A circle of ideas around the Gauss-Bonnet theorem

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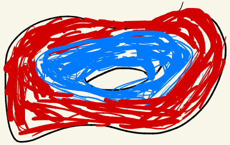
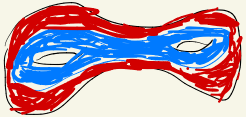
Thm For (M, g) a Riemannian surface, $K: M \to \mathbb{R}$ the Gaussian curvature,

$$\frac{1}{2\pi}\int_M K\,dA = \chi(M)$$

$\chi(S^2) = 2$



$\chi(M_1) = 0$



$\chi(M_2) = -2$

🟥 = positive curvature
🟦 = negative curvature

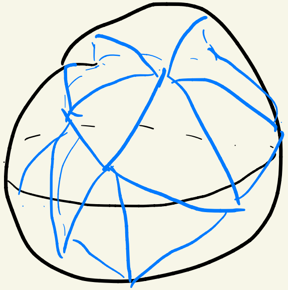
Thm: For (M,g) a $2n$-dimensional Riemannian manifold, let Ω^i_j be the curvature matrix relative to a frame. Then $Pf(\Omega) \in H^{2n}(M)$ is a form s.t

(1) The form itself does not depend on the choice of frame

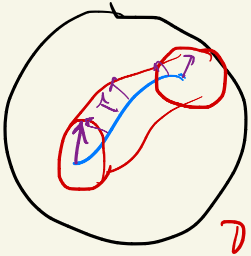
(2) It's integral does not depend on the metric:

$$\frac{1}{(2\pi)^n}\int_M Pf(\Omega) = \chi(M)$$

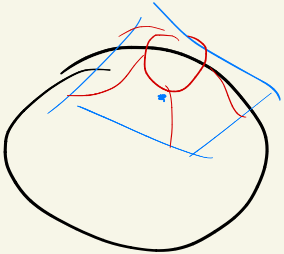
Plan:



holonomy becomes exact



Poincare Duality



global to local

Holonomy proof for surfaces

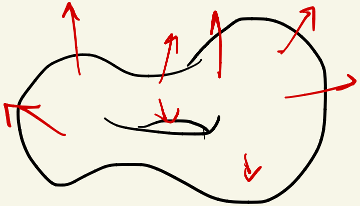
Transgression, Poincare-Hopf

Thom class construction

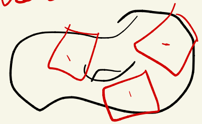
splitting & naturality

start here!

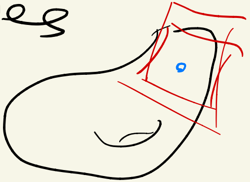
Gauss map Proof for surfaces $M \subset \mathbb{R}^3$



Grassmanians!

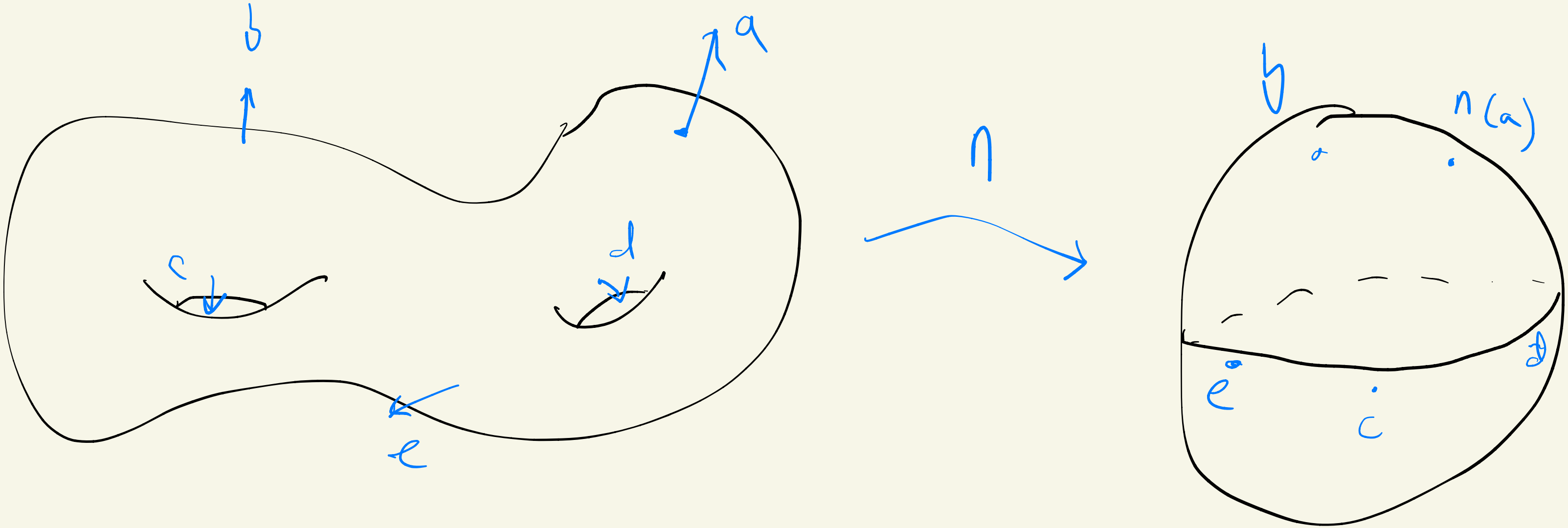


Higher order bundles



Gauss map for all embedded manifolds

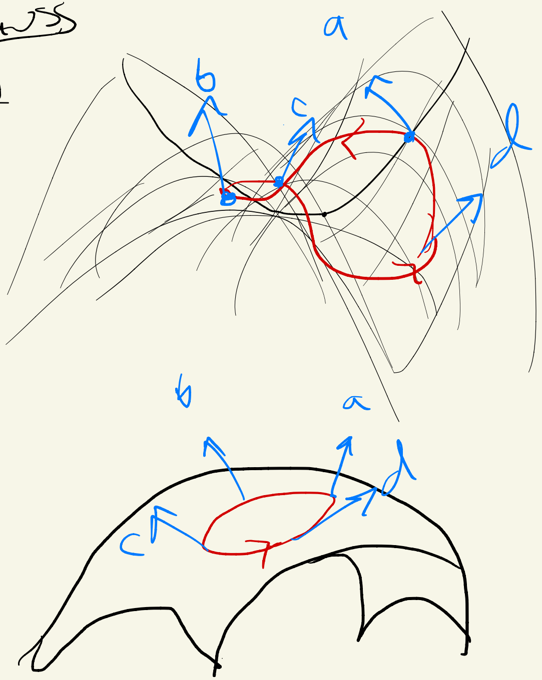
Gauss map



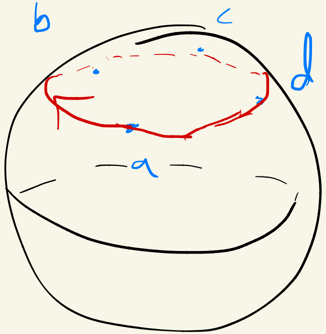
Theorem Egregium

$$\det dn = K$$

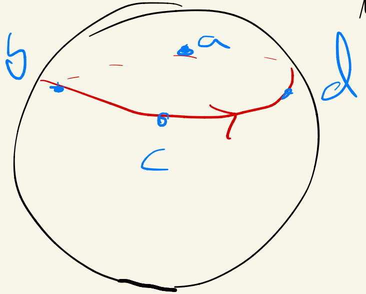
Gauss Map



Negative curvature, orientation reversed



Positive curvature, orientation maintained



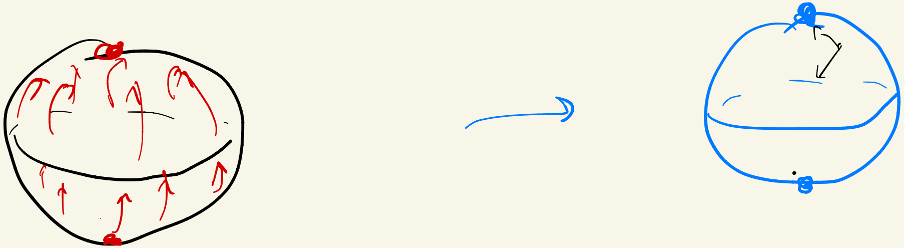
$$\det dn = K$$

By the degree formula,

$$\int_M K\,dA = \int_M (\det dn)\,dA = \int_M n^* dA_{S^2} = 4\pi \deg n$$

Prop: $\deg n = \frac{1}{2}\chi(M)$ $\Rightarrow$ $\int_M K\,dA = 2\pi\chi(M)$ ✓

Proof: Poincare - Hopf

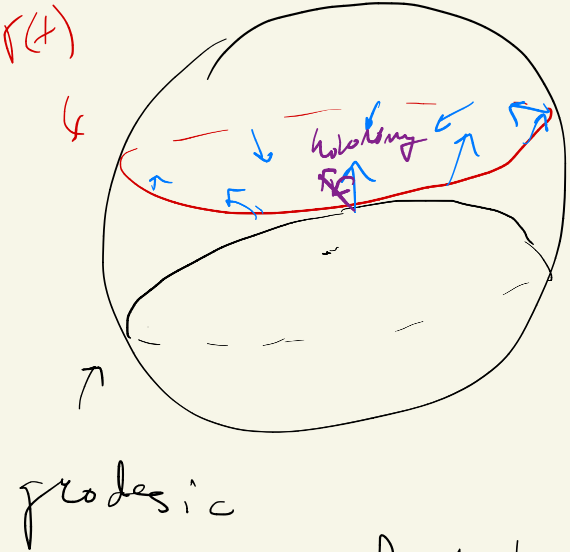


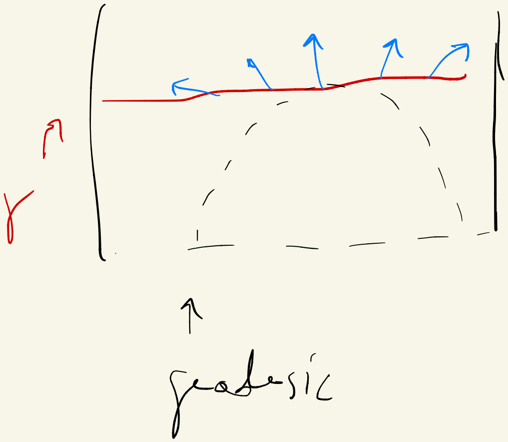
$$\chi(M) = \sum_{f^{-1}(v)} \operatorname{sgn} x + \sum_{f^{-1}(-v)} \operatorname{sgn} y$$

$$= 2\deg n$$

Switching gears to the local picture--- holonomy!

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

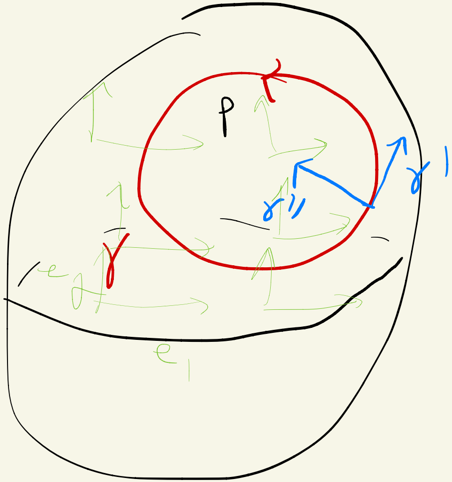




Parallel translation pulls vectors towards geodesics

Holonomy measures rotation due to parallel translation

Geodesic curvature measures deviation of a curve from being geodesic



$$K = dA(\gamma', \gamma'')$$

$$\omega = \langle \nabla e_1, e_2 \rangle$$

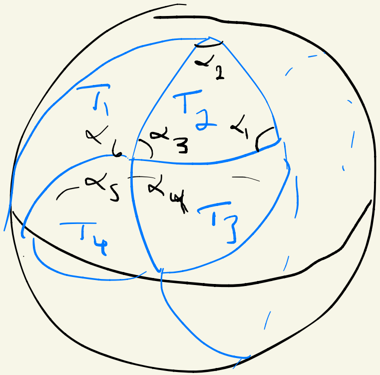
$$\int_\gamma \omega = 2\pi - \int_\gamma K dt$$

$$d\omega = K dA$$

$$\int_P K dA + \int_\gamma K dt = 2\pi$$

Local Gauss Bonnet

$$\int_P K\,dA + \int_\gamma k\,dt = 2\pi \longrightarrow \int_P K\,dA + \int_\gamma k\,dt = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$



$$\int_M K\,dA = \sum_v \alpha_v - \pi F$$

$$= 2\pi\left(V - \frac{F}{2}\right)$$

$$= 2\pi(V - E + F)$$

$$= 2\pi\chi(M)$$

Strange: proving the local Gauss-Bonnet theorem requires choosing a frame, even though the result does not

Reason: The equation $KdA = d\omega$ is only valid locally

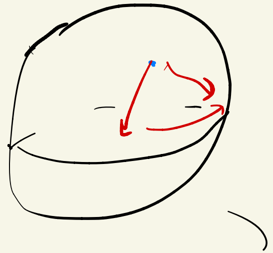
Transgression: $\Omega = KdA$ becomes exact globally when passing to a larger manifold

Let $SM = \{ v_p \in TM \mid \|v_p\| = 1 \}$

Define one form ψ on SM by "global angular form"

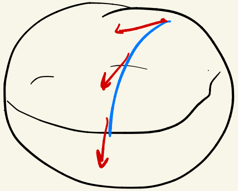
$$\psi\left(\frac{d}{dt} v_{\gamma(t)}(t)\right) = dA\left(D_{\gamma'} v, \gamma(0)\right)$$

$Z_{\gamma(t)}(t) \in SM$ a curve of unit vecs, $\int_Z \psi$ measures total rotation



$\psi\left(\frac{d}{dt} v\right) = 1$

$\gamma(t) = \text{const.}$, measures rotation



$\psi\left(\frac{d}{dt} v\right) = 0$

$v = \gamma'(t)$, measures geodesic curvature

Prop

$$\begin{tikzcd} SM \arrow[d, "\pi"] \\ M \end{tikzcd}$$

$\pi^* \Omega = d\psi$

Ω

curvature

Proof sketch

e_1, e_2 a frame

$\langle \cdot, e_1 \rangle = a : SM \to \mathbb{R}$, $\langle \cdot, e_2 \rangle = b : SM \to \mathbb{R}$

$a^2 + b^2 = 1 \Rightarrow ada + bdb = 0$

$\psi = adb - bda + \pi^* \omega^2_1$

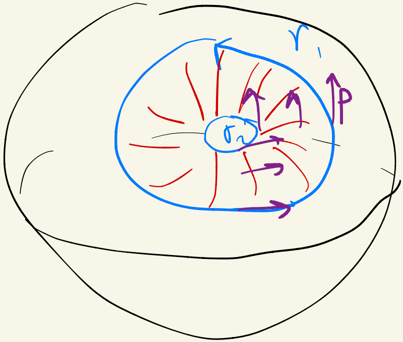
$\shortparallel$

$\langle \nabla e_1, e_2 \rangle$

$= 0$, da, db lin dep

$d\psi = dadb + \pi^* d\omega^2_1$

$= \pi^* \Omega$ ✓



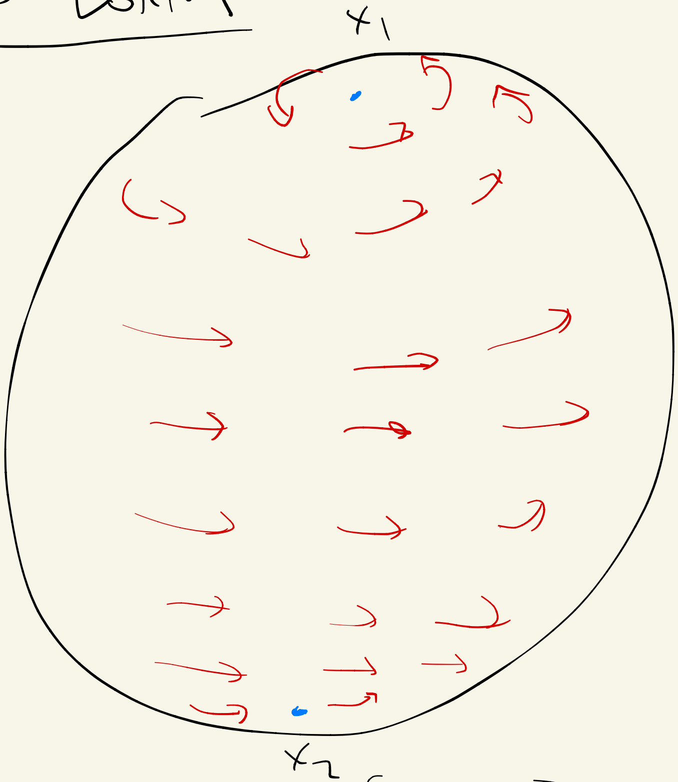
$$\int_{\gamma_1} K - \int_{\gamma_2} K$$

$$= \int_{p(R)} d\psi \quad \text{(Stokes)}$$

$$= \int_R \Omega$$

Recovers local Gauss-Bonnet

Gauss-Bonnet



$p : M - \{x_1, x_2\} \to SM$

$$\int_M \Omega =$$

$$\int_{p(M)} \pi^* \Omega = \int_{p(M)} d\psi$$

$$= 2\pi \sum_i \mathrm{ind}\, x_i$$

$$= 2\pi \chi(M)$$

by Poincare-Hopf

Thom class perspective

$$\begin{tikzcd} 0 \arrow[r] & \pi^* E \arrow[r] & TE \arrow[r] \arrow[l, bend right, dashed, "\nabla"'] & \pi^* TM \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \end{tikzcd}$$ a Riemannian plane bundle,

∇ a metric connection

Topological significance of a connection: gives a section, realizing $TE = \pi^* E \oplus \pi^* TM$

Thom isom: Exists a unique element $\Phi \in H^n_{cv}(M)$ s.t.

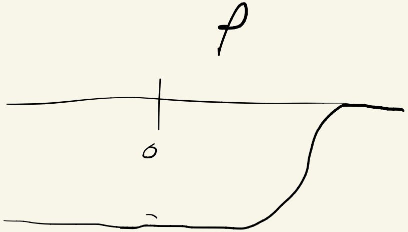
$$\int_{\pi^{-1}(x)} \Phi = 1 \quad \text{for all } x.$$

Euler class $e = s_0^* \Phi$

Let ψ be the global angular form on SE coming from ∇

$$\begin{tikzcd} E \arrow[d] \\ SE \arrow[d] & \|u\|=1 \\ M \end{tikzcd}$$

$\tilde{\psi}$ pullback to E



$r : E \to \mathbb{R}$ the length function

$$\Phi = d(\rho(r)\tilde{\psi}) = \underbrace{\rho' dr \wedge \tilde{\psi}}_{\text{supported away from } s_0(M)} + \rho(r)\underbrace{d\tilde{\psi}}_{= \pi^*\Omega}$$

from the prior work!

Thus $e = s_0^*\Phi = \frac{1}{2\pi}\Omega \rightsquigarrow$ Gauss-Bonnet

By functoriality we extend to all even rank $= 2n$ vector bundles.

For ∇ a connection, Ω^i_j a curvature matrix,

$$Pf(\Omega^i_j) = \sum_{\substack{\alpha \\ \text{a pairing of} \\ 1,\dots,2n}} \operatorname{sgn}\alpha \; \Omega^{i_1}_{j_1} \wedge \cdots \wedge \Omega^{i_n}_{j_n}$$

(1) $Pf\,\Omega$ does not depend on the choice of frame

(2) $d\,Pf\,\Omega = 0$ $\Rightarrow$ $e_g = \frac{1}{(2\pi)^n} Pf\,\Omega$

(3) $[Pf\,\Omega] \in H^{2n}(M)$ does not depend on the connection

Prop

(1) $e_g(E_1 \oplus E_2) = e_g(E_1) \wedge e_g(E_2)$

(2) $e_g(f^* E) = f^* e_g(E)$

Proof: Choose a compatible connection on $E_1 \oplus E_2$, $f^* E$

so:

$$\Omega_{E_1 \oplus E_2} = \begin{pmatrix} \Omega_{E_1} & \\ & \Omega_{E_2} \end{pmatrix} \Rightarrow \mathrm{Pf}\, \Omega_{E_1 \oplus E_2} = \mathrm{Pf}\, \Omega_{E_1} \wedge \mathrm{Pf}\, \Omega_{E_2}$$

$$\Omega_{f^* E} = f^* \Omega_E \Longrightarrow \mathrm{Pf}\, \Omega_{f^* E} = f^* \mathrm{Pf}\, \Omega_E$$

Prop (Splitting principle)

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \end{tikzcd}$$

Exists $f: N \to M$ s.t

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ N \arrow[r] & M \end{tikzcd}$$

(1) $f^*: H^\bullet(M) \to H^\bullet(N)$ is injective

(2) $f^*E = E_1 \oplus \cdots \oplus E_n$, E_j a plane bundle

$f^* e_g(E) = e_g(E_1 \oplus \cdots \oplus E_n) = e_t(E_1 \oplus \cdots \oplus E_n) = f^* e_t(E)$ ✓

As a conclusion:

$$\frac{1}{(2\pi)^n}\int_M \mathrm{Pf}\,\Omega = \chi(E),$$

$$\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$$

any even rnk v.b

Specialization to $E = TM$, ∇ = Levi-Civita connection

Gauss-Bonnet-Chern!

A more general proof for embedded manifolds

$M \subset \mathbb{R}^N$

$$\begin{tikzcd} E \arrow[d] \\ P = Gr(n,N) \end{tikzcd}$$

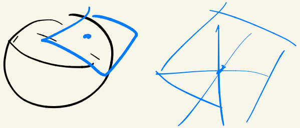
fibers are the planes

oriented n-planes

naturally Riemannian

$n : M \to Gr(n,N)$

$x \mapsto T_x M$



$TM = n^* E$

The connections are compatible, so

$$\mathrm{Pf}\,\Omega_M = f^*\mathrm{Pf}\,\Omega_P = 2 f^*(2\pi)^n\, dV$$

(constant b-c Gr is homogeneous)

$$e(Gr) = 2dV$$

so

$$e(M) = f^* e(Gr) = \frac{1}{(2\pi)^n} f^*\mathrm{Pf}\,\Omega_P = \frac{1}{(2\pi)^n}\mathrm{Pf}\,\Omega \quad \checkmark$$

Cool equation: $\det dn = \mathrm{Pf}\,\Omega$

Follows from almost no work

Thank you Prof. Neitzke!