out of reach at the moment its counterpart in finite characteristic is much better understood thanks to recent groundbreaking works of Kaledin. In particular, it is proven in [K4] that under some assumptions on a DG algebra $A_\bullet$ over a perfect field k of characteristic p, a lifting of $A_\bullet$ over the ring of second Witt vectors $W_2(k)$ specifies the structure of a Fontaine-Laffaille module on the periodic cyclic homology of $A_\bullet$. The purpose of this paper is to develop a relative version of Kaledin's theory for DG algebras over a base k-algebra R incorporating in the picture the Gauss-Manin connection on the relative periodic cyclic homology constructed by Getzler in [Ge]. Our main result, Theorem 1, asserts that, under some assumptions on $A_\bullet$, the Gauss-Manin connection on its periodic cyclic homology can be recovered from the Hochschild homology of $A_\bullet$ equipped with the action of the Kodaira-Spencer operator as the inverse Cartier transform ([OV]). As an application, we prove, using the reduction modulo p technique, that, for a smooth and proper DG algebra over a complex punctured disk, the monodromy of the Gauss-Manin connection on its periodic cyclic homology is quasi-unipotent.

1.1. **Relative Fontaine-Laffaille modules.** Let R be a smooth commutative algebra over a perfect field k of characteristic $p > 0$. Recall from ([Fa], §2 (c), [OV] §4.6) the notion of *relative Fontaine-Laffaille module* [1] over R. Fix a flat lifting $\tilde{R}$ of R over the ring $W_2(k)$ of second Witt vectors and a lifting $\tilde{F}: \tilde{R} \to \tilde{R}$ of the Frobenius morphism $F: R \to R$. Define the inverse Cartier transform

$$\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})} : \mathsf{HIG}(R) \to \mathsf{MIC}(R)$$

to be a functor from the category of Higgs modules *i.e.*, pairs (E, θ), where E is an R-module and $\theta : E \to E \otimes_R \Omega^1_R$ is an R-linear morphism such that the composition $\theta^2 : E \to E \otimes_R \Omega^1_R \to E \otimes_R \Omega^2_R$ equals 0[2], to the category of R-modules with integrable connection. Given a Higgs module (E, θ) we set

$$\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(E, \theta) := (F^*E, \nabla_{can} + C^{-1}_{(\tilde{R},\tilde{F})}(\theta)),$$

where ∇_{can} is the Frobenius pullback connection on F^*E and the map

$$C^{-1}_{(\tilde{R},\tilde{F})} : End_R(E) \otimes \Omega^1_R \to F_*(End_R(F^*E) \otimes_R \Omega^1_R) \tag{1.1}$$

takes $f \otimes \eta$ to $F^*(f) \otimes \frac{1}{p}\tilde{F}^*\tilde{\eta}$, with $\tilde{\eta} \in \Omega^1_{\tilde{R}}$ being a lifting of η. A relative Fontaine-Laffaille module over R consists of a finitely generated R-module M with an integrable connection ∇ and a Hodge filtration

$$0 = F^{l+1}M \subset F^l M \subset \cdots \subset F^m M = M$$

satisfying the Griffiths transversality condition, together with isomorphism in $\mathsf{MIC}(R)$:

$$\phi : \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(\mathrm{Gr}^\bullet_F M, \mathrm{Gr}_F \nabla) \xrightarrow{\sim} (M, \nabla).$$

Here $\mathrm{Gr}_F \nabla : \mathrm{Gr}^\bullet_F M \to Gr^{\bullet-1}_F M$ is the "Kodaira-Spencer" Higgs field induced by ∇. [3]

The category $\mathcal{MF}_{[m,l]}(\tilde{R}, \tilde{F})$(where $l \geq m$ are arbitrary integers) of relative Fontaine-Laffaille modules has a number of remarkable properties not obvious from the definition. It is proven by Faltings in ([Fa], Th. 2.1) that $\mathcal{MF}_{[m,l]}(\tilde{R}, \tilde{F})$ is abelian, every morphism between Fontaine-Laffaille modules is strictly compatible with the Hodge filtration, and, for every Fontaine-Laffaille module $(M, \nabla, F^\bullet M, \phi)$, the R-modules M and $Gr_F M$ are flat. Moreover, if $l - m < p$, the category $\mathcal{MF}_{[m,l]}(\tilde{R}, \tilde{F}) =: \mathcal{MF}_{[m,l]}(\tilde{R})$ is independent of the choice of the Frobenius lifting[4].

[1] In [Fa], Faltings does not give a name to these objects. In [OV], they are called Fontaine modules. The name suggested here is a tribute to [FL], where these objects were first introduced in the case when $R = k$.

[2] Equivalently, a Higgs module is a module over the symmetric algebra $S^\bullet T_R$.

[3] In [Fa], Faltings considers more general objects. In fact, what we call here a relative Fontaine-Laffaille module is the same as a p-torsion object in Faltings' category $\mathcal{MF}^\nabla_{[m,l]}(R)$

[4] Every two liftings $\tilde{R}$, $\overline{R}$ of R are isomorphic. A choice of such an isomorphism induces an equivalence $\mathcal{MF}_{[m,l]}(\tilde{R}) \xrightarrow{\sim} \mathcal{MF}_{[m,l]}(\overline{R})$. We refer the reader to ([OV] §4.6) for a construction of the category of Fontaine-Laffaille modules over any smooth scheme X over k equipped with a lifting $\tilde{X}$ over $W_2(k)$.

Fontaine-Laffaille modules arise geometrically: it is shown in ([Fa], Th. 6.2) that, for a smooth proper scheme $X \to spec\, R$ of relative dimension less than p, a lifting of X over $\tilde{R}$ specifies a Fontaine-Laffaille module structure on the relative de Rham cohomology $H^{\bullet}_{DR}(X/R)$.

1.2. **The Kodaira-Spencer class of a DG algebra.** Let $A_{\bullet}$ be a differential graded algebra over R. Denote by $HH^{\bullet}(A_{\bullet}, A_{\bullet})$ its Hochschild cohomology and by

$$\kappa \in HH^2(A_{\bullet}, A_{\bullet}) \otimes_R \Omega^1_R \tag{1.2}$$

the *Kodaira-Spencer class* of $A_{\bullet}$. This can be defined as follows. Choose a quasi-isomorphism $A_{\bullet} \xrightarrow{\sim} B_{\bullet}$, where $B_{\bullet}$ is a semi-free DG algebra over R ([Dr], §13.4) and a connection $\nabla' : \bigoplus B_i \to \bigoplus B_i \otimes \Omega^1_R$ on the graded algebra $\bigoplus B_i$ satisfying the Leibnitz rule with respect to the multiplication on $\bigoplus B_i$. Then the commutator

$$[\nabla', d] \in \prod Hom_R(B_i, B_{i+1}) \otimes \Omega^1_R \tag{1.3}$$

with the differential d on $B_{\bullet}$ commutes with d and it is a R-linear derivation of $B_{\bullet}$ with values in $B_{\bullet} \otimes \Omega^1_R$ of degree 1. Any such derivation determines a class in

$$HH^2(B_{\bullet}, B_{\bullet}) \otimes_R \Omega^1_R \xrightarrow{\sim} HH^2(A_{\bullet}, A_{\bullet}) \otimes_R \Omega^1_R.$$

The class corresponding to $[\nabla', d]$ is independent of the choices we made. This is the Kodaira-Spencer class (1.2).

1.3. **The Hodge filtration on the periodic cyclic homology.** Denote by $(CH_{\bullet}(A_{\bullet}, A_{\bullet}), b)$ the relative Hochschild chain complex of $A_{\bullet}$ and by $CP_{\bullet}(A_{\bullet}) = (CH_{\bullet}(A_{\bullet}, A_{\bullet})((u)), b + uB)$ the periodic cyclic complex. The complex $CP_{\bullet}(A_{\bullet})$ is equipped with the Hodge filtration

$$F^i CP_{\bullet}(A_{\bullet}) := (u^i CH_{\bullet}(A_{\bullet}, A_{\bullet})[[u]], b + uB),$$

which induces a Hodge filtration $F^{\bullet} HP_{\bullet}(A_{\bullet})$ on the periodic cyclic homology and a convergent Hodge-to-de Rham spectral sequence

$$HH_{\bullet}(A_{\bullet}, A_{\bullet})((u)) \Rightarrow HP_{\bullet}(A_{\bullet}). \tag{1.4}$$

The Gauss-Manin connection ∇ on the periodic cyclic homology (we recall its construction in §3) satisfies the Griffiths transversality condition

$$\nabla : F^{\bullet} HP_{\bullet}(A_{\bullet}) \to F^{\bullet - 1} HP_{\bullet}(A_{\bullet}) \otimes_R \Omega^1_R.$$

The Kodaira-Spencer class (1.2) acts on the Hochschild homology:

$$e_{\kappa} : HH_{\bullet}(A_{\bullet}, A_{\bullet}) \to HH_{\bullet - 2}(A_{\bullet}, A_{\bullet}) \otimes_R \Omega^1_R.$$

The operator e_{κ} is induced by the action of the Hochschild cohomology algebra on the Hochschild homology (referred to as the “interior product” action).

1.4. **Statement of the main result.** Recall that $A_{\bullet}$ is called homologically proper if $A_{\bullet}$ is perfect as a complex of R-modules. A DG algebra $A_{\bullet}$ is said to be homologically smooth if $A_{\bullet}$ is quasi-isomorphic to a DG algebra $B_{\bullet}$ which is termwise flat over R and $B_{\bullet}$ is perfect as a DG module over $B_{\bullet} \otimes_R B_{\bullet}{}^{op}$. The following is one of the main results of our paper.

Theorem 1. *Fix the pair $(\tilde{R}, \tilde{F})$ as in §1.1 and assume, in addition, that the characteristic p of k is odd. Let $A_\bullet$ be a homologically smooth and homologically proper DG algebra over R such that*

$$HH_m(A_\bullet, A_\bullet) = 0, \text{ for every } m \text{ with } |m| \geq p-2. \tag{1.5}$$

Then a lifting[5] *of $A_\bullet$ over $\tilde{R}$, if it exists, specifies an isomorphism*

$$\phi : \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(\mathrm{Gr}^\bullet_F HP_\bullet(A_\bullet), \mathrm{Gr}_F \nabla) \xrightarrow{\sim} (HP_\bullet(A_\bullet), \nabla) \tag{1.6}$$

giving $(HP_\bullet(A_\bullet), \nabla, F^\bullet HP_\bullet(A_\bullet))$ a Fontaine-Laffaille module structure. In addition, the Hodge-to-de Rham spectral sequence (1.4) degenerates at E_1 and induces an isomorphism of Higgs modules

$$(\mathrm{Gr}^\bullet_F HP_\bullet(A_\bullet), \mathrm{Gr}_F \nabla) \xrightarrow{\sim} (HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], u^{-1}e_\kappa). \tag{1.7}$$

Using (1.7), the isomorphism (1.6) takes the form

$$\phi : (F^* HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], \nabla_{can} + u^{-1} C^{-1}_{(\tilde{R},\tilde{F})}(e_\kappa)) \xrightarrow{\sim} (HP_\bullet(A_\bullet), \nabla), \tag{1.8}$$

where ∇_{can} is the Frobenius pullback connection and $C^{-1}_{(\tilde{R},\tilde{F})}$ is the inverse Cartier operator (1.1).

Remarks 1.1. (a) If $R = k$ the above result is due to Kaledin ([K4], Th. 5.1).

(b) The construction from Theorem 1 determines a functor from the category of homologically smooth and homologically proper DG algebras over $\tilde{R}$ satisfying (1.5) localized with respect to quasi-isomorphisms to the category of Fontaine-Laffaille modules. We expect, but do not check it in this paper, that this functor extends to the homotopy category of smooth and proper DG categories over $\tilde{R}$ satisfying the analogue of (1.5). When applied to the bounded derived DG category $D^b(Coh(\tilde{X}))$ of coherent sheaves on a smooth proper scheme $\tilde{X}$ over $\tilde{R}$ of relative dimension less than $p-1$, we expect to recover the Fontaine-Laffaille structure on
$$HP_0(D^b(Coh(X)) \xrightarrow{\sim} \bigoplus_i H^{2i}_{DR}(X)(i)$$
$$HP_1(D^b(Coh(X)) \xrightarrow{\sim} \bigoplus_i H^{2i+1}_{DR}(X)(i)$$
constructed by Faltings in ([Fa], Th. 6.2). Here X denotes the scheme over R obtained from $\tilde{X}$ by the base change and $H^*_{DR}(X)(i)$ the Tate twist of the Fontaine-Laffaille structure on the relative de Rham cohomology.

Let us explain some corollaries of Theorem 1. First, under the assumptions of Theorem 1 the Hochschild and cyclic homology of $A_\bullet$ is a locally free R-module. This follows from a general property of Fontaine-Laffaille modules mentioned above. Next, it follows, that under the same assumptions the p-curvature of the Gauss-Manin connection on $HP_\bullet(A_\bullet)$ is nilpotent[6]. In fact, there is a decreasing filtration,

$$V_i HP_\bullet(A_\bullet) \subset HP_\bullet(A_\bullet) \tag{1.9}$$

formed by the images under ϕ of

$$u^i F^* HH_\bullet(A_\bullet, A_\bullet)[u^{-1}] \subset F^* HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}]$$

[5] A lifting of $A_\bullet$ over $\tilde{R}$ is a termwise flat DG algebra $\tilde{A}_\bullet$ over $\tilde{R}$ together with a quasi-isomorphism $\tilde{A}_\bullet \otimes_{\tilde{R}} R \xrightarrow{\sim} A_\bullet$ of DG algebras over R.

[6] This suffices for our main application in characteristic 0: Theorem 3 below.

which is preserved by the connection and such that $\mathrm{Gr}^V_\bullet\, HP_\bullet(A_\bullet)$ has zero p-curvature:

$$(\mathrm{Gr}^V_\bullet\, HP_\bullet(A_\bullet), \mathrm{Gr}^V \nabla) \xrightarrow{\sim} (F^* HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], \nabla_{can}). \tag{1.10}$$

Moreover, using Theorem 1 we can express the p-curvature of ∇ on $HP_\bullet(A_\bullet)$ in terms of the Kodaira-Spencer operator e_κ: by ([OV], Th. 2.8), for any Higgs module (E, θ), such that the action of $S^p T_R$ on E is trivial, the p-curvature of $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(E, \theta)$, viewed as a R-linear morphism

$$\psi : F^*E \to F^*E \otimes F^*\Omega^1_R$$

is equal to $-F^*(\theta)$. In particular, under assumption (1.5), the p-curvature of $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(HH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], u^{-1}e_\kappa)$, equals $-u^{-1}F^*(e_\kappa)$. As a corollary, we obtain a version of the Katz formula for the p-curvature of the Gauss-Manin connection on the de Rham cohomology ([Katz2], Th. 3.2): by (1.10) the p-curvature morphism for $HP_\bullet(A_\bullet)$ shifts the filtration $V_\bullet$:

$$\psi : V_\bullet HP_\bullet(A_\bullet) \to V_{\bullet-1} HP_\bullet(A_\bullet) \otimes F^*\Omega^1_R.$$

Thus, ψ induces a morphism

$$\overline{\psi} : \mathrm{Gr}^V_\bullet\, HP_\bullet(A_\bullet) \to \mathrm{Gr}^V_{\bullet-1}\, HP_\bullet(A_\bullet) \otimes F^*\Omega^1_R.$$

Our version of the Katz formula asserts the commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{Gr}^V_i\, HP_j(A_\bullet) & \xrightarrow{\sim} & F^*HH_{j+2i}(A_\bullet, A_\bullet) \\ \Big\downarrow{\scriptstyle \overline{\psi}} & & \Big\downarrow{\scriptstyle -F^*(e_\kappa)} \\ \mathrm{Gr}^V_{i-1}\, HP_j(A_\bullet) \otimes F^*\Omega^1_R & \xrightarrow{\sim} & F^*HH_{j+2i-2}(A_\bullet, A_\bullet) \otimes F^*\Omega^1_R. \end{array} \tag{1.11}$$

1.5. **The co-periodic cyclic homology, the conjugate filtration, and a generalized Katz p-curvature formula.** Though, as explained above, formula (1.11) is an immediate corollary of Theorem 1, a version of the former holds for any DG algebra $A_\bullet$. What makes this generalization possible is the observation that although the morphism (1.8) does depend on the choice of a lifting of $A_\bullet$ over $\tilde{R}$ the induced ∇-invariant filtration (1.9) is canonical: in fact, it coincides with the *conjugate filtration* introduced by Kaledin in [K3].[7] However, in general, the conjugate filtration is a filtration on the *co-periodic cyclic homology* $\overline{HP}_\bullet(A_\bullet)$ defined Kaledin in *loc. cit.* to be the homology of the complex

$$\overline{CP}_\bullet(A_\bullet) = (CH_\bullet(A_\bullet, A_\bullet)((u^{-1})), b + uB).$$

For any $A_\bullet$, this comes together with the conjugate filtration $V_\bullet \overline{CP}_\bullet(A_\bullet)$ satisfying the properties

$$u : V_\bullet \overline{CP}_\bullet(A_\bullet) \xrightarrow{\sim} V_{\bullet+1} \overline{CP}_\bullet(A_\bullet)[2],$$
$$\mathrm{Gr}^V\, \overline{CP}_\bullet(A_\bullet) \xrightarrow{\sim} F^*C(A_\bullet, A_\bullet)((u^{-1})).$$

This yields a convergent *conjugate spectral sequence*

$$F^*HH_\bullet(A_\bullet, A_\bullet)((u^{-1})) \Rightarrow \overline{HP}_\bullet(A_\bullet), \tag{1.12}$$

whose E_∞ page is $\mathrm{Gr}^V_\bullet\, \overline{HP}_\bullet(A_\bullet)$. It is shown in [K3] that if $A_\bullet$ is smooth and homologically bounded then the morphisms

$$(CH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], b + uB) \longrightarrow (CH_\bullet(A_\bullet, A_\bullet)((u)), b + uB) \tag{1.13}$$

$$(CH_\bullet(A_\bullet, A_\bullet)[u, u^{-1}], b + uB) \longrightarrow (CH_\bullet(A_\bullet, A_\bullet)((u^{-1})), b + uB) \tag{1.14}$$

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[7] The terminology is borrowed from [Katz1], where the conjugate filtration on the de Rham cohomology in characteristic p was introduced.

are quasi-isomorphisms. In particular, for smooth and homologically bounded DG algebras one has a canonical isomorphism

$$\overline{HP}_\bullet(A_\bullet) \xrightarrow{\sim} HP_\bullet(A_\bullet). \tag{1.15}$$

For an arbitrary DG algebra $A_\bullet$ we introduce in §3 a Gauss-Manin connection on $\overline{HP}_\bullet(A_\bullet)$. It is compatible with the one on $HP_\bullet(A_\bullet)$ if $A_\bullet$ is smooth and homologically bounded. We show that ∇ preserves the conjugate filtration and the entire conjugate spectral sequence (1.12) is compatible with the connection (where the first page, $F^*HH_\bullet(A_\bullet, A_\bullet)((u^{-1}))$ is endowed with the Frobenius pullback connection). In particular, the p-curvature ψ of the connection on $\overline{HP}_\bullet(A_\bullet)$ is zero on $\mathrm{Gr}^V_\bullet \overline{HP}_\bullet(A_\bullet)$. Hence, ψ induces a morphism

$$\overline{\psi} : \mathrm{Gr}^V_\bullet \overline{HP}_\bullet(A_\bullet) \to \mathrm{Gr}^V_{\bullet-1} \overline{HP}_\bullet(A_\bullet) \otimes F^*\Omega^1_R.$$

In §3 we prove the following result, which is a generalization of formula (1.11).

Theorem 2. *Let $A_\bullet$ be a DG algebra over R and $\kappa \in HH^2(A_\bullet, A_\bullet) \otimes_R \Omega^1_R$ its Kodaira-Spencer class.*

(a) The morphism $u^{-1}F^*(e_\kappa) : F^*HH_\bullet(A_\bullet, A_\bullet)((u^{-1})) \to F^*HH_\bullet(A_\bullet, A_\bullet)((u^{-1})) \otimes F^*\Omega^1_R$ commutes with all the differentials in the conjugate spectral sequence (1.12) inducing a map

$$\mathrm{Gr}^V_\bullet \overline{HP}_\bullet(A_\bullet) \to \mathrm{Gr}^V_{\bullet-1} \overline{HP}_\bullet(A_\bullet) \otimes F^*\Omega^1_R,$$

which we also denote by $u^{-1}F^*(e_\kappa)$. With this notation, we have

$$u^{-1}F^*(e_\kappa) = \overline{\psi}. \tag{1.16}$$

(b) Assume that $HH_m(A_\bullet, A_\bullet) = 0$ for all sufficiently negative m. Then the p-curvature of the Gauss-Manin connection on $\overline{HP}_\bullet(A_\bullet)$ is nilpotent.

Corollary 1.2. *Let $A_\bullet$ be a smooth and proper DG algebra over R and let d be a non-negative integer d such that $HH_m(A_\bullet, A_\bullet) = 0$, for every m with $|m| > d$. Then the p-curvature of the Gauss-Manin connection on $HP_\bullet(A_\bullet)$ is nilpotent of exponent $\leq d+1$, i.e., there exists a filtration*

$$0 = V_0HP_\bullet(A_\bullet) \subset \cdots \subset V_{d+1}HP_\bullet(A_\bullet) = HP_\bullet(A_\bullet)$$

preserved by the connection such that, for every $0 < i \leq d+1$, the p-curvature of the connection on V_i/V_{i-1} is 0.

1.6. **An application: the local monodromy theorem.** As an application of the nilpotency of the p-curvature we prove, using a result from ([Katz1]), “the local monodromy theorem” for the periodic cyclic homology in characteristic 0.

Theorem 3. *Let S be a smooth irreducible affine curve over $\mathbb{C}$, $\overline{S}$ a smooth compactification of S, and let $A_\bullet$ be a smooth and proper DG algebra over $\mathcal{O}(S)$. Then the Gauss-Manin connection on the relative periodic cyclic homology $HP_*(A_\bullet)$ has regular singularities and, its monodromy around every point at $\overline{S} - S$ is quasi-unipotent.*

This result generalizes the Griffiths-Landman-Grothendieck theorem asserting that for a smooth proper scheme X over S the Gauss-Manin connection on the relative de Rham cohomology $H^*_{DR}(X)$ has regular singularities and that its monodromy at infinity is quasi-unipotent. The derivation of Theorem 3 from Corollary 1.2 is essentially due to Katz ([Katz1]); we explain the argument in §4.

1.7. **Proofs.** Let us outline the proofs of Theorems 1 and 2. Without loss of generality we may assume that $A_\bullet$ is a semi-free DG algebra over R. Let ${A_\bullet}^{\otimes p}$ denote the p-th tensor power of $A_\bullet$ over R. This is a DG algebra equipped with an action of the symmetric group S_p. In particular, it carries an action of the group $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} C_p \subset S_p$ of cyclic permutations. We denote by $T(C_p, {A_\bullet}^{\otimes p})$ the Tate cohomology complex of C_p with coefficients in ${A_\bullet}^{\otimes p}$. The algebra structure on ${A_\bullet}^{\otimes p}$ induces one on the Tate cohomology $\check{H}^\bullet(C_p, {A_\bullet}^{\otimes p})$. Moreover, choosing an appropriate "complete resolution" one can lift the cup product on the cochain level giving $T(C_p, {A_\bullet}^{\otimes p})$ the structure of a DG algebra over R. If $A_\bullet = A$ is an associative algebra then, for $p \neq 2$, one has a canonical isomorphism of algebras

$$\check{H}^*(C_p, {A_\bullet}^{\otimes p}) \xrightarrow{\sim} F^*A \otimes \check{H}^*(C_p, \mathbb{F}_p) \xrightarrow{\sim} F^*A[u, u^{-1}, \epsilon],$$

$\deg u = 2$ and $\deg \epsilon = 1$, $\epsilon^2 = 0$. In general, Kaledin defines an increasing filtration

$$\tau^{dec}_{\leq \bullet} T(C_p, {A_\bullet}^{\otimes p}) \subset T(C_p, {A_\bullet}^{\otimes p})$$

making $T(C_p, {A_\bullet}^{\otimes p})$ a filtered DG algebra equipped with a canonical quasi-isomorphism of graded DG algebras

$$\bigoplus_i \mathrm{Gr}^\tau_i\, T(C_p, {A_\bullet}^{\otimes p}) \xrightarrow{\sim} F^*A_\bullet \otimes \check{H}^*(C_p, \mathbb{F}_p), \tag{1.17}$$

where the grading on the right-hand side comes from the grading on $\check{H}^*(C_p, \mathbb{F}_p)$. Note that the right-hand side of (1.17) has a canonical connection - the Frobenius pullback connection. A key observation explained in §2.2 is that there is a canonical connection ∇ on the filtered DG algebra $T(C_p, {A_\bullet}^{\otimes p})$, which induces the Frobenius pullback connection on Gr^τ.

Denote by $T_{[m,l]}(C_p, {A_\bullet}^{\otimes p})$, $(m \leq l)$, the quotient of $\tau^{dec}_{\leq l} T(C_p, {A_\bullet}^{\otimes p})$ by $\tau^{dec}_{\leq m-1} T(C_p, {A_\bullet}^{\otimes p})$. The DG algebra

$$\mathcal{B}(A_\bullet) := T_{[-1,0]}(C_p, {A_\bullet}^{\otimes p}),$$

which is a square-zero extension of $F^*A_\bullet$

$$F^*A_\bullet[1] \xrightarrow{\mu} \mathcal{B}(A_\bullet) \longrightarrow F^*A_\bullet$$

with a compatible connection ∇, admits another description. Let $\hat{R}$ be a flat lifting of $\tilde{R}$ over $W(k)$, $\hat{i}_*$ the functor from the category of DG algebras over R to the category of DG algebras over $\hat{R}$, which carries a DG algebra over R to the same underlying DG ring with the action of $\hat{R}$ induced by the morphism $\hat{R} \to R$, and let $L\hat{i}^*$ be the left adjoint functor, which carries a DG algebra $C_\bullet$ over $\hat{R}$ to the derived tensor product $C_\bullet \overset{L}{\otimes}_{\hat{R}} R$. For any DG algebra $A_\bullet$ over R the composition $L\hat{i}^*\hat{i}_*A_\bullet$ is an algebra over $L\hat{i}^*\hat{i}_*R \xrightarrow{\sim} R[\mu]$, where $\deg \mu = -1$, $\mu^2 = 0$. One can easily check that the functor $L\hat{i}^*\hat{i}_*$ depends on $\tilde{R}$ only (in particular, every automorphism of $\hat{R}$, which restricts to the identity on $\tilde{R}$ acts trivially on $L\hat{i}^*\hat{i}_*$). Similarly, the morphism of crystalline toposes $\mathrm{Cris}(R/k) \to \mathrm{Cris}(R/W(k))$ induces a functor $\hat{i}^{cris}_*$ from the category of DG algebras in the category of crystals on $\mathrm{Cris}(R/k)$ (*i.e.*, the category of R-modules with integrable connections) to the category of DG algebras in the category of crystals on $\mathrm{Cris}(R/W(k))$ (*i.e.*, the category of p-adically complete $\hat{R}$-modules with integrable connections) and the left adjoint functor $L\hat{i}^*_{cris}$. A key step in our proof is the following result.

Theorem 4. *Let $A_\bullet$ be a term-wise flat DG algebra over R.*

(a) There is a canonical quasi-isomorphism of DG algebras with connections

$$(\mathcal{B}(A_\bullet), \nabla) \xrightarrow{\sim} L\hat{i}^{*cris}\hat{i}_{*cris}F^*A_\bullet.$$

(b) A lifting $(\tilde{R}, \tilde{F})$ of (R, F) over $W_2(k)$ gives rise to a canonical quasi-isomorphism of DG algebras with connections

$$(\mathcal{B}(A_\bullet), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(L\hat{i}^*\hat{i}_*A_\bullet, \mu\tilde{\kappa}).$$

Here $\tilde{\kappa}$ is the Kodaira-Spencer class of $A_\bullet$ regarded as a derivation of $A_\bullet$ with values in $A_\bullet \otimes \Omega^1_R$ of degree 1 (as defined by formula 1.3), $\mu\tilde{\kappa}$ the induced degree 0 derivation of $L\hat{i}^*\hat{i}_*A_\bullet$ with values in $(L\hat{i}^*\hat{i}_*A_\bullet) \otimes \Omega^1_R$, and $\mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}$ is the inverse Cartier transform.

(c) A lifting of $A_\bullet$ over $\tilde{R}$ gives rise to a canonical quasi-isomorphism of DG algebras with connections

$$(\mathcal{B}(A_\bullet), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(A_\bullet[\mu], \mu\tilde{\kappa}).$$

Remarks 1.3. (a) If R is a perfect field the above result is due to Kaledin ([K2], Prop. 6.13).

(b) The first part of the Theorem together with the projection formula gives a canonical isomorphism of DG algebras with connections

$$\hat{i}^{cris}_*\mathcal{B}(A_\bullet) \xrightarrow{\sim} \hat{i}^{cris}_*F^*A_\bullet \oplus \hat{i}^{cris}_*F^*A_\bullet[1],$$

where the right-hand side of the equation is the trivial square-zero extension with the Frobenius pullback connection. However, in general $\mathcal{B}(A_\bullet)$ does not split. For example, from the second part of the Theorem it follows that the p-curvature of ∇ on $\mathcal{B}(A_\bullet)$ equals $-\mu F^*(\mu\tilde{\kappa})$. In particular, it is not zero as long as $\tilde{\kappa}$ is not 0.

Next, we relate the cyclic homology of $\mathcal{B}(A_\bullet)$ together with the connection induced by the one on $\mathcal{B}(A_\bullet)$ with the periodic cyclic homology of $A_\bullet$ with the Gauss-Manin connection. The two-step fitration $F^*A_\bullet[1] \subset \mathcal{B}(A_\bullet)$ gives rise to a filtration $V_mCC(\mathcal{B}(A_\bullet)) \subset CC(\mathcal{B}(A_\bullet))$, $(m = 0, -1, -2, \cdots)$, on the cyclic complex of $\mathcal{B}(A_\bullet)$.

Theorem 5. *Let $A_\bullet$ be a term-wise flat DG algebra over R. We have a canonical quasi-isomorphism of filtered complexes with connections*

$$V_{[-p+2,-1]}CC(\mathcal{B}(A_\bullet))[1] \xrightarrow{\sim} V_{[-p+2,-1]}\overline{CP}(A_\bullet).$$

Moreover, the multiplication by u^{-1} on the right-hand side corresponds under the above quasi-isomorphism to the multiplication by the class $B\mu$ in the second negative cyclic homology group of the algebra $k[\mu]$.

Let us derive Theorem 1 from Theorems 5 and 4. Since the Cartier transform is a monoidal functor, we have by part 3 of Theorem 5

$$(V_{[-p+2,-1]}CC(\mathcal{B}(A_\bullet)), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(V_{[-p+2,-1]}CC(A_\bullet[\mu]), \mu\tilde{\kappa}).$$

We compute the right-hand side using the Künneth formula: with obvious notation we have a quasi-isomorphism of mixed complexes

$$V_{[-p+2,-1]}C(A_\bullet[\mu]) \xrightarrow{\sim} C(A_\bullet) \otimes V_{[-p+2,-1]}C(k[\mu]).$$

The Hochschild complex of $k[\mu]$ regarded as a mixed complex is quasi-isomorphic to the divided power algebra:

$$C(k[\mu], k[\mu]) \xrightarrow{\sim} k\langle \mu, B\mu \rangle$$

with zero differential and Connes' operator acting by the formulas: $B((B\mu)^{[m]}) = 0$, $B(\mu(B\mu)^{[m]}) = (m+1)(B\mu)^{[m+1]}$.

It follows, that

$$V_{[-p+2,-1]}CC(A_\bullet[\mu]) \xrightarrow{\sim} \bigoplus_{0\leq m\leq p-3} C(A_\bullet)\otimes \mu(B\mu)^{[m]}.$$

Setting $B\mu = u^{-1}$ and using the Cartan formula ([Ge]; see also §3 for a review), we find

$$(V_{[-p+2,-1]}CC(A_\bullet[\mu], \mu\tilde{\kappa})[-1] \xrightarrow{\sim} (C(A_\bullet)\otimes k[u^{-1}]/u^{2-p}, u^{-1}\iota_{\tilde{\kappa}}).$$

Summarizing, we get

$$(V_{[-p+2,-1]}\overline{CP}(A_\bullet), \nabla) \xrightarrow{\sim} \mathcal{C}^{-1}_{(\tilde{R},\tilde{F})}(C(A_\bullet)\otimes k[u^{-1}]/u^{2-p}, u^{-1}\iota_{\tilde{\kappa}})[2]$$

This implies the desired result. The derivation of Theorem 2 is similar.

2. The Tate cohomology complex of $A_\cdot^{\otimes p}$

In this section we construct a connection on the Tate complex $T(C_p, A_\cdot^{\otimes p})$ and prove Theorem 4.

2.1. The Tate cohomology complex. Let G be a finite group. A complete resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is an acyclic complex of free $\mathbb{Z}[G]$-modules

$$\longrightarrow \cdots P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$$

together with an isomorphism of $\mathbb{Z}[G]$-modules

$$\epsilon : \mathbb{Z} \xrightarrow{\sim} \ker(d : P^0 \longrightarrow P^1).$$

One can show for any two complete resolutions $(P^\bullet, \epsilon)$, $(P'^\bullet, \epsilon')$ there exists a morphism $f_\bullet : P^\bullet \to P'^\bullet$ of complexes of $\mathbb{Z}[G]$-modules such that $f_0\circ\epsilon = \epsilon'$ and such $f_\bullet$ is unique up to homotopy (in fact $Hom(P^\bullet, P'^\bullet)$ in the homotopy category of complexes of $\mathbb{Z}[G]$-modules is canonically isomorphic to $\mathbb{Z}$). Fix a complete resolution $(P^\bullet, \epsilon)$. For a complex $M_\bullet$ of $\mathbb{Z}[G]$-modules we define its Tate cohomology complex $T(G, M_\bullet)$ to be

$$T(G, M_\bullet) := (M_\bullet \otimes_{\mathbb{Z}} P^\bullet)^G.$$

This defines a DG functor $T(G, \dot{})$ from the DG category $C(Mod(\mathbb{Z}[G]))$ of complexes of $\mathbb{Z}[G]$-modules to the DG category of complexes of abelian groups. By construction, $T(G, \dot{})$ commutes with arbitrary direct sums. Also, it easy to check that $T(G, \dot{})$ carries bounded complexes of free $\mathbb{Z}[G]$-modules and bounded acyclic complexes to acyclic complexes.[8]

2.2. Connection on the Tate complex. Denote $spec\, R$ by X. The following construction is essentially contained in [K1], and it does not depend on the fact that X is affine. By $X^{[2]}$ we denote the first infinitesimal thickening of the diagonal $\Delta \subset X\times X$ and $p_1, p_2 : X^{[2]} \to X$ are projections.

We will construct a connection on the DG algebra $T(C_p, A_\bullet{}^{\otimes p})$, that is a quasi-isomorphism of DG algebras

$$\nabla : p_1^* T(C_p, A_\bullet{}^{\otimes p}) \cong p_2^* T(C_p, A_\bullet{}^{\otimes p})$$

[8] Both statements may fail for unbounded complexes. For example, ϵ induces a quasi-isomorphism $T(G,\mathbb{Z}) \xrightarrow{\sim} T(G, P^\bullet)$. Thus, $T(G, \dot{})$ does not respect arbitrary quasi-isomorphisms.

which is, when restricted to Δ is equal to identity in the category of DG algebras localized in quasi-isomorphisms. There is an exact sequence of sheaves on $X \times X$

$$0 \to \Omega^1_\Delta \to \mathcal{O}_{X\times X}/I^2 \to \mathcal{O}_\Delta \to 0$$

which induces an exact sequence of complexes

$$0 \to A_\bullet \otimes \Omega^1_X \xrightarrow{\beta} p_{1*}p_2^*A_\bullet \xrightarrow{\alpha} A_\bullet \to 0 \tag{2.1}$$

making $p_{1*}p_2^*A_\bullet$ into a square-zero extension of the DG algebra $A_\bullet$. Giving connection on $A_\bullet$ is equivalent to providing a splitting of this extension in the category of DG algebras localized in quasi-isomorphisms. We are going to construct such a splitting. Consider the following two-term filtration on $p_{1*}p_2^*A_\bullet : G^2 = 0, G^1 = A_\bullet \otimes \Omega^1_X, G^0 = p_{1*}p_2^*A_\bullet$. It induces a filtration on $(p_{1*}p_2^*A_\bullet)^{\otimes p}$, we will denote it also by $G^\bullet$.

Lemma 2.1. *For a term-wise flat DG algebra $A_\bullet$ the morphism α induces the following isomorphism of DG algebras with the action of C_p*

$$G^0(p_{1*}p_2^*A_\bullet)^{\otimes p}/G^1(p_{1*}p_2^*A_\bullet)^{\otimes p} \xrightarrow{\sim} A_\bullet{}^{\otimes p}$$

and β induces the following isomorphism of complexes with the action of C_p

$$A_\bullet{}^{\otimes p} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathbb{Z}} \mathbb{Z}[C_p] \xrightarrow{\beta} G^1(p_{1*}p_2^*A_\bullet)^{\otimes p}/G^2(p_{1*}p_2^*A_\bullet)^{\otimes p}$$

Proof. It is enough to prove the statements locally on X so we may choose a connection on the graded module $\bigoplus A_i$ and get a splitting(non-compatible with the differentials) $p_{1*}p_2^*A_\bullet = A_\bullet \oplus A_\bullet \otimes_{\mathcal{O}_X} \Omega^1_X$, hence

$$\begin{aligned}(p_{1*}p_2^*A_\bullet)^{\otimes p} = A_\bullet{}^{\otimes p} \oplus \bigoplus_{i=0}^{p-1} (A_\bullet \otimes \ldots \otimes (A_\bullet \overset{i}{\otimes} \Omega^1_X) \otimes \cdots \otimes A_\bullet) \oplus \\ \bigoplus_{i\neq j}^{p-1} (A_\bullet \otimes \ldots \otimes (A_\bullet \overset{i}{\otimes} \Omega^1_X) \otimes \cdots \otimes (A_\bullet \overset{j}{\otimes} \Omega^1_X) \otimes \cdots \otimes A_\bullet) \oplus \ldots\end{aligned} \tag{2.2}$$

α projects $(p_{1*}p_2^*A_\bullet)^{\otimes p}$ on the first summand of this decomposition while β embeds $A_\bullet{}^{\otimes p} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathbb{Z}} \mathbb{Z}[C_p]$ onto the second (the summand $(A_\bullet \otimes \ldots \otimes (A_\bullet \overset{i}{\otimes} \Omega^1_X) \otimes \cdots \otimes A_\bullet)$ corresponds to $A_\bullet{}^{\otimes p} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathbb{Z}} \mathbb{Z}[C_p] \otimes \sigma^i$) so they indeed induce isomorphisms on the graded quotients. $\square$

By adjunction, we have a map $m : (p_{1*}p_2^*A_\bullet)^{\otimes p} \to p_{1*}p_2^*(A_\bullet{}^{\otimes p})$. Since $X \to X^{[2]}$ is a square-zero extension, m factors through G^2, so we get the following diagram of complexes of C_p-modules in which the top row is a distinguished triangle

$$\begin{array}{ccccc} A_\bullet{}^{\otimes p} \otimes \Omega^1_X \otimes \mathbb{Z}[C_p] & \xrightarrow{i} & G^0/G^2 & \xrightarrow{\pi} & A_\bullet{}^{\otimes p} \longrightarrow \\ & & \downarrow{\scriptstyle m} & & \\ & & p_{1*}p_2^*(A_\bullet{}^{\otimes p}) & & \end{array} \tag{2.3}$$

Lemma 2.2. *$T(C_p, \mathbb{Z}[C_p])$ is contractible.*

Proof. We will prove it using the resolution

$$\ldots \xrightarrow{1-\sigma} \overset{2k-1}{\mathbb{Z}[C_p]} \xrightarrow{N} \overset{2k}{\mathbb{Z}[C_p]} \xrightarrow{1-\sigma} \overset{2k+1}{\mathbb{Z}[C_p]} \xrightarrow{N} \ldots$$

where $N = 1+\sigma+\cdots+\sigma^{p-1}$. Put $h_0(\sigma^i) = \delta_{i,p-1}$ and $h_1(\sigma^i) = -(1+\sigma+\cdots+\sigma^{i-1})$ for $0 \leq i \leq p-1$. Then $Nh_0+h_1(1-\sigma) = Id, h_0N+(1-\sigma)h_1 = Id$ so h given by h_0 at even degrees and by h_1 at odd degrees is a contracting homotopy for $T(C_p, \mathbb{Z}[C_p])$. □

It follows that Tate cohomology complex of ${A_\bullet}^{\otimes p} \otimes \Omega^1_X \otimes \mathbb{Z}[C_p]$ is contractible so π turns into a homotopy equivalence after taking Tate complexes. Finally, put $s = m\pi^{-1}$. It is a section of α and, by adjunction, induces a connection $\nabla : p_1^* T(C_p, {A_\bullet}^{\otimes p}) \cong p_2^* T(C_p, {A_\bullet}^{\otimes p})$.

2.3. **The connection on the truncated Tate complex.** Let $A_\bullet$ be term-wise flat DG algebra over R. Equip ${A_\bullet}^{\otimes p}$ with the stupid filtration rescaled by p. It induces a filtration on the Tate complex $T(C_p, {A_\bullet}^{\otimes p})$. In the following we denote by τ the filtered truncation functors(cf. Definition 6.4 in [K2]). Unlike Kaledin, throughout the paper we use cohomological grading for the filtrations.

As in the introduction, by $\hat{i}^{cris}$ we denote the inclusion of crystalline toposes $\mathrm{Cris}(R/k) \to \mathrm{Cris}(R/W(k))$. In this section we prove that

Theorem 6. *There is a quasi-isomorphism of DG algebras with connection*

$$\mathcal{B}(A_\bullet) := \tau_{[-1,0]} T(C_p, {A_\bullet}^{\otimes p}) \cong L\hat{i}^{cris*} \hat{i}_{cris*} Fr^* A_\bullet =: T^{cris}(A_\bullet) \tag{2.4}$$

Now let $\tilde{R}$ be the lifting of R over $W_2(k)$ and choose a lifting $\tilde{F}$ of the Frobenius morphism on $\tilde{R}$. Choose also a lifting $\mathcal{O}$ of $\tilde{R}$ over $W(k)$. Consider the functors

$$\begin{array}{ll} i_* : D(Mod-R) \to D(Mod-\mathcal{O}) & Li^* : D(Mod-\mathcal{O}) \to D(Mod-R) \\ \tilde{i}_* : D(Mod-R) \to D(Mod-\tilde{R}) & L\tilde{i}^* : D(Mod-\tilde{R}) \to D(Mod-R) \end{array} \tag{2.5}$$

Note that as a complex of R-modules $T^{cris}(A_\bullet)$ is quasi-isomorphic to $Li^* i_* Fr^* A_\bullet$.

Theorem 7. *A lifting of $A_\bullet$ to a DG algebra $\tilde{A}_\bullet/\tilde{R}$ gives a quasi-isomorphism of DG algebras with connection*

$$T^{cris}(A_\bullet) \cong (Fr^* A_\bullet \oplus Fr^* A_\bullet[1], \nabla_{can} + \mu C^{-1}_{\tilde{F}}(\tilde{\kappa})) \tag{2.6}$$

2.4. **Proof of Theorem 7.** We start from the proof of Theorem 7, since we will partially use it in the proof of Theorem 6. Fix a connection ∇' on the algebra $\bigoplus A_i$. It might not be compatible with the differential – the Kodaira-Spencer class measures this incompatibility: $\tilde{\kappa} = [\nabla', d]$.

Lemma 2.3. *For a module $B/\tilde{R}$ a connection ∇_0 on $\tilde{i}^* B$ gives rise to a connection on $\widetilde{Fr}^*(B)$ which reduces to the canonical connection on $Fr^* B$ under $\tilde{i}^*$.*

Proof. Lift ∇_0 to a map of $W_2(k)$-modules $\nabla_0' : B \to B \otimes \Omega^1_{\tilde{R}/W_2 k}$. Then define a connection $\widetilde{\nabla}$ on B as the pullback of ∇_0' under $\widetilde{Fr}$. Namely, for $f \otimes x \in \tilde{R} \otimes_{\widetilde{Fr}, \tilde{R}} B$ put

$$\widetilde{\nabla}(f \otimes x) = x \otimes df + f \cdot \widetilde{Fr}^*(\nabla_0'(x)) \tag{2.7}$$

This is indeed a connection which does not depend on the choice of ∇_0' because the value of $\widetilde{Fr}^*(\omega)$ depends only on $\tilde{i}^*\omega$ since $\tilde{i}^*\widetilde{Fr}$ is zero on 1-forms. □

Applying this lemma to $B = \bigoplus \tilde{A}_i$ and ∇', we get a connection $\widetilde{\nabla}$. Since $\widetilde{\nabla}$ and $\widetilde{d}$ commute modulo p, we get the following map

$$\frac{[\widetilde{\nabla},\widetilde{d}]}{p} : \widetilde{Fr}^*\tilde{A}_i \to \tilde{i}_*Fr^*A_{i+1}\otimes\Omega^1_{\tilde{R}/W_2(k)} \tag{2.8}$$

We are now ready to prove the theorem. Put $\mathcal{F} = cone(\tilde{i}_*Fr^*A_\bullet \xrightarrow{p} \widetilde{Fr}^*\tilde{A}_\bullet)$. It is a complex of $\tilde{R}$-modules with terms

$$\mathcal{F}^i = \widetilde{Fr}^*\tilde{A}^i \oplus \tilde{i}_*Fr^*A^{i+1}$$

and the differential given by $(x,y)\mapsto (d_{\tilde{A}}x + (-1)^i py, d_{A_\bullet}y)$. Let $r:\mathcal{F}\to\tilde{i}_*Fr^*A_\bullet$ be the morphism which maps $(x,y)\in\mathcal{F}^i$ to the reduction of x modulo p in $\tilde{i}_*Fr^*A^i$. It is a morphism of complexes because $p\in\tilde{R}$ acts by zero on $\tilde{i}_*Fr^*A_\bullet$.

Lemma 2.4. *(i) r is a quasi-isomorphism. (ii) Considering further $\mathcal{F}$ as a complex of $\mathcal{O}$-modules, the canonical map $L\hat{i}^*\mathcal{F}\to\hat{i}^*\mathcal{F}$ is a quasi-isomorphism.*

Proof. (i) is clear as r is term-wise surjective and its kernel is isomorphic to $cone(\tilde{i}_*Fr^*A_\bullet \xrightarrow{id} \tilde{i}_*Fr^*A_\bullet)$ which has zero cohomology.

(ii) Terms of $\mathcal{F}$ are not flat over $\mathcal{O}$ so, a priori, there might be non-zero higher derived functors of i^*. Pick $\bigoplus\mathcal{A}^i$ – a lifting of the graded algebra $\bigoplus\widetilde{Fr}^*\tilde{A}^i$ to a free graded algebra over $\mathcal{O}$. Pick also a lifting δ of the differential $\widetilde{d}$(δ is not a differential anymore – its square need not be zero). It enables us to right down the following resolution of $\hat{i}_*A_\bullet$. Put

$$C^i = \mathcal{A}^i\oplus\mathcal{A}^{i+1}; d_C = \begin{pmatrix} \delta & (-1)^i p \\ (-1)^i\frac{\delta^2}{p} & \delta\end{pmatrix} \tag{2.9}$$

δ^2 is divisible by p because $d^2=0$ on $Fr^*A_\bullet$ and modules $\mathcal{A}^i$ are free over $\mathcal{O}$. Reduction maps $C^i\to\mathcal{F}^i$ give a morphism of complexes $\rho: C^\bullet\to\mathcal{F}$ (reduction maps commute with the differentials because $\frac{\delta^2}{p}$ is actually divisible by p). Actually, ρ is a quasi-isomorphism. Indeed, composing it with r we get a term-wise surjective morphism of complexes with kernel given by $K^i = p\mathcal{A}^i\oplus\mathcal{A}^{i+1}$ and the differential restricted from $C^\bullet$. For any $(x,y)\in K^i$ such that $d_c(x,y)=0$ we have $(x,y) = d_C(0,(-1)^{i-1}\frac{x}{p})$ so $K^\bullet$ is acyclic and $C^\bullet$ is an $\mathcal{O}$-flat resolution of $\mathcal{F}$. We get a commutative diagram

$$\begin{array}{ccc} L\hat{i}^*C^\bullet & \xrightarrow{\sim} & \hat{i}^*C^\bullet \\ \downarrow & & \downarrow \\ L\hat{i}^*\mathcal{F} & \longrightarrow & \hat{i}^*\mathcal{F}\end{array}$$

Left vertical arrow is a quasi-isomorphism because $C^\bullet\to\mathcal{F}$ is a quasi-isomorphism and the right vertical arrow is an isomorphism because both $C^\bullet,\mathcal{F}$ reduce modulo p to the complex $Fr^*A_\bullet\oplus Fr^*A_\bullet[1]$. Thus, the lower arrow is a quasi-isomorphism. □

We will now give $\mathcal{F}$ a structure of a DG algebra with connection. Let DG algebra structure to be that of the trivial square-zero extension of $\widetilde{Fr}^*\tilde{A}_\bullet$ by the bimodule $\tilde{i}_*Fr^*A_\bullet[1]$. To see that this algebra structure is compatible with the differential it is enough to check that $D:(x,y)\mapsto((-1)^ipy,0)$ is a derivation because the diagonal part $(x,y)\mapsto(d_{\tilde{A}}x, d_{A_\bullet}y)$ is a derivation by default. For $(a_1,b_1)\in\mathcal{F}^i,(a_2,b_2)\in\mathcal{F}^j$ we have $D((a_1,b_1)(a_2,b_2)) = ((-1)^{i+j}p((-1)^jb_1a_2+(-1)^ia_1b_2),0) =$

$((-1)^i pb_1, 0)(a_2, b_2) + (a_1, b_1)((-1)^j pb_2, 0) = D((a_1, b_1))(a_2, b_2) + (a_1, b_1)D((a_2, b_2))$, q. e. d.

Next, define a connection as

(2.10)
$$\nabla_{\mathcal{F}} = \begin{pmatrix} \widetilde{\tilde{\nabla}} & 0 \\ (-1)^i \frac{[\tilde{\nabla}, \tilde{d}]}{p} & \tilde{i}_* \nabla^{can} \end{pmatrix} : \widetilde{Fr}^* \tilde{A}_i \oplus \tilde{i}_* Fr^* A_{i+1} \to (\widetilde{Fr}^* \tilde{A}_i \oplus \tilde{i}_* Fr^* A_{i+1}) \otimes_{\tilde{R}} \Omega^1_{\tilde{R}/W_2(k)}$$

The entry below the diagonal is chosen so that this connection commutes with the differential on the DG algebra. To ensure that this connection respects the algebra structure it is, as above, enough to check that $(x, y) \mapsto (0, (-1)^i \frac{[\tilde{\nabla}, \tilde{d}]}{p} x)$ is a derivation which follows from $\frac{[\tilde{\nabla}, \tilde{d}]}{p}$ being a commutator of derivations. Finally, it is clear that our connection is integrable.

Also, quasi-isomorphism r is compatible with connection because $\widetilde{\tilde{\nabla}}$ reduces to ∇^{can} modulo p. In other words, $\hat{i}_{cris*} Fr^* A_\bullet$ is quasi-isomorphic to $(\mathcal{F}, \nabla_{\mathcal{F}})$. Thus, $T^{cris}(A_\bullet) \cong L\hat{i}^{cris*}((\mathcal{F}, \nabla_{\mathcal{F}}))$. By the virtue of Lemma 2.4, $L\hat{i}^{cris*}(\mathcal{F}, \nabla_{\mathcal{F}})$ is quasi-isomorphic to $(i^*\mathcal{F}, i^*\nabla_{\mathcal{F}})$. The latter complex of R-modules with integrable connection is given by

$$\begin{pmatrix} \nabla^{can} & 0 \\ (-1)^i \frac{[\tilde{\nabla}, \tilde{d}]}{p} & \nabla^{can} \end{pmatrix} : Fr^* A^i \oplus Fr^* A^{i+1} \to (Fr^* A^i \oplus Fr^* A^{i+1}) \otimes \Omega^1_{R/k} \tag{2.11}$$

So, Theorem 7 follows after we check that

Lemma 2.5.
$$\frac{[\widetilde{\nabla}, \widetilde{d}]}{p} = C^{-1}(\tilde{\kappa}) \tag{2.12}$$

Proof. By definition $\tilde{\kappa} = [\nabla', d]$. Recall that $\tilde{\nabla}_i$ on $\widetilde{Fr}^* \tilde{A}_i$ is given by the formula $\tilde{\nabla}_i(f \otimes x) = df \otimes x + f \otimes \tilde{Fr}^*(\nabla'_i(x))$. Hence,

(2.13)
$$\frac{[\widetilde{\nabla}, \widetilde{d}]}{p}(f \otimes x) = \frac{df \otimes \widetilde{d}(\widetilde{x}) + f \otimes \tilde{Fr}^*(\nabla'_i(dx)) - df \otimes \widetilde{d}(\widetilde{x}) - f \otimes d\widetilde{Fr}^*(\nabla'_i(x))}{p} =$$
$$= f \otimes \frac{\widetilde{Fr}^*([\nabla'_i, d])}{p}$$

and $\frac{\widetilde{Fr}^*}{p}$ is exactly the Cartier isomorphism by [DI].

□

Remark 2.6. Of course, we could have computed $T^{cris}(A_\bullet)$ in one step using the resolution (2.9) but we deal with non-liftability of $A_\bullet$ over $W(k)$ and non-existence of a connection on $A_\bullet$ separately for the sake of exposition.

2.5. **Proof of Theorem 6.** Choose a lifting $\hat{Fr} : \mathcal{O} \to \mathcal{O}$ of the Frobenius endomorphism.

Lemma 2.7. *Let M be a flat $\mathcal{O}$-module. For any $n \in \mathbb{Z}$ we have $\hat{H}^{2n-1}(C_p, M^{\otimes p}) = 0$ and $\hat{H}^{2n}(C_p, M^{\otimes p})$ is canonically isomorphic to $i_* Fr^* i^* M$, where C_p, as usual, acts on $M^{\otimes p}$ by cyclic permutations.*

Proof. The proof is similar to that of the Lemma 6.9 in [K3]. By peridoicity, it is enough to consider the case $n = 0$. So, we should compute cohomology of the following canonical truncation of the Tate complex

$$(M^{\otimes p})_{C_p} \xrightarrow{N} (M^{\otimes p})^{C_p}$$

Lemma 6.9 from [K3] gives for any flat R-module N a map $(N^{\otimes p})^{C_p} \to Fr^* N$. Composing this map for $N = i^* M$ with the inclusion $i^*(M^{\otimes p})^{C_p} \to (i^* M^{\otimes p})^{C_p}$ we get a map $\psi : i^*(M^{\otimes p})^{C_p} \to Fr^* i^* M$ which, by adjointness, gives a map of complexes

$$[(M^{\otimes p})_{C_p} \to (M^{\otimes p})^{C_p}] \to i_* Fr^* i^* M$$

Since any flat module is a filtered colimit of free modules, it is enough to prove that this map is a quasi-isomorphism for finitely-generated free modules. Fixing a basis S in a free module M, we get a decomposition of C_p-modules

$$M^{\otimes p} = M_1 \oplus M_2$$

where M_1 is generated by $s^{\otimes p}$ for $s \in S$ and M_2 is generated by all other tensors. So, M_1 is a trivial C_p-module, while M_2 is free and ψ factors through projection on M_1. So, to prove the lemma it is left to check that $\hat{H}^{-1}(C_p, \mathcal{O}) = 0, \hat{H}^0(C_p, \mathcal{O}) = i_* R$. The standard Tate complex for trivial module $\mathcal{O}$ takes the following form

$$\ldots \xrightarrow{0} \mathcal{O} \xrightarrow{p} \mathcal{O} \xrightarrow{0} \ldots \tag{2.14}$$

So, $\hat{H}^0(C_p, \mathcal{O}) = \mathcal{O}/p\mathcal{O} = i_* R, \hat{H}^{-1}(C_p, \mathcal{O}) = 0$ because multiplication by p is injective on $\mathcal{O}$. ∎

In what follows, for any DG algebra $B_\bullet$ we write $T(B_\bullet)$ for the algebra $T(C_p, {B_\bullet}^{\otimes p})$.

Proposition 2.8. *Let $\hat{A}_\bullet$ be a lifting of $A_\bullet$ to $\hat{R}$, namely a DG algebra over $\mathcal{O}$ such that $L\hat{i}^* \hat{A}_\bullet$ is quasi-isomorphic to $A_\bullet$. The choice of a lifting gives a quasi-isomorphism of DG algebras*

$$\tau_{[-1,0]} T(A_\bullet) \cong L\hat{i}^* \hat{i}_* Fr^* A_\bullet \tag{2.15}$$

Proof. By definition, $\tau_{[-1,0]} T(A_\bullet) \cong \tau_{[-1,0]} L\hat{i}^* T(\hat{A}_\bullet) = L\hat{i}^* \tau_{[-1,0]} T(\hat{A}_\bullet)$. Replacing in the proof of Proposition 6.10 from [K3] their Lemma 6.9 by our 2.7 we get that $\tau_{[-1,-1]} T(\hat{A}_\bullet) = 0, \tau_{[0,0]} T(\hat{A}_\bullet) = \hat{i}_* Fr^* \hat{i}^* \hat{A}_\bullet = \hat{i}_* Fr^* A_\bullet$. The vanishing of $\tau_{[-1,-1]}$ implies that $\tau_{[-1,0]} T(\hat{A}_\bullet) \to \tau_{[0,0]} T(\hat{A}_\bullet) \cong \hat{i}_* Fr^* A_\bullet$ is an isomoprhism. Applying $L\hat{i}^*$ we get the statement.

∎

For a liftable $A_\bullet$ the above proposition can be reformulated as $\tau_{[-1,0]} T(A_\bullet) \cong Fr^* A_\bullet \oplus Fr^* A_\bullet[1]$ because $\hat{Fr}^* \hat{A}_\bullet$ is a lifting of $Fr^* A_\bullet$ which splits $L\hat{i}^* \hat{i}_* Fr^* A_\bullet$ by Theorem 7.

Next, if $A_\bullet$ is arbitrary, apply the proposition to $L\hat{i}^* \hat{i}_* A_\bullet$ putting $\hat{A}_\bullet$ to be a semi-free resolution of $\hat{i}_* A_\bullet$. We get

$$\tau_{[-1,0]} T(L\hat{i}^* \hat{i}_* A_\bullet) \cong L\hat{i}^* \hat{i}_* Fr^* A_\bullet \oplus L\hat{i}^* \hat{i}_* Fr^* A_\bullet[1] \tag{2.16}$$

Consider the morphism $L\hat{i}^*\hat{i}_*A_\bullet \to \hat{i}^*\hat{i}_*A_\bullet = A_\bullet$. It induces $T(L\hat{i}^*\hat{i}_*A_\bullet) \to T(A_\bullet)$. So we get the following diagram

$$L\hat{i}^*\hat{i}_*Fr^*A_\bullet \longrightarrow L\hat{i}^*\hat{i}_*Fr^*A_\bullet \oplus L\hat{i}^*\hat{i}_*Fr^*A_\bullet[1] \cong \tau_{[-1,0]}T(L\hat{i}^*\hat{i}_*A_\bullet) \longrightarrow \tau_{[-1,0]}T(A_\bullet)$$

Denote the composition by φ. First,

Lemma 2.9. $\varphi : L\hat{i}^*\hat{i}_*Fr^*A_\bullet \to \tau_{[-1,0]}T(A_\bullet)$ *is a quasi-isomorphism of DG algebras.*

Proof. Clearly, φ is a morphism of DG algebras, so it is enough to check that it is a quasi-isomorphism of complexes of R-modules. Since functors $\hat{i}^*, \hat{i}_*, Fr^*, A_\bullet \mapsto \mathcal{F}$ commute with filtered colimits, we may assume that $A_\bullet$ is a perfect complex (any complex is a direct limit of perfect complexes). Next, it is enough to check that it is a quasi-isomorphism over all the localizations $R_\mathfrak{m}$ at maximal ideals $\mathfrak{m} \subset R$. Finally, by Nakayama lemma, it is enough to verify the statement over residue fields $R/\mathfrak{m}$.

Note that for any $A_\bullet$, the following square is commutative

$$\begin{array}{ccc} L\hat{i}^*\hat{i}_*Fr^*A_\bullet & \xrightarrow{\varphi} & \tau_{[-1,0]}T(A_\bullet) \\ \downarrow & & \downarrow \\ Fr^*A_\bullet & = & Fr^*A_\bullet \end{array}$$

Put $k' = R/\mathfrak{m}$. $L\hat{i}^*\hat{i}_*Fr^*k'$ and $\tau_{[-1,0]}T(k')$ are both non-canonically split, i.e. quasi-isomorphic to $k' \oplus k'[1]$ and φ induces an isomorphism on zeroth cohomology. We should prove that it is also an isomorphism on (-1)-st cohomology. Assume it is not, i.e. is zero on H^{-1}. Then φ factors through $L\hat{i}^*\hat{i}_*Fr^*k' \to Fr^*k'$ so induces a splitting of $\tau_{[-1,0]}T(k')$. Since, φ is compatible with direct sums, $\tau_{[-1,0]}(V)$ is also canonically split for any k'-vector space V. In other words, the following extension of polynomial functors $Vect_{k'} \to Vect_{k'}$ is split

$$0 \to Fr^*V \to (V^{\otimes p})_{C_p} \to (V^{\otimes p})^{C_p} \to Fr^*V \to 0$$

This extension is equivalent to a similar one with C_p replaced by the symmetric group S_p

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Fr^*V & \longrightarrow & (V^{\otimes p})_{C_p} & \xrightarrow{N_{C_p}} & (V^{\otimes p})^{C_p} & \longrightarrow & Fr^*V & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle av_p} & & \| & & \\ 0 & \longrightarrow & Fr^*V & \longrightarrow & (V^{\otimes p})_{S_p} & \xrightarrow{N_{S_p}} & (V^{\otimes p})^{S_p} & \longrightarrow & Fr^*V & \longrightarrow & 0 \end{array}$$

Here π_p is the projection and av_p is the averaging over left cosets of $C_p \subset S_p$ that is $av_p(x) = \frac{1}{(p-1)!}\sum\limits_{gC_p \in S_p/C_p} g(x)$(note that this does not depend on the choice of representatives of cosets). From Corollary 4.7($r = j = 1$) and Lemma 4.12 from [FS] follows that the latter extension is non-split. Hence, φ must induce an isomorphism on (-1)-st cohomology so it is a quisi-isomorphism for any $A_\bullet$. $\square$

We have constructed a map $\varphi : T^{cris}(A) \to \mathcal{B}(A_\bullet)$ of complexes of R-modules. To finish the proof of the theorem we need to prove that

Lemma 2.10. φ *is compatible with connection. Namely, the following square is commutative in the derived category of R-modules*

$$
\begin{tikzcd}
T^{cris}(A) \arrow[r] \arrow[d, "\varphi"] & T^{cris}(A) \otimes_R \Omega^1_{R/k} \arrow[d, "\varphi\otimes\mathrm{id}"] \\
\mathcal{B}(A_\bullet) \arrow[r] & \mathcal{B}(A_\bullet) \otimes_R \Omega^1_{R/k}
\end{tikzcd}
\tag{2.17}
$$

Proof. First, assume that lemma is proven for liftable DG algebras, in particular for $L\hat{i}^*\hat{i}_*A_\bullet$. Theorem 7 implies that embedding $L\hat{i}^*\hat{i}_*Fr^*A_\bullet \to L\hat{i}^*\hat{i}_*Fr^*L\hat{i}^*\hat{i}_*A_\bullet$ is compatible with connection because the Kodaira-Spencer class of $Fr^*A_\bullet$ vanishes. The morphism $\tau_{[-1,0]}T(L\hat{i}^*\hat{i}_*A_\bullet) \to \tau_{[-1,0]}T(A_\bullet)$ is also compatible with the connection because, by definition, connection on the Tate complex is functorial in the DG algebra. So, φ is a composition of morphisms compatible with connection.

So, we may assume that $A_\bullet$ is liftable. We claim even more – that $\tau_{[-1,0]}T(\hat{A}_\bullet) \to \hat{i}_*Fr^*A_\bullet$ is compatible with connection. The map $\tau_{[-1,0]}T(\hat{A}_\bullet) \to \tau_{[0,0]}T(\hat{A}_\bullet)$ is obviously compatible, so we need to check that the isomorphism $\tau_{[0,0]}T(\hat{A}_\bullet) \cong \hat{i}_*Fr^*A_\bullet$ is compatible. Applying $\tau_{[0,0]}$ to the diagram (2.3) used in the definition of connection, we get

$$
\begin{tikzcd}
\tau_{[0,0]}T(C_p, F^0/F^2(\hat{i}_*p_{1*}p_2^*A_\bullet)^{\otimes p}) \arrow[r, "\pi"] \arrow[d, "m"] & \tau_{[0,0]}T(C_p, (i_*A_\bullet)^{\otimes p}) \\
\tau_{[0,0]}T(C_p, p_{1*}p_2^*(\hat{i}_*A_\bullet)^{\otimes p}) &
\end{tikzcd}
$$

By the proof of 2.8, $\tau_{[0,0]}T(C_p, (\hat{i}_*A_\bullet)^{\otimes p}) = \hat{i}_*Fr^*A_\bullet$ and, similarly, π induces and isomorphism, because the kernel of $\pi : F^0/F^2((\hat{i}_*p_{1*}p_2^*A_\bullet)^{\otimes p}) \to (\hat{i}_*A_\bullet)^{\otimes p}$ is a complex of free C_p-modules, whose Tate complex is contractible. Finally, since $p_{1*}p_2^*$ commutes with T and $\tau_{[0,0]}$, we get

$$
\begin{tikzcd}
i_*Fr^*A_\bullet \arrow[r, "\mathrm{Id}"] \arrow[d, "m"] & \hat{i}_*Fr^*A_\bullet \\
p_{1*}p_2^*\hat{i}_*Fr^*A_\bullet &
\end{tikzcd}
$$

So, indeed, $\tau_{[0,0]}T(\hat{A})$ is isomorphic to the i_* of the canonical connection on $Fr^*A_\bullet$. □

3. The Gauss-Manin connection on the (co-)periodic cyclic homology

In this section we review Getzler's and Kaledin's constructions of the Gauss-Manin connection, check that the two constructions agree, show that the Gauss-Manin connection preserves the conjugate filtration, and prove Theorem 5.

3.1. **Getzler's construction.** Let R be a smooth commutative algebra over a field k, and let $A_\bullet$ be a semi-free differential graded algebra over R ([Dr], §13.4). Denote by $(CH_\bullet(A_\bullet, A_\bullet), b)$ the relative Hochschild chain complex of $A_\bullet$ over R [9] and by

[9] Here "relative over R" means that all the tensor products in the standard complex are taken over R.

$CP_\bullet(A_\bullet) = (CH_\bullet(A_\bullet, A_\bullet)((u)), b + uB)$ the periodic cyclic complex. Getzler defined in [Ge] a connection on $CP_\bullet(A_\bullet)$

$$\nabla : CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet) \otimes_R \Omega^1_R.$$

His construction can be explained as follows: choose a connection $\nabla' : \bigoplus A_i \to \bigoplus A_i \otimes \Omega^1_R$ on the graded algebra $\bigoplus A_i$ satisfying the Leibnitz rule with respect to the multiplication on $\bigoplus A_i$. Then the commutator

$$\tilde{\kappa} = [\nabla', d] \in \prod Hom_R(A_i, A_{i+1}) \otimes \Omega^1_R$$

with the differential d on $A_\bullet$ commutes with d and it is a R-linear derivation of $A_\bullet$ (with values in $A_\bullet \otimes \Omega^1_R$) of degree 1. [10] As a derivation, $\tilde{\kappa}$ acts on $CH_\bullet(A_\bullet, A_\bullet)$ by the Lie derivative

$$\mathcal{L}_{\tilde{\kappa}} : CH_\bullet(A_\bullet, A_\bullet) \to CH_\bullet(A_\bullet, A_\bullet) \otimes \Omega^1_R[1], \quad [\mathcal{L}_{\tilde{\kappa}}, B] = 0$$

and the "interior product" operator

$$e_{\tilde{\kappa}} : CH_\bullet(A_\bullet, A_\bullet) \to CH_\bullet(A_\bullet, A_\bullet) \otimes \Omega^1_R[2].$$

The operators $\mathcal{L}_{\tilde{\kappa}}, e_{\tilde{\kappa}}, B$ satisfy the Cartan formula up to homotopy: there is a canonical operator

$$E_{\tilde{\kappa}} : CH_\bullet(A_\bullet, A_\bullet) \to CH_\bullet(A_\bullet, A_\bullet) \otimes \Omega^1_R[2], \quad [E_{\tilde{\kappa}}, B] = 0$$

such that $[e_{\tilde{\kappa}}, B] = \mathcal{L}_{\tilde{\kappa}} - [E_{\tilde{\kappa}}, b]$ ([L], §4.1.8). One defines

$$\nabla := \nabla' - u^{-1}\iota_{\tilde{\kappa}}, \tag{3.1}$$

where the first summand is the connection on $\bigoplus CP_i(A_\bullet)$ induced the connection ∇' on $\bigoplus A_i$ and $\iota_{\tilde{\kappa}} : \bigoplus CP_i(A_\bullet) \to \bigoplus CP_i(A_\bullet) \otimes \Omega^1_R$ is an $R((u))$ linear map given by the formula $\iota_{\tilde{\kappa}} = e_{\tilde{\kappa}} + uE_{\tilde{\kappa}}$. By construction, ∇ commutes with $b + uB$. Thus, it induces a connection on $CP_\bullet(A_\bullet)$. Getzler showed that up to homotopy ∇ does not depend on the choice of ∇'.[11] He also proved that the induced connection on $HP_\bullet(A_\bullet)$ is flat. However, we do not know how to make ∇ on $CP_\bullet(A_\bullet)$ flat up to coherent homotopies[12].

By construction, the connection ∇ satisfies the Griffiths transversality property with respect to the Hodge filtration $F^iCP_\bullet(A_\bullet) := (u^iCH_\bullet(A_\bullet, A_\bullet)[[u]], b + uB)$:

$$\nabla : F^iCP_\bullet(A_\bullet) \to F^{i-1}CP_\bullet(A_\bullet) \otimes_R \Omega^1_R.$$

---

[10] Denote by $Der^\bullet_R(A_\bullet)$ the DG Lie algebra of R-linear derivations of $A_\bullet$: $Der^i_R(A_\bullet)$ is the R-module of R-linear derivations of the graded algebra $\bigoplus A_i$; the differential on $Der^\bullet_R(A_\bullet)$ is given by the commutator with d. The cohomology class $\kappa \in H^1(Der^\bullet_R(A_\bullet)) \otimes \Omega^1_R$ of $\tilde{\kappa}$ does not depend on the choice of ∇'. (Indeed, any two connections differ by an element of $Der^0_R(A_\bullet)$.) Recall that the Hochschild cochain complex of $A_\bullet$ is quasi-isomorphic to the cone of the map $A_\bullet \to Der^\bullet_R(A_\bullet)$ which takes an element of A_i to the corresponding inner derivation. We refer to the image $\overline{\kappa}$ of κ under the induced morphism $H^1(Der^\bullet_R(A_\bullet)) \to HH^2(A_\bullet, A_\bullet)$ as the Kodaira-Spencer class of $A_\bullet$.

[11] One can rephrase the above construction to make this fact obvious: let $Der^\bullet_k(R \to A_\bullet)$ be the DG Lie algebra of k-linear derivations which take the subalgebra $R \subset A_0$ to itself. Then $Der^\bullet_R(A_\bullet)$ is a Lie ideal in $Der^\bullet_k(R \to A_\bullet)$. Denote by $\widetilde{Der_k(R)}$ the cone of the morphism $Der^\bullet_R(A_\bullet) \to Der^\bullet_k(R \to A_\bullet)$. The restriction morphism $\widetilde{Der_k(R)} \to Der_k(R)$ a homotopy equivalence of DG Lie algebras: a choice of ∇' as above yields a homotopy inverse map. Next, we have a canonical morphism of complexes $\widetilde{Der_k(R)} \otimes_R CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet)$ given by the formulas $\theta \otimes c \mapsto u^{-1}\iota_\theta(c)$, for $\theta \in Der^\bullet_R(A_\bullet)$, and $\zeta \otimes c \mapsto \mathcal{L}_\zeta(c)$, for $\zeta \in Der^\bullet_R(R \to A_\bullet)$. This yields a morphism $Der_k(R) \otimes_R CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet)$ well defined up to homotopy.

[12] The problem is that, in general, the canonical morphism $\widetilde{Der_k(R)} \otimes_R CP_\bullet(A_\bullet) \to CP_\bullet(A_\bullet)$ is not a Lie algebra action.

Thus, ∇ induces a degree one R-linear morphism of graded complexes

$$Gr^F\nabla : Gr^F CP_\bullet \to Gr^F CP_\bullet \otimes_R \Omega^1_R.$$

Abusing terminology, we refer to $Gr^F\nabla$ as the Kodaira-Spencer operator. Under the identification $Gr^F CP_\bullet = (CH_\bullet(A_\bullet, A_\bullet)((u)), b)$ the Kodaira-Spencer operator is given by the formula

$$Gr^F\nabla = u^{-1}e_{\tilde{\kappa}}.$$

3.2. **Kaledin's definition.** Following ([K1], §3), we extend the argument from §2.2 to give another definition of the Gauss-Manin connection which will be used in our proofs. Consider a two-term filtration on $p_{1*}p_2^*A_\bullet$ given as $I^0 = p_{1*}p_2^*A_\bullet, I^1 = A_\bullet \otimes \Omega^1_X, I^2 = 0$. Note that $I^0/I^1 = A_\bullet$. Taking tensor powers of the filtered complex $p_{1*}p_2^*A_\bullet$, we obtain a filtration on the cyclic object $(p_{1*}p_2^*A_\bullet)^\#$. This gives rise a filtration I^i on the periodic cyclic complex of $p_{1*}p_2^*A_\bullet$ such that $I^0CP_\bullet(p_{1*}p_2^*A_\bullet)/I^1CP_\bullet(p_{1*}p_2^*A_\bullet) = CP_\bullet(A_\bullet)$. So we get a diagram with the upper row being a distinguished triangle

$$\begin{tikzcd}
I^1/I^2 \arrow[r, "i"] & I^0/I^2 \arrow[r, "\pi"] \arrow[d, "m"] & CP_\bullet(A_\bullet) \arrow[r] & {} \\
& p_{1*}p_2^*CP_\bullet(A_\bullet) & &
\end{tikzcd}$$

Lemma 3.1. *$I^1CP_\bullet(p_{1*}p_2^*A_\bullet)/I^2CP_\bullet(p_{1*}p_2^*A_\bullet)$ is contractible*

Proof. By [K1] §3, the cyclic object $I^1(p_{1*}p_2^*A_\bullet)^\#/I^2(p_{1*}p_2^*A_\bullet)^\#$ is free generated by $A^\# \otimes \Omega^1$ so its periodic cyclic complex is contractible. □

Hence, π is a quasi-isomorphism and the connection is defined as $\nabla = m\pi^{-1} : CP_\bullet(A_\bullet) \to p_{1*}p_2^*CP_\bullet(A_\bullet)$

Proposition 3.2. *Kaledin's connection is equal to Getzler's connection as a morphism $CP_\bullet(A_\bullet) \to p_{1*}p_2^*CP_\bullet(A_\bullet)$ in the derived category.*

Proof. We will show that Getzler's formula comes from a section of π on the level of complexes.

∇' gives rise to a section φ of $\pi : \bigoplus CP_i(p_{1*}p_2^*A_\bullet) \to \bigoplus CP_i(A_\bullet)$ because ∇' yields a connection on any $A^{i_1} \otimes \cdots \otimes A^{i_k}$ by the Leibnitz rule. Note that

(3.2)

$$\begin{aligned}
&[\varphi, b](a_0 \otimes \cdots \otimes a_n) = (\sum 1 \otimes \cdots \otimes \overset{i}{\nabla'} \otimes \cdots \otimes 1)(\sum a_0 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_n + \\
&+ \sum (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) - b(\sum a_0 \otimes \cdots \otimes \nabla' a_i \otimes \cdots \otimes a_n) = \\
&= \sum a_0 \otimes \cdots \otimes (\nabla' d - d\nabla')a_i \otimes \cdots \otimes a_n
\end{aligned}$$

This computation shows that $[\varphi, b + uB] = \mathcal{L}_{\tilde{\kappa}}$ (because, clearly $[\varphi, B] = 0$) where $\mathcal{L}_{\tilde{\kappa}} : CP_\bullet(A_\bullet) \to CP_\bullet(I^1(p_{1*}p_2^*A_\bullet)^\#/I^2(p_{1*}p_2^*A_\bullet)^\#)$. By [L], §4.1.11 we have $[u^{-1}\iota_{\tilde{\kappa}}, b + uB] = \mathcal{L}_{\tilde{\kappa}}$. Hence, $\varphi - u^{-1}\iota_{\tilde{\kappa}}$ is a morphism of complexes and a section of π so, in the derived category, $\pi^{-1} = \varphi - u^{-1}\iota_{\tilde{\kappa}}$. Applying m we get precisely the 3.1 considered as a map $CP_\bullet(A_\bullet) \to p_{1*}p_2^*CP_\bullet(A_\bullet)$.

□

3.3. **Proof of Theorem 5.** As explained in ([K4], §3.3 and §5.1) we have a canonical morphism

$$\mathcal{B}(A_\bullet)^\natural \to \pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural \tag{3.3}$$

in $D(\Lambda, R)$. This induces a morphism of cyclic complexes

$$CC_\bullet(\mathcal{B}(A_\bullet)) = CC_\bullet(\mathcal{B}(A_\bullet)^\natural) \to CC_\bullet(\pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural),$$

$$V_{-1}CC_\bullet(\mathcal{B}(A_\bullet)) \to CC_\bullet(\pi^\flat_{(-2(p-1),-1]} i_p^* A_\bullet{}^\natural) \xrightarrow{\sim} V_{[-p+2,-1]}\overline{C}P_\bullet(A_\bullet) \tag{3.4}$$

We have to check that (3.4) factors through $V_{[-p+2,-1]}CC_\bullet(\mathcal{B}(A_\bullet))$ and that the resulted morphism is a quasi-isomorphism.

Recall that any complete resolution has the structure of an E_∞ operad. This makes $\mathcal{B}(R)^\natural$ and $\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural$ into E_∞ algebras in the category of complexes over $Fun(\Lambda, R)$ and $\mathcal{B}(A_\bullet)^\natural$, $\pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural$ are modules over these algebras respectively. The morphism 3.3 can be promoted to

$$\mathcal{B}(A_\bullet)^\natural \overset{L}{\otimes}_{\mathcal{B}(R)^\natural} \pi^\flat_{(-2(p-1),0]} i_p^* R^\natural \to \pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural \tag{3.5}$$

Moreover, if we endow the left-hand side of (3.5) with the filtration induced by the canonical filtration on $\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural$ and the right-hand side with τ^{dec}, then (3.5) is a filtered quasi-isomorphism. Pass to mixed complexes:

$$C(\mathcal{B}(A_\bullet)) \overset{L}{\otimes}_{C(\mathcal{B}(R))} C(\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural) \to C(\pi^\flat_{(-2(p-1),0]} i_p^* A_\bullet{}^\natural) \tag{3.6}$$

Now Theorem 5 follows from an easy Lemma below.

Lemma 3.3. *The homomorphism of E_∞ algebras*

$$C(\mathcal{B}(R)) \to C(\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural)$$

induces a quasi-isomorphism

$$\tau_{(-2(p-1),0]} C(\mathcal{B}(R)) \xrightarrow{\sim} C(\pi^\flat_{(-2(p-1),0]} i_p^* R^\natural).$$

4. The local monodromy theorem

In this section we prove Theorem 3 in a stronger and more general form. We start by recalling some results of Katz from ([Katz1]).

4.1. **Katz's Theorem.** Let S be a smooth geometrically connected complete curve over a field K of characteristic 0, $K(S)$ the field of rational functions on S, and let E be a finite-dimensional vector space over $K(S)$ with a K-linear connection

$$\nabla : E \to E \otimes \Omega^1_{K(S)/K}.$$

Recall that ∇ is said to have regular singularities if E can be extended to a vector bundle $\mathcal{E}$ over S such that ∇ extends to a connection on $\mathcal{E}$, which has at worst simple poles at some finite closed subset $D \subset S$:

$$\overline{\nabla} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_S(\log D).$$

One says that the local monodromy of (E, ∇) is quasi-unipotent if the pair $(\mathcal{E}, \overline{\nabla})$ as above can be chosen so that the residue of $\overline{\nabla}$

$$\operatorname{Res} \overline{\nabla} : \mathcal{E}_{|D} \to \mathcal{E}_{|D}$$

has rational eigenvalues[13]. Let $\operatorname{Res}\overline{\nabla} = D + N$, with $[D, N] = 0$, be the Jordan decomposition of $\operatorname{Res}\overline{\nabla}$ as a sum of a semi-simple operator D and a nilpotent operator N. If the local monodromy of (E, ∇) is quasi-unipotent we say its exponent of nilpotence is $\leq \nu$ if $N^\nu = 0$.

If $K = \mathbb{C}$ then the category of finite-dimensional $K(S)$-vector spaces with K-linear connections with regular singularities and quasi-unipotent local monodromy is equivalent to the category of local systems (in the topological sense) over S take off finitely many points whose local monodromy around every puncture is quasi-unipotent (*i.e.*, all its eigenvalues are roots of unity). The exponent of nilpotence of local monodromy is the size of its largest Jordan block.

In ([Katz1], Th. 13.0.1), Katz proved the following result.

Theorem (Katz)**.** *Let C be a smooth scheme of relative dimension 1 over a domain R which is finitely generated (as a ring) over $\mathbb{Z}$, with fraction field K of characteristic zero. Assume that the generic fiber of C is geometrically connected. Let (M, ∇) be a locally free $\mathcal{O}_C$-module with a connection $\nabla : M \to M \otimes \Omega^1_{C/R}$. Assume that (M, ∇) is globally nilpotent of nilpotence ν, that is, for any prime number p, the $\mathcal{O}_{C\otimes\mathbb{F}_p}$-module $M \otimes \mathbb{F}_p$ with $R \otimes \mathbb{F}_p$-linear connection admits a filtration*

$$0 = V_0(M \otimes \mathbb{F}_p) \subset \cdots \subset V_\nu(M \otimes \mathbb{F}_p) = M \otimes \mathbb{F}_p$$

such that the p-curvature of each successive quotient V_i/V_{i-1} is 0. Then the pullback $M \otimes_{\mathcal{O}(C)} K(C)$ of M to the generic point of C has regular singularities and quasi-unipotent local monodromy of exponent $\leq \nu$.

4.2. **Monodromy Theorem.** Now we can prove the main result of this section.

Theorem 8. *Let $A_\bullet$ be a smooth and proper DG algebra over $K(S)$ and let d be a non-negative integer such that*

$$HH_m(A_\bullet, A_\bullet) = 0, \text{ for every } m \text{ with } |m| > d. \tag{4.1}$$

Then the Gauss-Manin connection on the relative periodic cyclic homology $HP_*(A_\bullet)$ has regular singularities and quasi-unipotent local monodromy of exponent $\leq d + 1$.

Proof. Using Theorem 1 from [Toën], there exists a finitely generated $\mathbb{Z}$-algebra $R \subset K$, a smooth affine scheme C of relative dimension 1 over R with a geometrically connected generic fiber, and a smooth proper DG algebra $B_\bullet$ over $\mathcal{O}(C)$ together with an open embedding $C \otimes_R K \hookrightarrow S$ of curves over K and a quasi-isomorphism $A_\bullet = B_\bullet \otimes_{\mathcal{O}(C)} K(S)$ of DG algebras over $K(S)$. We can choose $B_\bullet$ to be term-wise flat over $\mathcal{O}(C)$. Since the Hochschild homology $\bigoplus_i HH_i(B_\bullet, B_\bullet)$ of a smooth proper DG algebra is finitely generated over $\mathcal{O}(C)$ replacing C by a dense open subscheme we may assume that $\bigoplus_i HH_i(B_\bullet, B_\bullet)$ and $HP_*(B_\bullet, B_\bullet)$ are free $\mathcal{O}(C)$-modules of finite rank. It follows that

$$HH_i(B_\bullet, B_\bullet) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\sim} HH_i(B_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p, B_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p),$$

$$HH_i(B_\bullet, B_\bullet) \otimes_{\mathcal{O}(C)} K(S) \xrightarrow{\sim} HH_i(A_\bullet, A_\bullet).$$

Using the Hodge-to-de Rham spectral sequence it follows that the periodic cyclic homology also commutes with the base change. Then by Cor. 1.2 $(M, \nabla) = (HP_*(B_\bullet), \nabla_{GM})$ satisfies the assumptions of the theorem of Katz with $\nu = d + 1$ and we are done. $\square$

[13] One can show (see *e.g.*, [Katz1], §12) that if $\operatorname{Res}\overline{\nabla}$ has rational eigenvalues for one extension then it has rational eigenvalues for every extension $(\mathcal{E}, \overline{\nabla})$ of (E, ∇).

References

[BO] P. Berthelot, A. Ogus, *Notes on crystalline cohomology,* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[BvB] A. Bondal, M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry,* Mosc. Math. J. 3 (2003), no. 1, 1–36, 258.

[BB] A. Braverman, R. Bezrukavnikov, *Geometric Langlands correspondence for D-modules in prime characteristic: the $GL(n)$ case,* Pure Appl. Math. Q. 3 (2007), no. 1, Special Issue: In honor of R. MacPherson. Part 3, 153–179.

[CV] D. Calaque, M. Van den Bergh, *Hochschild cohomology and Atiyah classes,* Adv. Math. 224 (2010), no. 5, 1839–1889.

[DI] P. Deligne, L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham,* Invent. Math. 89 (1987), no. 2, 247–270.

[Dr] V. Drinfeld, *DG quotients of DG categories*, J. Algebra 272 (2004), no. 2, 643–691.

[Fa] G. Faltings, *Crystalline cohomology and p-adic Galois-representations,* Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[FL] J-M. Fontaine, G. Laffaille, *Construction de représentations p-adiques.* Ann. Sci. Ecole Norm. Sup. (4) 15 (1982), no. 4, 547–608 (1983).

[FS] E. M. Friedlander, A. Suslin, *Cohomology of finite group schemes over a field* Invent. Math. 127 (1997), no. 2, 209–270.

[Ge] E. Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology,* Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), 65-78, Israel Math. Conf. Proc., 7, Bar-Ilan Univ., Ramat Gan, 1993.

[K1] D. Kaledin, *Cyclic homology with coefficients,* Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 23–47, Progr. Math., 270, Birkhauser Boston, Inc., Boston, MA, 2009.

[K2] D. Kaledin, *Bokstein homomorphism as a universal object,* http://lanl.arxiv.org/pdf/1510.06258v1

[K3] D. Kaledin, *Co-periodic cyclic homology,* arXiv:1509.08784

[K4] D. Kaledin, *Spectral sequences for cyclic homology,* http://lanl.arxiv.org/pdf/1601.00637

[Katz1] N. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin,* Inst. Hautes Études Sci. Publ. Math. No. 39 (1970), 175–232.

[Katz2] N. Katz, *Algebraic solutions of differential equations (p-curvature and the Hodge filtration),* Invent. Math. 18 (1972), 1–118.

[Keller] B. Keller, *On differential graded categories,* International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich (2006).

[L] J. Loday, *Cyclic homology,* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 301. Springer-Verlag, Berlin, 1992.

[Laz] M. Lazard *Commutative formal groups,* Lecture Notes in Math. no.443, Springer-Verlag, 1975

[OV] A. Ogus, V. Vologodsky, *Nonabelian Hodge theory in characteristic p.* Publ. Math. Inst. Hautes Études Sci. No. 106 (2007), 1-138.

[Toën] B. Toën, *Finitude homotopique des dg-algebres propres et lisses,* Proc. Lond. Math. Soc. (3) 98 (2009), no. 1, 217–240.