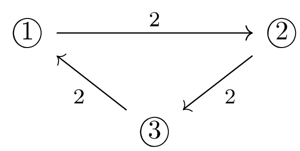
Cluster Algebras I: Main Exercises

math.mit.edu/~amandabu/
NDclusteralgebras.html

Exercise 1.1. Consider the *Markov quiver* Q:



(a) Show that this quiver stays the same under mutations. (The cluster variables do change though!)

(b) Compute the new cluster variables in $\mathcal{A} = \mathcal{A}((x, y, z), Q)$ after the mutations μ_1, then μ_2, then μ_3.

(c) Suppose that x, y, z are cluster variables in some seed and x' is obtained from x by mutation. Prove that

$$\frac{x^2 + y^2 + z^2}{xyz} = \frac{(x')^2 + y^2 + z^2}{x'yz}.$$

(d) Suppose that x_s, y_s, z_s are cluster variables in some seed s written as Laurent polynomials in the initial cluster variables x, y, z. Prove that the specializations of x_s, y_s, z_s at $x = y = z = 1$ give a solution of the Markov equation[1]

$$a^2 + b^2 + c^2 = 3abc.$$

Exercise 1.2. Suppose we place variables x_1, x_2, x_3 down the diagonal of a frieze pattern with 3 nontrivial rows and express all other entries as elements of (x_1, x_2, x_3).

(a) Prove that every element of the frieze is an element of the A_3 cluster algebra

$$\mathcal{A}((x_1, x_2, x_3), \bullet \to \bullet \to \bullet).$$

Hint: draw a quiver $x_1 \to x_2 \to x_3$. Mutate at x_1. What is the new cluster variable? Where does it live in the frieze? Continue this argument by performing more source/sink mutations

(b) Generalize your argument to the case where we place x_1, x_2, $\ldots$, x_n down the diagonal of a frieze with n nontrivial rows. Show that this yields only elements of the A_n cluster algebra

$$\mathcal{A}\left((x_1, x_2, \ldots, x_n), \underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{n}\right).$$

(c) Further generalize your argument to the case where x_1, x_2, $\ldots$, x_n are placed in any lightning bolt pattern, with one variable in each row of the frieze such that each x_j and x_{j+1} are part of the same diamond.

[1]The integers that arise in solutions to this equation (or equivalently, the specializations of the cluster variables x_s,y_s, and z_s) are the famous *Markov numbers*! The first few Markov numbers are 1, 2, 5, 13, 29, 34, 89, 169, …, and these have been the source of much study in number theory and beyond. There is a 113 year old *unicity conjecture* that each Markov number shows up exactly once as the maximum value in a Markov triple (a, b, c).

Cluster Algebras I: Additional Exercises

Exercise 1.3. We defined mutation for quivers. A quiver (without frozen vertices) can be encoded as an $n \times n$ skew-symmetric[2] matrix $B = (b_{ij})$, where

$$b_{ij} = \#(\text{arrows } i \to j) - \#(\text{arrows } j \to i)\,.$$

Mutation μ_k on the matrix B (called the *exchange matrix*) can then be equivalently define via

$$b_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0, \\ b_{ij} & \text{otherwise.} \end{cases}$$

We can equally well apply mutation to *skew-symmetrizable matrices*, matrices B such that $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$.

(a) Show that the matrix $B = \begin{bmatrix} 0 & 3 & -1 \\ -2 & 0 & 1 \\ 2 & -3 & 0 \end{bmatrix}$ is skew-symmetrizable.

(b) Compute the mutation μ_2 of B. Show that it is also skew-symmetrizable.

(c) Verify that the mutation of a skew-symmetrizable matrix is always skew-symmetrizable.

Exercise 1.4. Return to the Markov quiver Q from Exercise 1.2 and consider the cluster algebra $\mathcal{A} = \mathcal{A}((x, y, z), Q)$

(a) Prove that $\mathcal{A}$ is non-negatively graded with all cluster variables (in all seeds) of degree 1.

(b) Use (a) to prove that $\mathcal{A}$ is not finitely generated.

(c) Use Exercise 1.1(c) to prove that $m = \frac{x^2+y^2+z^2}{xyz}$ can be written as a Laurent polynomial in any cluster of $\mathcal{A}$ (we'll learn tomorrow that this means m is in the *upper cluster algebra* $\mathcal{U} \supseteq \mathcal{A}$). Then prove that m is not in $\mathcal{A}$. Hint: what is the degree of m?

Open Problem 1.1. A quiver is *acyclic* if it has no oriented cycles. Give a combinatorial proof that if two acyclic quivers Q and Q' are mutation-equivalent, then Q can be transformed into Q' by only mutating at sources and sinks.

There is a proof that goes through category theory, but no other proof is known!

Exercise 1.5. Which orientations of an n-cycle quiver are mutation-equivalent to each other?

Feel free to assume the result of Open Problem 1.1.

[2] *skew-symmetric* here means that $B = -B^\top$