Fractal Uncertainty for Discrete 2D Cantor Sets

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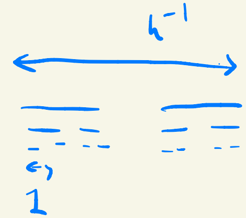
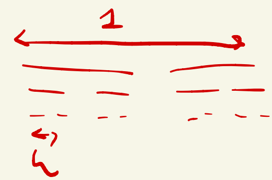
FUP = Fractal Uncertainty Principle

Thm (Bourgain & Dyatlov):

Suppose $f \in L^2(\mathbb{R})$, Supp $f \subset X$ =



Then $\|f 1_Y\|_2 \lesssim h^\beta$ Y =



some $\beta > 0$

- Developed for applications to quantum chaos (spectral gaps, mass of eigenfunctions)

- Major problem: extend to 2D

- We study a simpler discrete model

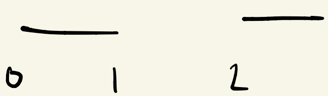
Fix an integer M.

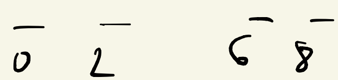
Let $\mathcal{A} \subset \mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$ be an alphabet

$$X_K \subset \mathbb{Z}_{M^K} = \mathbb{Z}_N \quad (N := M^K)$$

$$= \{a_0 + a_1 M + \cdots + a_{K-1} M^{K-1} : a_j \in \mathcal{A}\}$$

<u>Ex</u> $M = 3$, $\mathcal{A} = \{0, 2\}$

$X_1 =$ 

$X_2 =$ 

$\vdots$

These are discrete cantor sets

$$|\mathcal{A}| = M^\delta \implies |X_K| = M^{K\delta} = N^\delta$$

$\delta < 1$ is the dimension

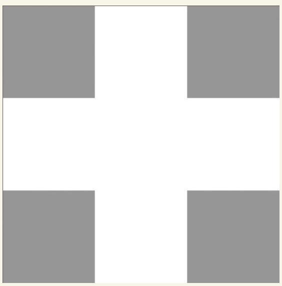
$A \subset \mathbb{Z}_M \times \mathbb{Z}_M$. $\quad |A| = M^\delta, \; 0 < \delta < 2.$

$X_k \subset \mathbb{Z}_N \times \mathbb{Z}_N \qquad N = M^k$

$$= \left\{ \left(a_0 + a_1 M + \cdots + a_{k-1} M^{k-1},\; b_0 + b_1 M, \ldots, b_{k-1} M^{k-1}\right) : (a_j, b_j) \in A \text{ for all } j \right\}$$

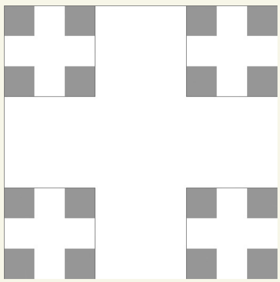
Ex

A



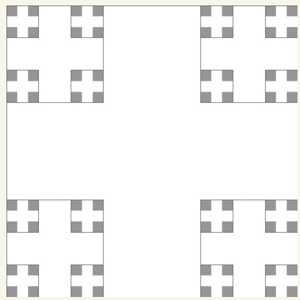
$\mathbb{Z}_3^2$

X_2

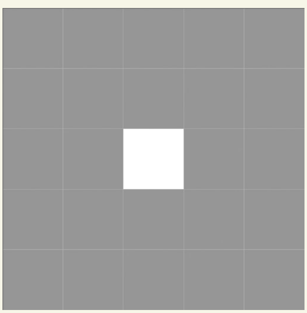


$\mathbb{Z}_9^2$

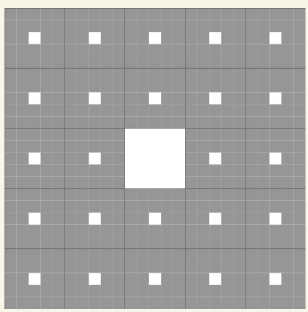
X_3



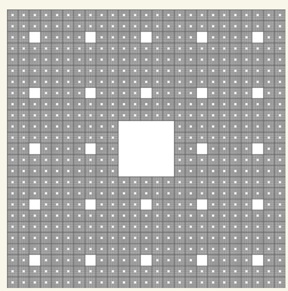
$\mathbb{Z}_{27}^2$



$\mathbb{Z}_5^2$



$\mathbb{Z}_{25}^2$



$\mathbb{Z}_{125}^2$

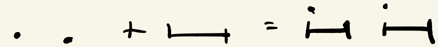
Main result: classify which 2D cantor sets have a FUP

Fourier transform $\mathcal{F} : L^2(\mathbb{Z}/N\mathbb{Z}) \to L^2(\mathbb{Z}/N\mathbb{Z})$

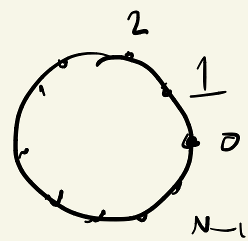
$$\hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}_N} f(x) e^{-\frac{2\pi i}{N} x \cdot k}, \quad f(x) = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \hat{f}(k) e^{\frac{2\pi i}{N} x \cdot k}$$

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \quad \widehat{(f * g)} = \hat{f}\hat{g}$$

$\hookrightarrow \operatorname{supp} \widehat{fg} \subset \operatorname{supp} \hat{f} + \operatorname{supp} \hat{g}$



If $z = e^{2\pi i \frac{x}{N}}$,



$$f(x) = \frac{1}{\sqrt{N}} \sum_k \hat{f}(k) z^k$$

Think of $f : S^1 \to \mathbb{C}$ as a trigonometric polynomial (in the variable $z = e^{2\pi i \frac{x}{N}}$)

$A \subset \mathbb{Z}_M$ $\quad$ $B \subset \mathbb{Z}_M$ $\quad$ $N := M^k$

$X_k \subset \mathbb{Z}_N$ $\quad$ $Y_k \subset \mathbb{Z}_N$

Thm (Dyatlov-Jin)

If $\operatorname{supp} f \subset X_k$, $\|\hat{f}\, 1_{Y_k}\|_2 \lesssim M^{-k\beta}$

or: $\|1_{Y_k} \mathcal{F} 1_{X_k}\|_{2\to 2} \lesssim M^{-k\beta}$ some $\beta > 0$

Submultiplicativity:

$$\|1_{Y_{k+r}} \mathcal{F} 1_{X_{k+r}}\|_{2\to 2} \leq \|1_{Y_k} \mathcal{F} 1_{X_k}\|_{2\to 2} \|1_{Y_r} \mathcal{F} 1_{X_r}\|_{2\to 2}$$

Suffices to show $\|1_{Y_k} \mathcal{F} 1_{X_k}\|_{2\to 2} < 1$ for some k.

Goal: For some k, impossible for $\operatorname{supp} f \subset X_k$, $\operatorname{supp} \hat{f} \subset Y_k$

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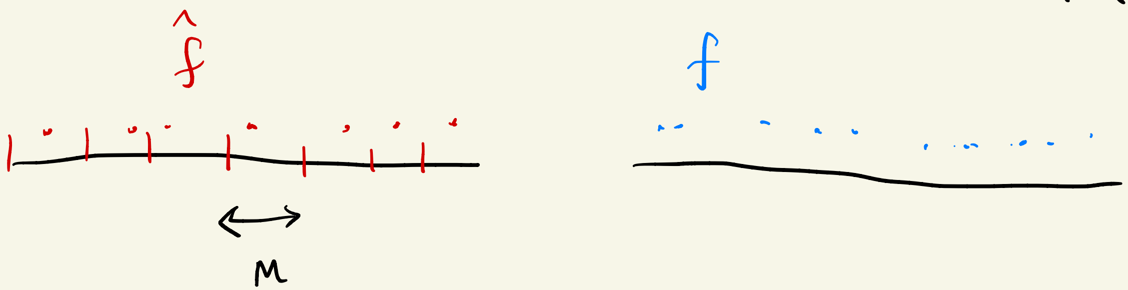
(very special to the discrete case!)

Prop: $f \in L^2(\mathbb{Z}_N)$

If $|\operatorname{supp} f| = M$, then $\operatorname{supp} \hat{f} \cap [a, a+M)$ nonempty for all a



Pf of FUP assuming prop:

If $\operatorname{supp} f \subset X_k$, then $|\operatorname{supp} f| \le M^{kf}$

If $\operatorname{supp} \hat{f} \subset Y_k$, then $\operatorname{supp} \hat{f}$ has gaps of size $\frac{N}{M}$ ∎

Or: If $\operatorname{supp} f$ has a large gap, then $|\operatorname{supp} \hat{f}|$ is large

Pf of prop

$\operatorname{supp} f = \{x_1, \dots, x_m\}$

Let $$h\left(z = e^{\frac{2\pi i}{N} x}\right) = \prod_{j=0}^{m-1} \left(z - e^{\frac{2\pi i}{N} x_j}\right) = \sum_{k=0}^{m-1} a_k z^k$$

(1) $fh = c\,\delta_{x_m}$

$\widehat{fh}$ has full support

(2) $\operatorname{supp} \hat{f} \subset [0, m)$

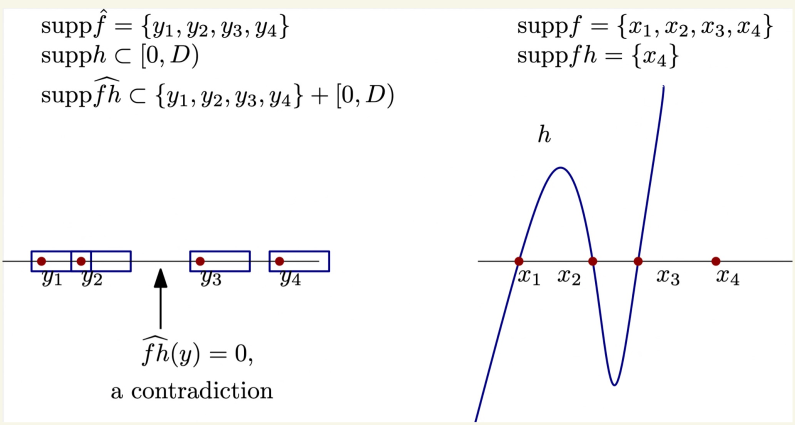
b.c $\hat{f}(k) = \sqrt{N}\, a_k$.

Because $\widehat{fh} = \hat{f} * \hat{h}$, $\operatorname{supp} \widehat{fh} \subset \operatorname{supp} \hat{f} + \operatorname{supp} \hat{h}$

$= \operatorname{supp} \hat{f} + [0, m)$

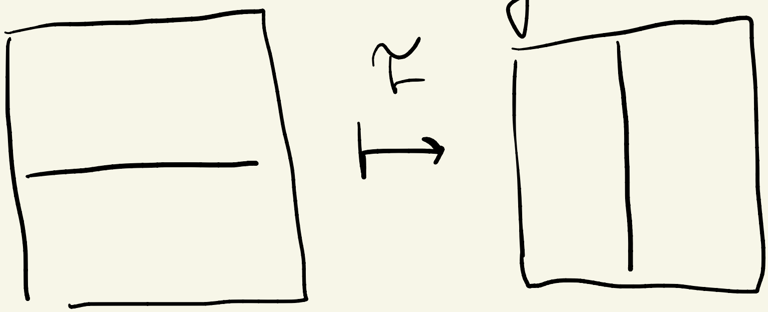
"m-neighborhood"

$\operatorname{supp} \hat{f} + [0, m) = \mathbb{Z}/N\mathbb{Z} \Rightarrow \hat{f}$ has support in every $[a, a+m)$

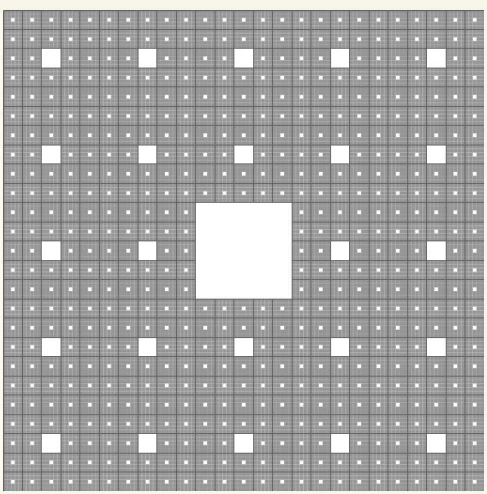


Moving on to 2D...

$$\mathcal{F} 1_{x=0} = 1_{y=0}$$



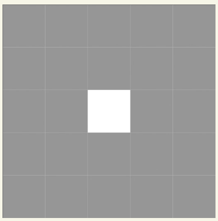
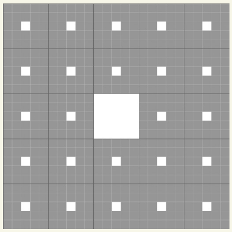
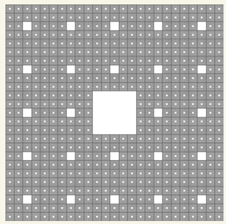
If $\{(t,0) : t \in \mathbb{Z}_N\} \subset X_K$ then no FUP!
$\{(0,t) : t \in \mathbb{Z}_N\} \subset Y_K$



This "pair of orthogonal lines" is the only obstruction!

Let X_k, Y_k be Cantor sets

$$\underline{X}, \underline{Y} \subset \mathbb{T}^2 \quad \text{the drawings}$$

 $\cap$  $\cap$  $\cap \cdots = \underline{X}$

<u>Thm</u> As long as $\underline{X}$, $\underline{Y}$

don't contain a pair of orthogonal lines

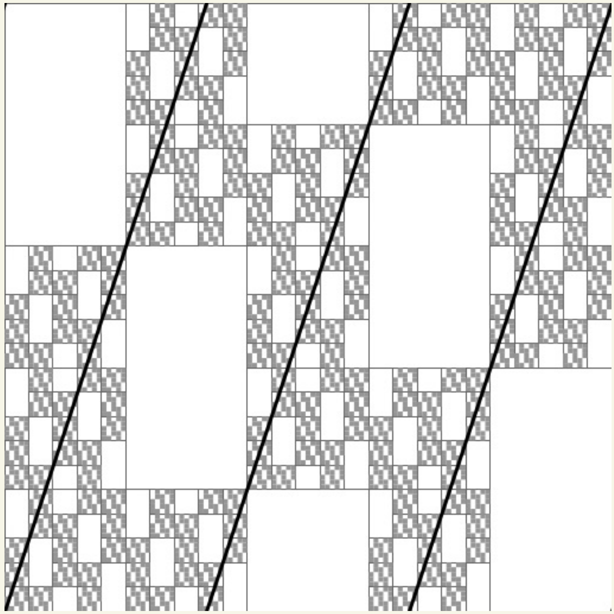
$$\mathbb{R}v + p \subset \underline{X}, \quad \mathbb{R}v^{\perp} + q \subset \underline{Y}$$

then

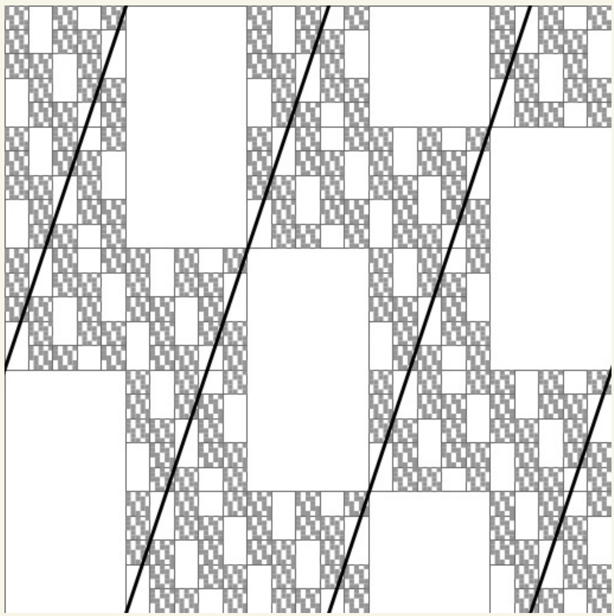
$$\| 1_{Y_k} F 1_{X_k} \|_{2\to 2} \lesssim M^{-k\beta}$$

some $\beta > 0$.

This cantor set contains a line



This one doesn't



By submultiplicativity, want to show

there is no f with

$$\operatorname{supp} f \subset \mathcal{X}_K \qquad \operatorname{supp} \hat{f} \subset \mathcal{Y}_K$$

General Q: which sets S, T

arise as

$$S = \operatorname{supp} f \qquad T = \operatorname{supp} \hat{f}$$

Generically, expect $|S| + |T| \geq N^2 \quad \left(= |\mathbb{Z}_N^2|\right)$

Donoho-Stark: $|S| \cdot |T| \geq N^2$

$S = \{(t, 0)\}$

$T = \{(0, t)\}$

linear structure

If S, T lack linear structure, improvement.

Goal: Can't have

$$\operatorname{supp} f \subset X_K$$
$$\operatorname{supp} \hat{f} \subset Y_K$$

Suppose $\underline{X}$ contains no lines

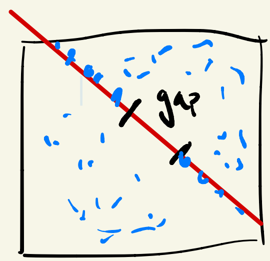
Vague prop: Suppose

$|\operatorname{supp} f|$ is small. Then

$\operatorname{supp} \hat{f}$ is dense along some line.

Or: If $\operatorname{supp} f$ avoids all lines,

then $|\operatorname{supp} \hat{f}|$ is large



need a gap on every line

stronger than "not too many points on a line"

Strategy

$\underline{X}$ has no lines $\Rightarrow X_K$ avoids lines in a quantitative sense $\Rightarrow$ If $\operatorname{supp} f \subset X_K$,

$|\operatorname{supp} \hat{f}| \sim cN^2$

$\gg |Y_K|$ □

The vague prop follows from the following

Main Lemma: Given $S \subset \mathbb{Z}_N^2$, there exists

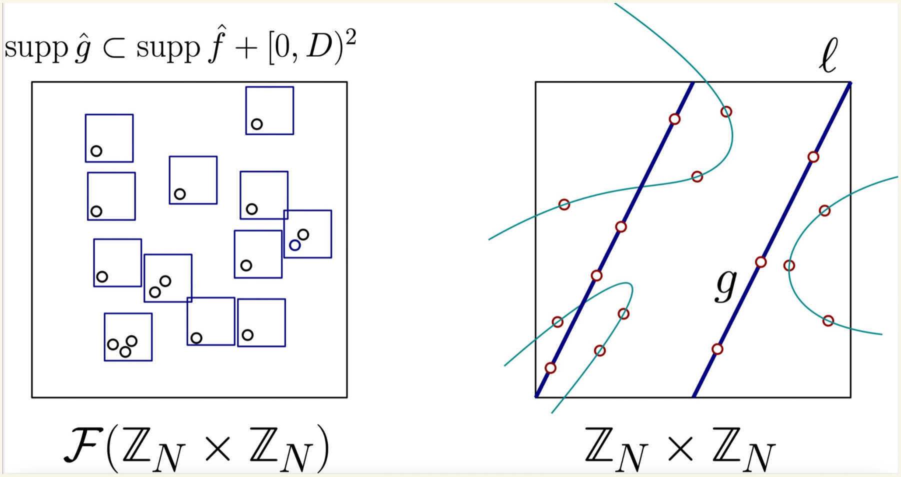
$$h(x,y) = \sum_{0 \le k, l \le D} a_{kl} z^k w^l \qquad \text{supp}\, \hat{h} \subset [0,D] \times [0,D]$$

$$z = e^{\frac{2\pi i}{N} x}, \quad w = e^{\frac{2\pi i}{N} y}$$

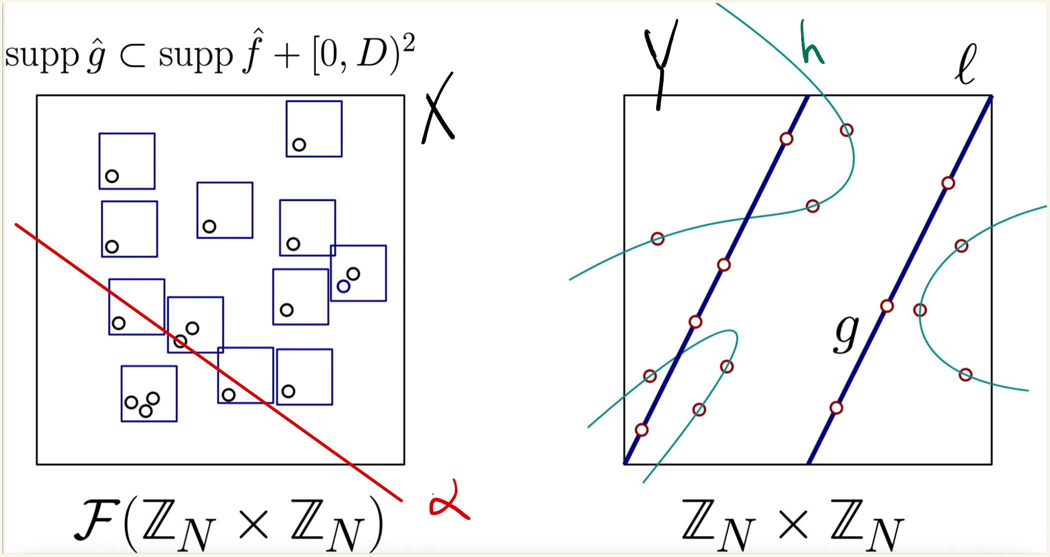
(trigonometric polynomial)

such that h vanishes on all of S except a line $\ell = \mathbb{Z} v + b$, $v, b \in \mathbb{Z}_N^2$

(and h does not vanish on all of S)



Vague proof of vague prop



Consider $g = hf$. Then $\operatorname{supp} g \subset \ell$,

$\operatorname{supp}\hat{h} \subset [0,D) \times [0,D)$

so $|\hat{g}| = \text{const.}$ on dual lines α

Some dual line α has full support. So

$$\alpha \subset \operatorname{supp}\widehat{gf} = \operatorname{supp}\hat{f} + \operatorname{supp}\hat{g} = \operatorname{supp}\hat{f} + [0,D) \times [0,D)$$

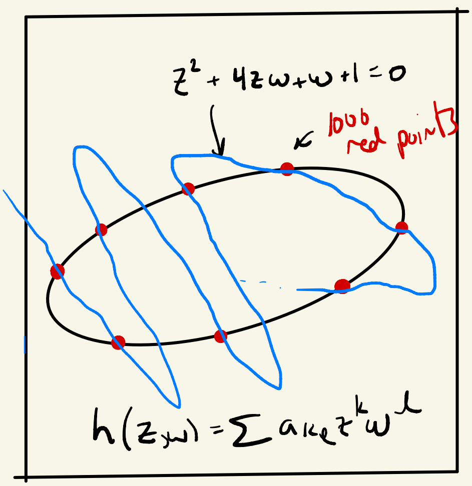
$= D$-nbhd of $\operatorname{supp}\hat{f}$

So X is dense along the line α $\square$

Q: What is the obstacle to the main lemma?

A: Polynomials passing through many cyclotomic points.

$$F(z,w) = z^2 + 4zw + w + 1 = 0$$



$$Z_N(F) = \left\{(x,y) \in \mathbb{Z}_N^2 : F\left(e^{\frac{2\pi i}{N}x}, e^{\frac{2\pi i}{N}y}\right) = 0\right\}.$$

Bezout:

$$Z_N(F) \cap Z_N(G) \leq \deg F \cdot \deg G$$

A choice of h has $Z_N(F) \cap Z_N(h) \sim |Z_N(F)|$

If $|Z_N(F)| \sim 1000 \implies \deg h \geq \frac{1000}{2} >> \sqrt{1000}$

↑ desired bound

Given $S \subset \mathbb{Z}_N^2$, exists

$$F = \sum_{0 \le k,l \le D} a_{kl} z^k w^l, \quad S \subset Z_N(F), \quad D \le \sqrt{|S|}$$

(D^2 variables a_{kl}) ($|S|$ lin. eqs. in a_{kl})

<u>Theorem</u> (Ruppert, Beukers & Smyth)

Suppose $F = \sum_{0 \le k,l \le D} a_{kl} z^k w^l$ irreducible ($F \ne GH$)

Then either:

(1) $|Z_N(F)| \le 22 D^2$

$$\left(= \#\{(x,y) \in \mathbb{Z}_N^2 : F(e^{\frac{2\pi i}{N}x}, e^{\frac{2\pi i}{N}y}) = 0\} \right)$$

(2) $F = z^a - e^{\frac{2\pi i}{N}c} w^b$ or $F = z^a w^b - e^{\frac{2\pi i}{N}c}$,

So $Z_N(F) = \{(x,y) \in \mathbb{Z}_N^2 : ax + by = c\} = \ell$

a line!!

This is a quantitative form of Lang's conjecture from number theory

Pf of main lemma: the obstruction we pointed out is the only one

· can't happen b.c Beukers & Smyth

$h = 1$, $S_0 = S \subset \mathbb{Z}_N^2$

while S_k doesn't lie on a line & $|S_k| > 200$ (cases 1+2)

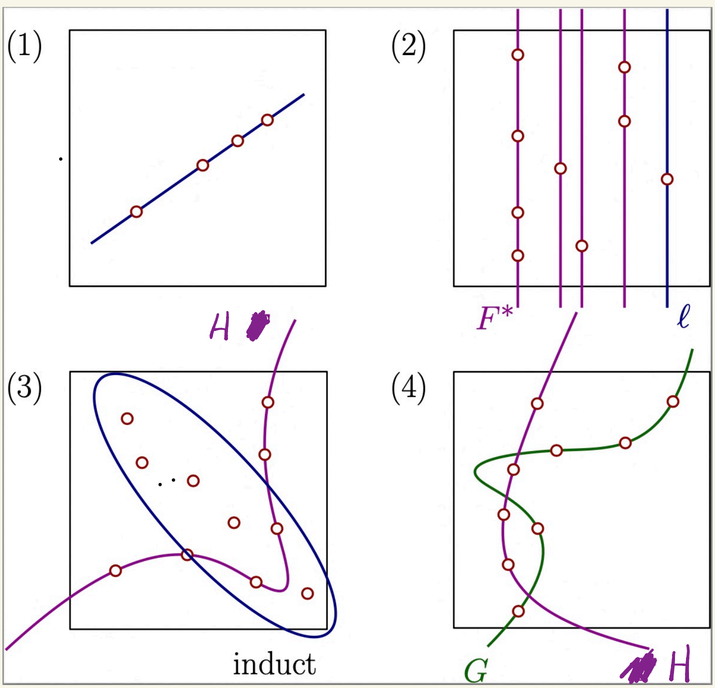
- $F = \sum_{0 \le k, l \le D} a_{kl} z^k w^l$, D minimal

(4) If $F = GH$ (reducible), $h := h \cdot H$, $S_{k+1} = S_k \setminus Z_N(H)$, repeat

(3) Otherwise, because S_k doesn't lie on a line, $\deg F \ge \sqrt{\frac{|S_k|}{22}}$ BY THM.

Let H have $\deg < \deg F$. Can pass through $\ge c|S_k|$ pts. And not all of S_k, b.c $\deg F$ minimal.

$h := h \cdot H$, $S_{k+1} = S_k \setminus Z_N(H)$



To Recap:

Thm: Cantor sets X_k, Y_k have a FUP unless there's a pair of orthogonal lines.

- Submultiplicativity: impossible for: $\operatorname{supp} f \subset X_k$

  $\operatorname{supp} \hat{f} \subset Y_k$

suppose for simplicity X_k has <u>no</u> lines (so X_k, Y_k has a FUP for all Y_k).

- If $\operatorname{supp} f \subset X_k$, then $|\operatorname{supp} \hat{f}| \sim cN^2 >> |Y_k|$

  ↑ b.c it avoids all lines

Prove this by a multiplier argument, using

Lemma: $S \subset \mathbb{Z}_N^2$, $\exists$ low degree poly vanishing on $S \setminus \ell$

Lemma is true because quantitative Lang conj.

Ruppert, Beukers & Smyth: Poly's can't pass through $\gtrsim$ expected number of $\mathbb{Z}_N^2$ pts, unless it's a line.

Challenge for continuous case of $\mathbb{R}^2$

looking at zero locus of $F(z,w)$ not good enough. Need to control

$$|F(z,w)| \text{ for all } z,w$$

Much harder!

Key ingredient for Bourgain & Dyatlov 1D FUP:

Let $\omega : [0,R] \to \mathbb{R}$ be

(1) $\displaystyle\int \frac{\omega(x)}{1+x^2}\,dx < \infty$,

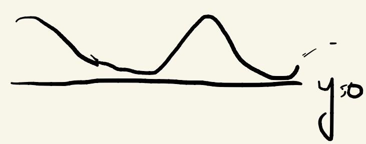
(2) ω is regular enough

$$\|\omega'\|_{L^\infty}, \|H[\omega']\|_{L^\infty} \le C$$

then $\exists\, T = \displaystyle\sum_{0 \le k \le 100R} a_k z^k$, $\quad \left|\omega - \log\left|T\left(e^{\frac{2\pi i}{R}x}\right)\right|\right| < C.$

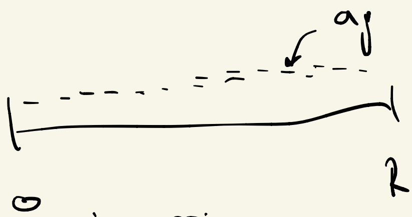
$z = e^{2\pi i \frac{x}{R}}$

$\omega =$ 

$T =$ 

<u>Idea:</u> write T in terms of roots

$$T(z) = c\prod_{j=1}^{100R}(z-\alpha_j), \quad \alpha_j = e^{\frac{2\pi i}{R}a_j}$$



$$\log|T(x)| = \log c + \sum_j \log\left|e^{\frac{2\pi i}{R}x} - e^{\frac{2\pi i}{R}a_j}\right|$$

$$= \log c + \sum_j \log 4\sin^2\left(2\pi\,\frac{x-a_j}{R}\right)$$

$$= \log c + \left(\log \sin^2\left(2\pi\frac{x}{R}\right)\right) * \left(\sum \delta_{a_j}\right)$$

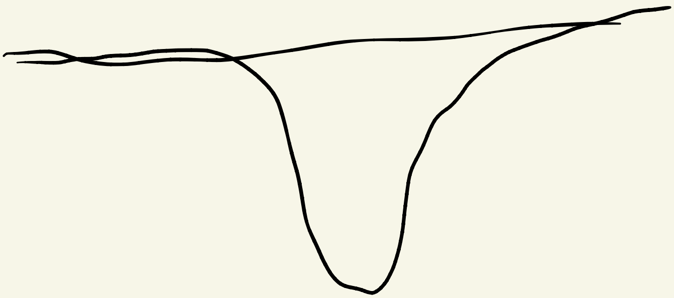
$$\widehat{\log\left(\sin^2\left(2\pi\frac{x}{R}\right)\right)}(k) = \frac{1}{|k|}$$

$$(\log\text{'' \quad ''}) * \rho = \sum_k \frac{1}{|k|}\hat{\rho}(k)\, e^{2\pi i\frac{k}{R}x} = \Delta^{-\frac{1}{2}}\rho$$

Want $\omega \sim \Delta^{-\frac{1}{2}}v$. Set $v \sim \Delta^{1/2}\omega = -H[\omega']$

(v: discrete set of points; $-H[\omega']$: continuous distribution)

Ex

$\omega =$ 

$\rho = H[\omega'] =$ 

__Very__ delicate problem of choosing location of zeros.

Norm is $\|\Delta^{-1/2}\rho\|_{L^\infty}$, super hard to work with.

Explicit formula $\rho = \Delta^{1/2}\omega$ is __essential__

Can we find a more robust proof?