Cluster Algebras II: Main Exercises

math.mit.edu/~amandabu/NDclusteralgebras.html

Exercise 2.1. Consider the Somos-4 recurrence, given by $z_1 = z_2 = z_3 = z_4 = 1$ and

$$z_m = \frac{z_{m-1}z_{m-3} + z_{m-2}^2}{z_{m-4}}.$$

We will try to work backwords to construct a cluster algebra that has this recurrence as its exchange relation.

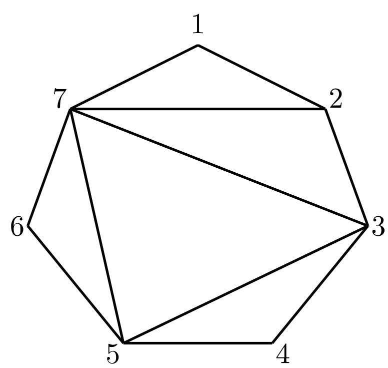
(a) Construct a rank 4 quiver such that:

- The quiver is isomorphic to itself (the same up to a relabeling of vertices) after every mutation in the sequence $1, 2, 3, 4, 1, 2, 3, 4, \ldots$
- At each mutation in this sequence, the exchange relation is as above (where when mutating the cluster variable z_k, we replace it with z_{k+4}).

Hint: Let z_m be the cluster variable we mutate. What does the relation above tell you about some arrows in the quiver? Use the first condition to determine the other arrows.

(b) Explain how to use the Laurent phenomenon and Laurent positivity to prove that all z_m are positive integers.

Exercise 2.2. Let T be the following triangulation of a heptagon.



Let T' be obtained from T by replacing the diagonal $\overline{37}$ with $\overline{25}$.

(a) Construct the quivers Q corresponding to T and T'. Verify that they differ by a single mutation.

(b) Write down the exchange relation for the mutation in part (a) and compare it to the Plücker relation.

Exercise 2.3. Prove that filling any lightning bolt with 1's yields a frieze with positive integer entries.
Hint: Use the fact that all entries are cluster variables from Exercise 1.2.

Cluster Algebras II: Additional Exercises

Exercise 2.4. Consider the set of triangulations of an $(n+3)$-gon.

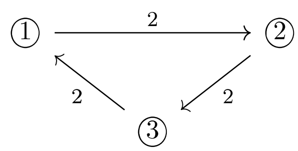
(a) Which triangulations correspond to a quiver whose mutable part is an orientation of a path on n-vertices?

(b) Prove that any type A_n quiver that is not an orientation of a path on n-vertices contains an oriented 3-cycle.

Exercise 2.5. In $\mathsf{Gr}(2,n)$, prove the three-term Plücker relation

$$P_{ik}P_{j\ell} = P_{ij}P_{k\ell} + P_{i\ell}P_{jk}\,.$$

Exercise 2.6. Return to the Markov quiver from Exercise 1.1.



Prove that seeds in the corresponding cluster algebra are in bijection with positive integers triples (a,b,c) of solutions to the Markov equation

$$a^2 + b^2 + c^2 = 3abc\,.$$

Exercise 2.7. Prove that the cluster algebra for the Markov quiver is not equal to the upper cluster algebra.

Hint: Use Exercise 1.4(c) to find a candidate element that is inside $\mathcal{U}$ but not inside $\mathcal{A}$.

Exercise 2.8. Prove that the distinct seeds in the A_n cluster algebra (up to relabeling of the vertices) are in bijection with triangulations of the $(n+3)$-gon.

Exercise 2.9. Consider the A_3 cluster algebra $\mathcal{A}((x,y,z), \bullet \to \bullet \to \bullet)$.

(a) Find all 14 distinct seeds (up to renumbering the quiver vertices). You can either use Keller's mutation app or look at triangulations of the hexagon up to diagonal flips.

(b) Draw a graph with 14 vertices (each representing one of the seeds you found in (a)), and add an edge between any pair of vertices whose quivers are related by a single mutation.

(c) What do you notice about this graph? [3]

Exercise 2.10. Read Section 3.3 (page 43-48) of "Introduction to Cluster Algebras" by Fomin–Williams–Zelevinsky to see a proof of the Laurent Phenomenon.

[3] In fact, this is the 1-skeleton (meaning a representation of the vertices and edges) of a polytope called the *associahedron*.