Fractal Uncertainty for Discrete 2D Cantor Sets

FUP: Fractal Uncertainty Principle

Thm (Bourgain & Dyatlov):
Suppose $f \in L^2(\mathbb{R})$, $\text{Supp } f \subset \mathcal{K}$.

Then $\|f\|_2 \leq h^\beta$ for some $\beta > 0$.

- Developed for applications to quantum chaos
  - spectral gaps,
  - mass of eigenfunctions

- Major problem: extend to 2D

- We study a simpler discrete model
Fix an integer $M$. Let $A \subset \mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$ be an alphabet.

\[ \mathcal{X}_k \subset \mathbb{Z}_{M^k} = \mathbb{Z}_N \quad (N := M^k) \]

\[ = \left\{ a_0 + a_1 M + \cdots + a_{k-1} M^{k-1} : a_i \in A \right\} \]

**Ex** $M = 3, \ A = \{0, 2\}$

\[ \mathcal{X}_1 = \frac{0 \ 1 \ 2}{\text{is the dimension}} \]

\[ \mathcal{X}_2 = \frac{0 \ 2 \ 5}{\text{is the dimension}} \]

These are discrete cantor sets

\[ |A| = M^S \quad \Rightarrow \quad |\mathcal{X}_k| = M^{kS} = N^S \]

$S < 1$ is the dimension
Main result: classify which 2D cantor sets have a FUP
Fourier transform $\mathcal{F} : L^2(\mathbb{Z}/N\mathbb{Z}) \to L^2(\mathbb{Z}/N\mathbb{Z})$.

\[ \hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f(x) e^{\frac{-2\pi i}{N} x k}, \quad f(x) = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \hat{f}(k) e^{\frac{2\pi i}{N} x k} \]

\[ \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \quad (f * g) = \hat{f} \hat{g} \]

\[ \Rightarrow \text{supp} \hat{f} \hat{g} \subseteq \text{supp} \hat{f} + \text{supp} \hat{g} \]

If $z = e^{\frac{2\pi i}{N}}$, then

\[ f(x) = \frac{1}{\sqrt{N}} \sum_{k} \hat{f}(k) z^k \]

Think of $f : S^1 \to C$ as a trigonometric polynomial (in the variable $z = e^{\frac{2\pi i}{N}}$).
Thm (Dyatlov - Jin)

If supp\(f\) \(\subset X_k\), \(\|\text{supp}\ f\|_2 \leq M^{-k\beta}\)

or: \(\|1_{Y_k} \varphi 1_{X_k}\|_2 \leq M^{-k\beta}\)

Submultiplicativity:

\[
\|1_{Y_{k+r}} \varphi 1_{X_{k+r}}\|_2 \leq \|1_{Y_k} \varphi 1_{X_k}\|_2 \\quad \|1_{Y_r} \varphi 1_{X_r}\|_2 \leq 1
\]

Suffices to show \(\|1_{Y_k} \varphi 1_{X_k}\|_2 < 1\)

for some \(k\).

Goal: For some \(k\), impossible for \(\text{supp} f \subset X_k\)

\(\text{supp} f \subset Y_k\)
Goal: For some $k$, impossible for

$$\text{supp} f \subset Y_k$$

(very special to the discrete case!)

Prop: $f \in L^2(\mathbb{Z}/N\mathbb{Z})$

If $|\text{supp} f| = M$, then $\text{supp} f \cap [a, a + M]$ nonempty for all $a$

\[\hat{f} \leftrightarrow f\]

Pt of FUP assuming Prop:

If $\text{supp} f \subset Y_k$, then $|\text{supp} f| \leq M$

If $\text{supp} \hat{f} \subset Y_k$, then $\text{supp} \hat{f}$

\[\text{has gaps of size } \frac{N}{M}\]

or: If $\text{supp} f$ has a large gap, then $|\text{supp} f|$ is large

\[\blacksquare\]
pf of prop \[ \text{supp } f = \{ x_1, \ldots, x_m \} \]

Let \[ h(z = e^{\frac{2\pi i}{N} x}) = \prod_{j=0}^{m-1} (z - e^{\frac{2\pi i}{N} x_j}) = \sum_{k=0}^{m-1} a_k z^k \]

(1) \[ f h = c \delta x_m \]
(2) \[ \text{supp } \hat{f} \subset [0, m) \]

\( \hat{f} \) has full support \( \hat{f}(k) = \sqrt{n} a_k \).

Because \( \hat{f} h = \hat{f} \ast \hat{h} \), \( \text{supp } f h \subset \text{supp } f + \text{supp } h \)

\[ \text{supp } f h \subset [0, m) \]

"m-neighbourhood"

\[ \text{supp } f + [0, m) = \mathbb{Z}_N / \mathbb{Z} \Rightarrow \hat{f} \text{ has support in every } [a, a + m)\]

\[ \text{supp } \hat{f} = \{ y_1, y_2, y_3, y_4 \} \]
\[ \text{supp } h \subset [0, D) \]
\[ \text{supp } f h \subset \{ y_1, y_2, y_3, y_4 \} + [0, D) \]

\[ \hat{f} h (y) = 0, \]
a contradiction
Moving on to 2D...

\[ T 1_{x=0} = 1_{y=0} \]

This "pair of orthogonal lines" is the only obstruction.

If \( \{(t,0) : t \in \mathbb{Z}_N\} \subset Y_k \) then no FUP!

\( \{(0,t) : t \in \mathbb{Z}_N\} \subset Y_k \)
Let $X_k, Y_k$ be Cantor sets

\[ X, Y \subseteq \mathbb{T}^2 \]

the drawings

\[ n \]

Thm. As long as $X, Y$

don't contain a pair of orthogonal lines

\[ R_{v+p} \subseteq X, R_{v+q} \subseteq Y \]

then

\[ \| 1 y_k - 1 x_k \|_2 \leq M \text{ for some } \beta > 0. \]
This cantor set contains a line

This one doesn't
By submultiplicativity, want to show there is no \( f \) with \( \text{supp} f \subset X_k \) \( \text{supp} f \subset Y_k \).

General Q: Which sets \( S,T \) arise as

\[ S = \text{supp} f \quad T = \text{supp}^\wedge f \]

Generically, expect \(|S| + |T| \geq N^2\) (= \(|Z_N^2|\))

Donoho - Stark: \(|S| \cdot |T| \geq N^2\)

If \( S,T \) lack linear structure, improvement.

\[ S = \{(+,0)\} \]
\[ T = \{(0,+}\} \]
Strategy

If $X$ has no lines, \( X \) avoids lines in a qualitative sense.

If \( \text{supp} f \) avoids all lines, then \( \text{supp} f \) is large.

Or: If \( \text{supp} f \) is large, then \( \text{supp} f \) avoids all lines.

\[ \text{Goal: Can't have} \quad \text{supp} f \subset X. \]

Suppose \( X \) contains no lines.

Suppose \( \text{supp} f \) avoids lines.

\( \text{Goal: Can't have} \quad \text{supp} f \subset X. \)

\[ \text{Need a gap on every line.} \]
The vague prop follows from the following

**Main Lemma:** Given \( S \subset \mathbb{Z}_N^2 \), there exists

\[
h(x, y) = \sum_{0 \leq k, l \leq D} a_{k,l} e^{2\pi i k x/N} e^{2\pi i l y/N} \quad \text{supp} \hat{h} \subset [0, D] \times [0, D]
\]

such that \( h \) vanishes on all of \( S \) except a line \( \ell = \mathbb{Z}v + b \), \( v, b \in \mathbb{Z}_N \)

(and \( h \) does not vanish on all of \( S \))
Consider \( g = h f \). Then \( \text{supp } g \subset \text{supp } f + [0, D)^2 \).

Some dual line \( \alpha \) has full support. So
\[
\alpha = \text{supp } \hat{g} f = \text{supp } \hat{f} + \text{supp } \hat{g} = \text{supp } \hat{f} + [0, D) \times [0, D)
\]

So \( X \) is dense along the line \( \alpha \). \( \Box \)
Q: What is the obstacle to the main lemma?

A: Polynomials passing through many cyclotomic points.

\[ F(z,w) = z^2 + 4z w + w + 1 = 0 \]

\[ Z_N(F) = \{(x,y) \in \mathbb{Z}_N^2 : F\left(e^{\frac{2\pi i x}{N}}, e^{\frac{2\pi i y}{N}}\right) = 0\}. \]

\[
\text{Bezout:} \quad Z_N(F) \cap Z_N(G) \leq \deg F \cdot \deg G
\]

A choice of \( h \) has \( Z_N(F) \cap Z_N(h) \sim |Z_N(F)| \).

If \( |Z_N(F)| > 1000 \Rightarrow \deg h > \frac{1000}{2} \Rightarrow \sqrt{1000} \) desired bound
Given $S \subset \mathbb{Z}/N^2$, exists

$$F = \sum a_{k,e} z_k w^e, \quad S \subset \mathbb{Z}/N(F), \quad D \leq \sqrt{|S|}$$

$D^2$ variables $a_{k,e}$

Theorem (Ruppert, Beukers & Smyth)

Suppose $F = \sum a_{k,e} z_k w^e$ irreducible ($F \neq GH$)

Then either:

1. $|\mathbb{Z}_N(F)| \leq 22D^2$

   $$= \#\{(x,y) \in \mathbb{Z}_N^2 : F(e^{\frac{2\pi i x}{N}}, e^{\frac{2\pi i y}{N}}) = 0\}$$

2. $F = z^a - e^{\frac{2\pi i c}{N}} w^b$ or $F = z^a w^b - e^{\frac{2\pi i c}{N}}$

So $\mathbb{Z}_N(F) = \{(x,y) \in \mathbb{Z}_N^2 : ax + by = c\}$ = a line!

This is a quantitative form of Lang's conjecture from number theory.
PF of main lemma: the obstruction we pointed out is the only one can’t happen b.c Beckers & Smyth

$h = 1$, $S_0 = S < \mathbb{Z}_N^2$

while $S_K$ doesn’t lie on a line & $|S_K| > 200$ (case 1.2)

• $F = \sum_{0 \leq k \leq D} q_k z^k w^l$, D minimal

(4) If $F = GH$ (reducible), $h := h \cdot H$, $S_{K+1} = S_K \cdot \mathbb{Z}_N(H)$, repeat

(3) Otherwise, because $S_K$ doesn’t lie on a line, $\deg F = \sqrt{|S_K|}$ BY THM.

Let $H$ have $\deg < \deg F$. Can pass through $\geq c |S_K| \text{ pts}$. And not all of $S_K$, b.c $\deg F$ minimal.

$h := h \cdot H$, $S_{K+1} = S_K \cdot \mathbb{Z}_N(H)$
To Recap:

**Thm:** Cantor sets $X_k, Y_k$ have a FUP unless there's a pair of orthogonal lines.

- **Submultiplicativity:** impossible for $\text{supp } f \subseteq X_k$
- $\text{supp } f \subseteq Y_k$

Suppose for simplicity $Y_k$ has no lines (so $X_k \neq Y_k$ has a FUP for all $Y_k$).

- If $\text{supp } f \subseteq X_k$, then $|\text{supp } f| \leq \frac{c}{n^2} \gg |Y_k|$
  \[\text{b.c. it avoids all lines}\]

Prove this by a multiplier argument, using

**Lemma:** $S \subseteq \mathbb{Z}^2$ low degree poly vanishing on $S \setminus I$

Lemma is true because quantitative long can.

Support, Beukers-Smyth: Poly's can't pass though a expected number of $\mathbb{Z}^2$ pts, unless it's a line.
Challenge for continuous case of $\mathbb{R}^2$

Looking at zero locus of $F(z, w)$

Not good enough. Need to control $|F(z, w)|$ for all $z, w$

Much harder!

Key ingredient for Bourgain & Dyatlov 1D FUP:

Let $w: [0, 2] \to \mathbb{R}$ be

$$ (1) \quad \int \frac{w(x)}{1 + x^2} \, dx < \infty, \quad (2) \quad w \text{ is regular enough} $$

$$ \|w\|_{L^2}, \quad \|H\|_{L^2} \leq C $$

Then

$$ \sum_{0 \leq k \leq 100R} \frac{x^{2k}}{k} $$

$$ z = c \frac{\log T}{t} $$

$$ w = \ldots $$

$$ T = \ldots $$
Idea: Write $T$ in terms of roots

$$T(z) = \frac{100R}{\prod (z - \alpha_j)}$$

$$= \prod \left( 1 - e^{-\frac{2\pi i}{R} \alpha_j} \right)^{ \frac{1}{2} }$$

$$\log |T(x)| = \log c + \sum \log \left| e^{\frac{2\pi i}{R} \alpha_j} - e^{-\frac{2\pi i}{R} \alpha_j} \right|$$

$$= \log c + \sum \log 4 \sin^2 \left( \frac{2\pi x}{R} \right)$$

$$\left( \log \left( \frac{\sin^2 \left( \frac{2\pi x}{R} \right)}{\sqrt{\sin^2 \left( \frac{2\pi x}{R} \right) - \frac{1}{2}}} \right) \right)(K) = \frac{1}{144}$$

$$\left( \log \left( \sin^2 \left( \frac{2\pi x}{R} \right) \right) \right), \frac{2\pi i}{R} \alpha_j$$

Wait $\omega \sim \Delta^2 \nu$. Set $\nu \sim \Delta^2 \omega = -\frac{1}{2} \Delta \rho$
$E_x$

$\psi = \frac{1}{x^2}$ tails

Very delicate problem of choosing location of zeros.

Norm is $\| \Delta^{-\frac{1}{2}} \|_{L^2}$, super hard to work with.

Explicit formula $p = \Delta^{-\frac{1}{2}} \omega$ is essential

Can we find a more robust proof?