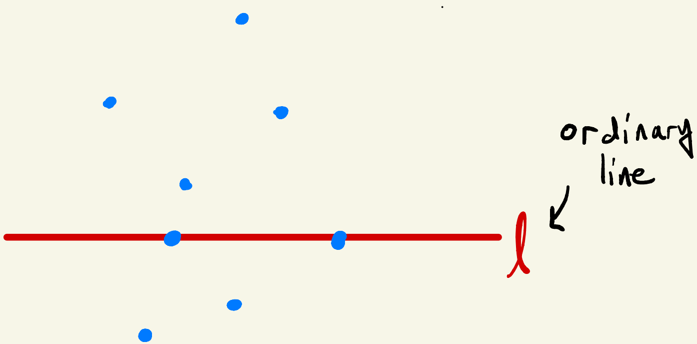
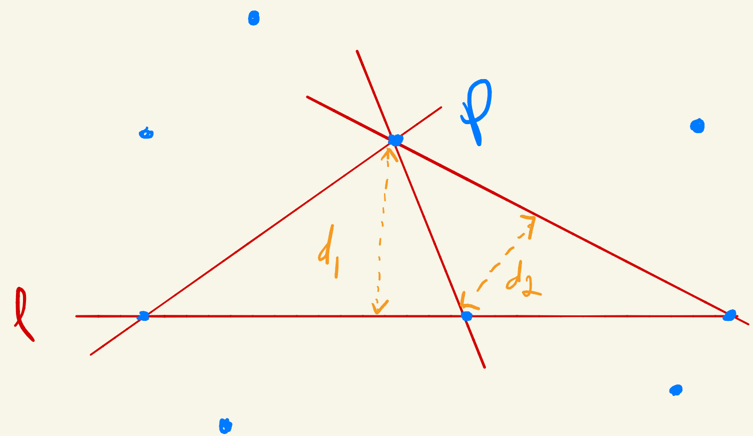
A Sylvester–Gallai theorem for lines in $\mathbb{C}^2$

Alex Cohen

<u>Thm</u> (Sylvester–Gallai): Let $S \subset \mathbb{R}^2$ be a finite set of points not all on one line. There exists a line ℓ passing through exactly 2 points of S.



<u>Pf</u> Let (p, ℓ) minimize distance. Then ℓ is ordinary.

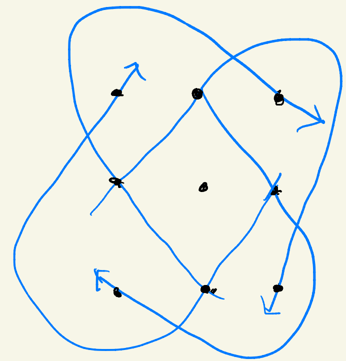


Sylvester-Gallai originated from recreational math...

renewed interest in related questions as discrete geo flourishes

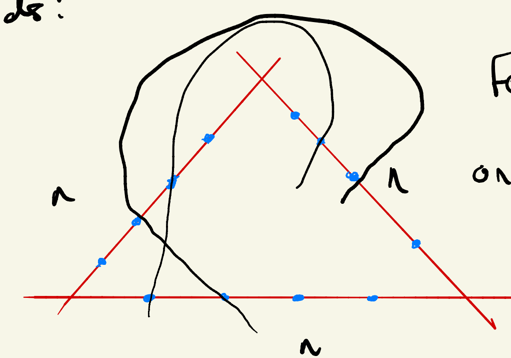
"Sylvester-Gallai configuration"

- How few ordinary lines can you have? (Green & Tao)
- S-G for circles, curves, etc..
- ☆ Over complex numbers, other fields?



Hesse configuration

9 inflection points of an elliptic curve



Fermat configurations on $3n$ points

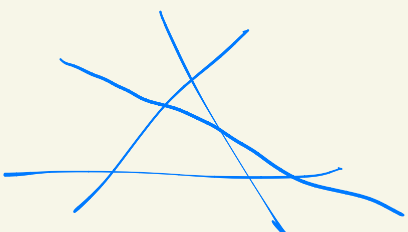
S-G fails over $\mathbb{C}^2$!

Q: We only know of a few examples of complex S-G configurations---

- Can we classify them?
- Rule out special cases?

Main difficulty: Topology of $\mathbb{C}^2$ much more complicated than $\mathbb{R}^2$.

$\mathbb{R}^2$

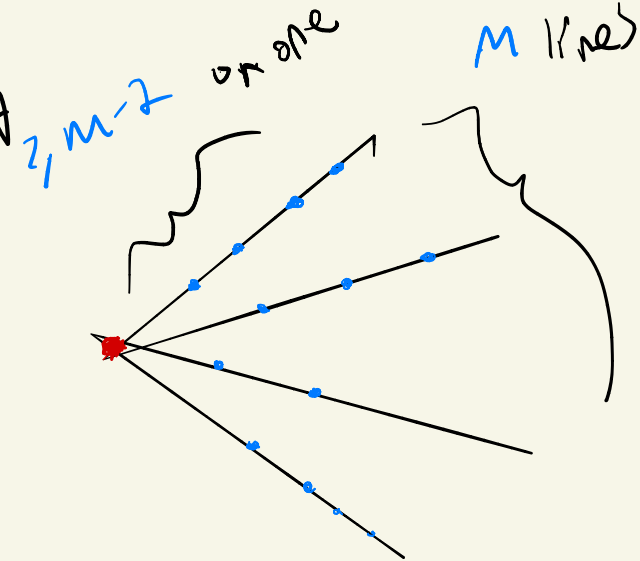


ℓ: 1D in 2D

$\mathbb{C}^2$

ℓ: 2D in 4D

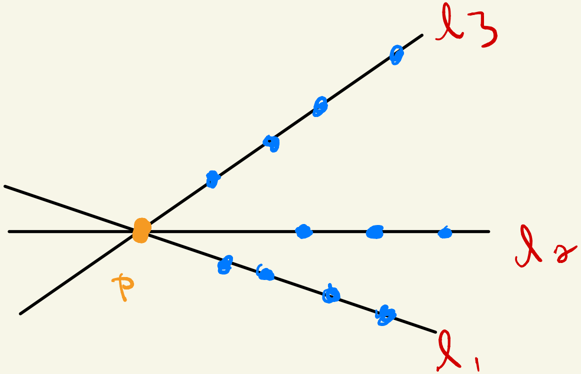
Main Thm: $S \subset \mathbb{C}^2$ lies on a family of m concurrent lines. If some line contains $\geq m-2$ points... then $\exists$ an ordinary line.



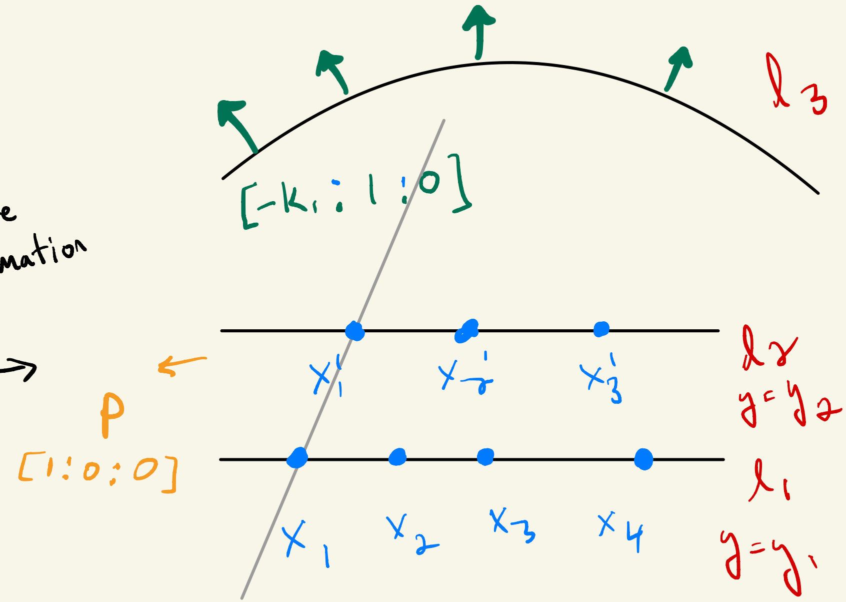
- This result is sharp
- Very strong hypothesis
- Comes up in some applications (answers a Conjecture of Frank de Zeeuw)
- Not many results of this type
- In this special case, we can use the topology of $\mathbb{C}$ rather than $\mathbb{C}^2$

Pf

$m=3$



projective transformation



$\mathrm{Re}(x_1) < \mathrm{Re}(x_2) < \mathrm{Re}(x_3) < \mathrm{Re}(x_4)$

Lines through $[-k:1:0]$: $x+ky=d$

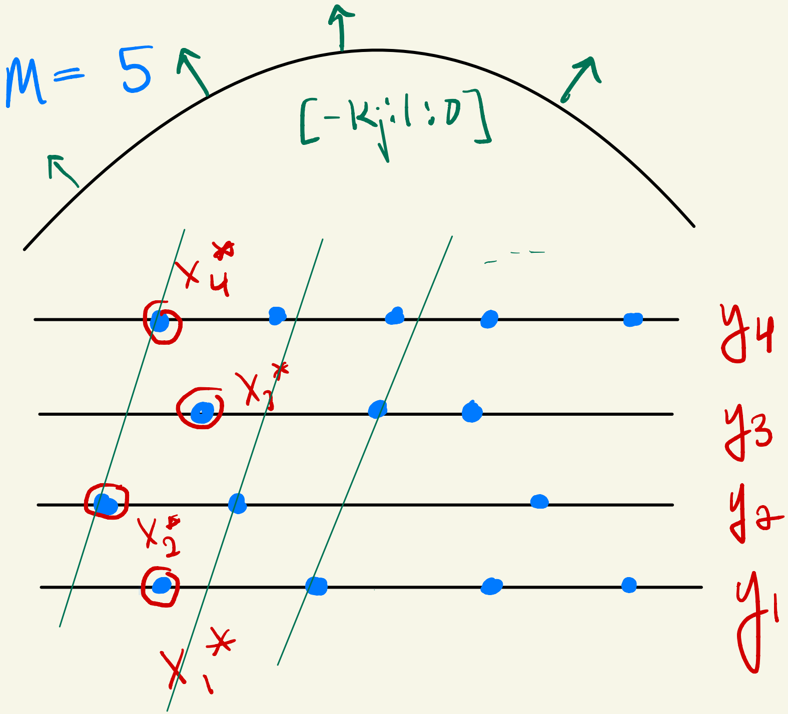
For fixed k, minimize $\mathrm{Re}(d)$

$$x_1+ky_1 = x_1' + ky_2 = d$$

$\Rightarrow k = -\dfrac{x_1-x_1'}{y_1-y_2}$, 1 pt on ℓ_3 ✓

$\mathrm{Re}(x_1+ky_1) < \mathrm{Re}(x_j+ky_1)$

$j>1$



For every $\bullet = [-k_j : 1 : 0] \in \ell_5$

Lines through $\bullet$ are

$$x + k_j y = d$$

minimize d --- goes through

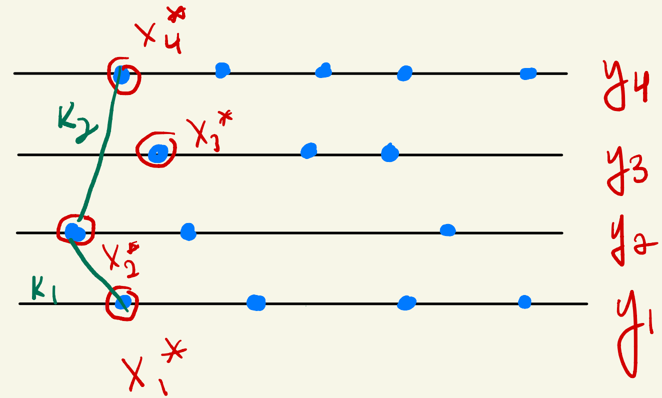
(x_i^*, y_i), (x_j^*, y_j)

$$k = -\frac{y_i - y_j}{x_i^* - y_j^*}$$

$\Longrightarrow \leq \binom{M-1}{2}$ choices for k

I promised $\leq M-2$ choices! why?

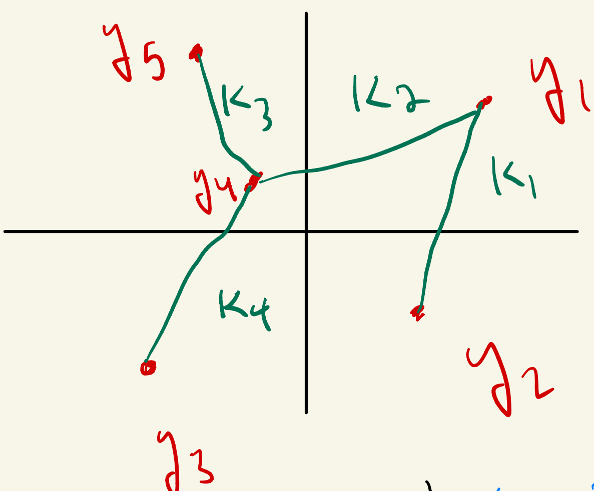
$S = \{(x_1^*, y_1), \dots, (x_{m-1}^*, y_{m-1})\}$



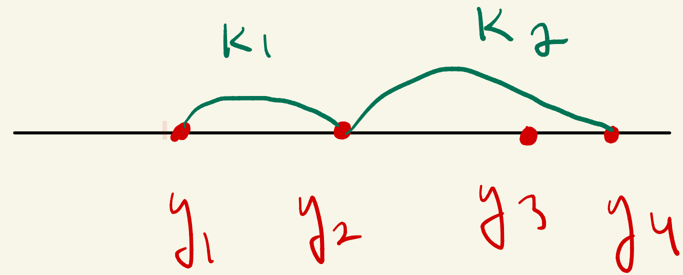
Draw an edge for $k_j \in \ell_m$

If $x + k_j y = \ell$, minimal ℓ, through pair

$\mathbb{C}$



$\mathbb{R}$



A path graph

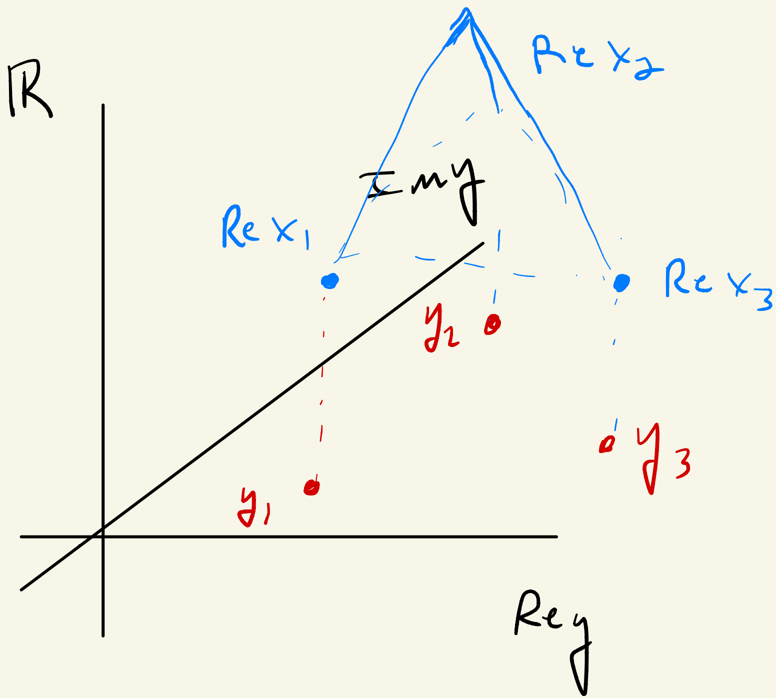
Prop An acyclic graph! $\leq m-2$ edges ✓

A sort of convex hull---

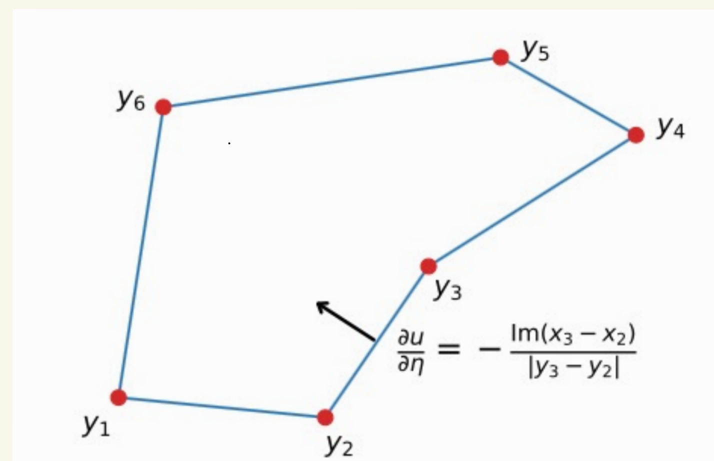
$y_i \sim y_j$ if for k, d s.t.

$$x_i + k y_i = x_j + k y_j = d,$$

$$\mathrm{Re}(x_s + k y_s) > d$$



Convexity $\Rightarrow$ graph is planar



Q:

- Can we prove that a general complex S–G config...
  - Has some line through many pts?
  - Not all lines through 3 pts?
  - Something else?

Thank you Frank de Zeeuw & the Baruch REU