Lower bounds for incidences

10/10/24

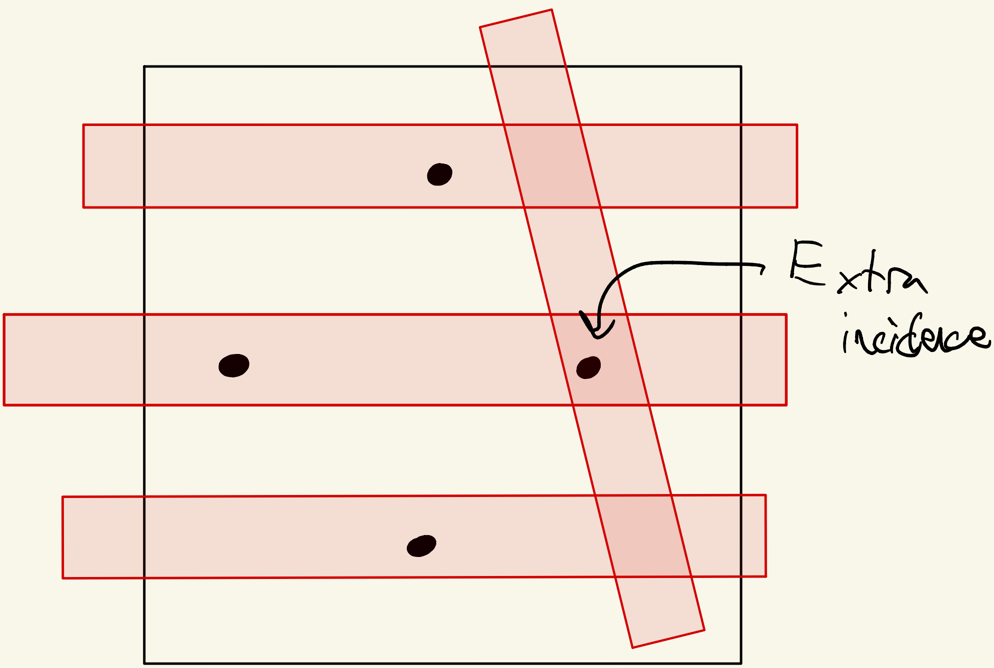
Yale

Main Result

Let $p_1, \dots, p_n \in [0,1]^2$

T_j is a δ-tube through p_j for $j = 1 \dots n$

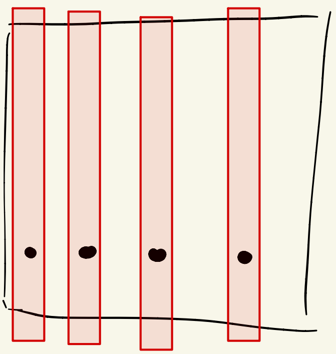
Q How big does 'n' need to be to guarantee there is some extra incidence (EI)?



Easy upper bound: If $n > \delta^{-2}$, exists EI

If $n > \delta^{-2}$, some two points at dist $< \delta$.

Easy lower bound $\exists$ construction w/ $n = \delta^{-1}$ and no EI.



Thm If $n > \delta^{-3/2 - o(1)}$, then there is an extra incidence

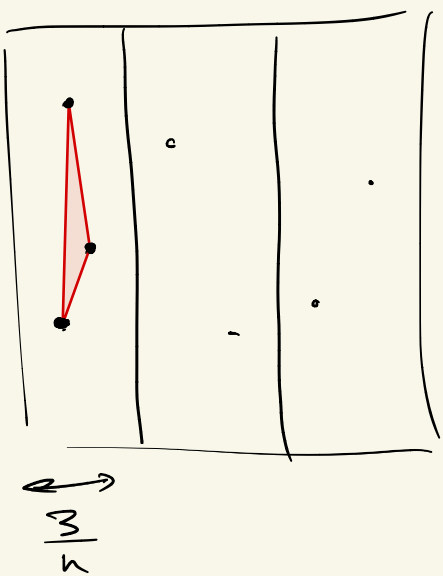
Heilbronn's triangle problem

$$\Delta(n) = \max_{\substack{|P|=n \\ P\subset[0,1]^2}} \left(\text{smallest area triangle formed by 3 pts in } P\right)$$

Rephrase

Q Asymptotic upper bounds for $\Delta(n)$?

Easy bound: $\Delta(n) \lesssim n^{-1}$



Prior work

Easy bound $\quad \Delta \lesssim n^{-1}$

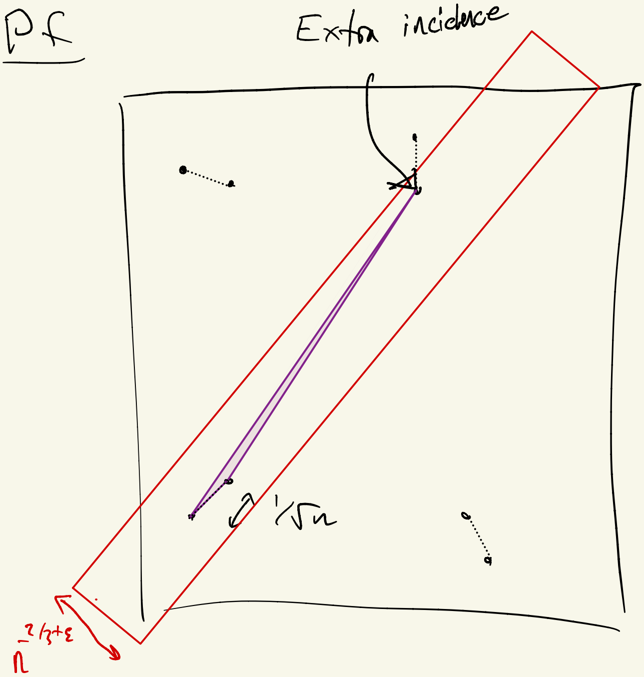
Roth 1972 $\quad \Delta \lesssim n^{-1.1}$

KPS 1981 $\quad \Delta \lesssim n^{-8/7}$

CPZ 2023 $\quad \Delta \lesssim n^{-8/7 - 1/2000}$

Cor In any set of n points in the unit square, three form a triangle w/ area $\Delta \lesssim_{\varepsilon} n^{-\frac{7}{6}+\varepsilon}$

Pf



Form $\sim n$ pairs at dist $\sim 1/\sqrt{n}$

Form a $\delta = n^{-2/3+\varepsilon}$ tube around each point.

Because $n > \delta^{-3/2-\varepsilon}$, there is some

extra incidence

$$\Longrightarrow \Delta = n^{-1/2} \times n^{-2/3+\varepsilon} = n^{-7/6+\varepsilon}.$$

[Note that Heilbronn's problem is generally about incidence lower bounds]

General setup

$P \subset [0,1]^2$ a set of pts

$\mathbb{T}$ a set of δ-tubes

$$I(P,\mathbb{T}) = \#\{(p,T) \in P \times \mathbb{T} : p \in T\}$$

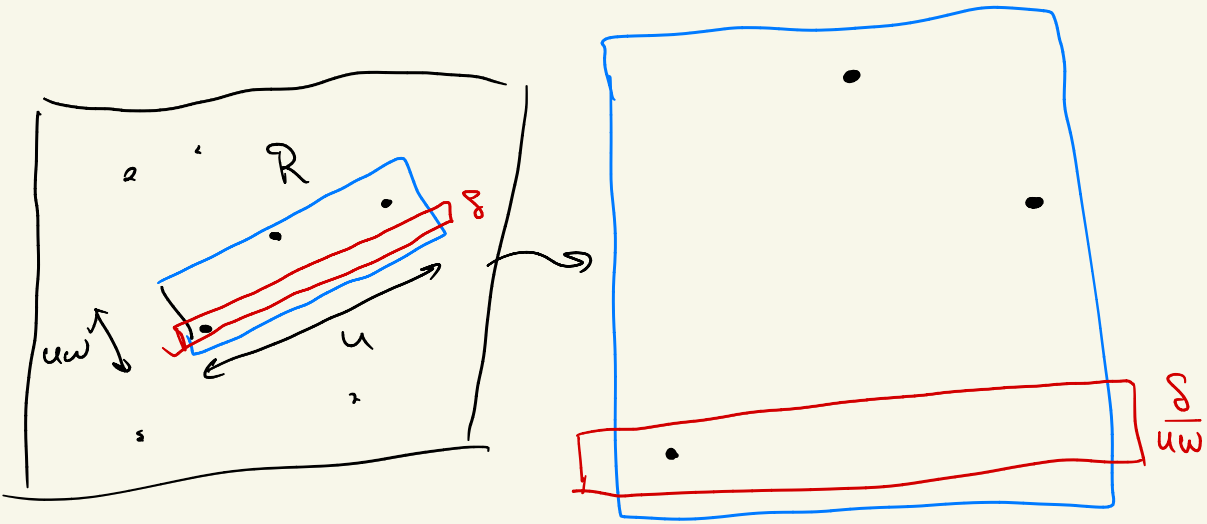
<u>Goal</u> $I(P,\mathbb{T}) > n$

upper bounds vs. lower bounds

Key observation : Rescaling

$$X = \{(p_j, \ell_j)\}_{j=1}^{n}$$

$$X \cap R = \{(p_j, \ell_j) : p_j \in R \quad \ell_j \text{ goes through short ends of } R\}$$



$|X|$ pts

δ-tubes

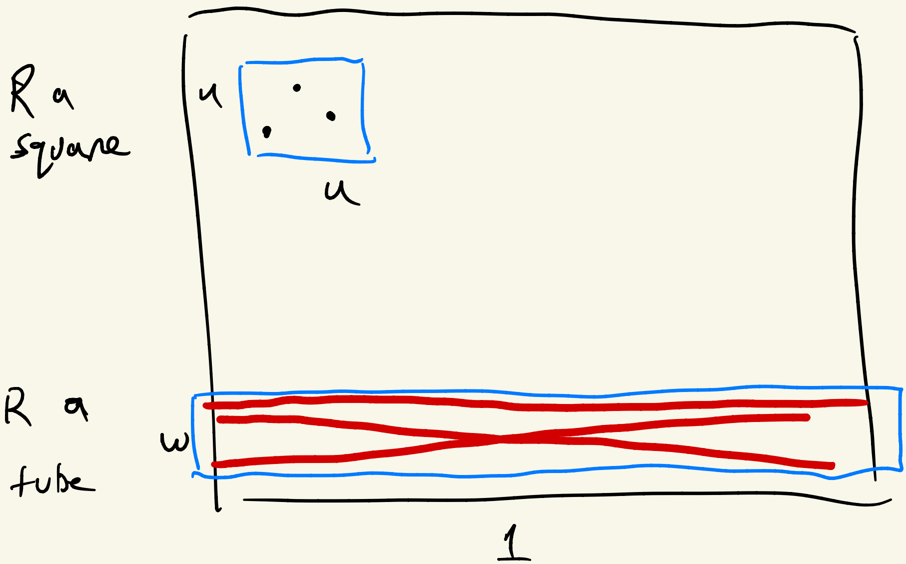
no extra incidence

$|X \cap R|$ pts

δ/uw - tubes

no extra incidence

Special case:



Goal: $|X| \leq \delta^{-\gamma}$ for any $\gamma > 3/2$

Inductive hyp: Goal true for $\delta' \geq 2\delta$

$\Rightarrow$ R is $u \times uw$, $\delta \leq uw < 1/2$

$\star$ $$|X \cap R| \leq \left(\frac{\delta}{uw}\right)^{-\gamma} \leq u^\gamma w^\gamma |X|$$

In particular, Frostman cond

$|P \cap Q| \leq u^\gamma |P|$ for every $u \times u$-square

$|L \cap T| \leq w^\gamma |P|$ for every $1 \times w$-tube

$\star$ Is a Frostman regularity condition over rectangles.

Thm If X satisfies (γ) for $\gamma > 3/2$, then

$$I(\delta\,;\,P[X],\,L[X]) \gtrsim_\varepsilon \delta^{1+\varepsilon}|P[X]|\,|L[X]|$$

$\mathbb{E}$ number if placed unif. at random

$$\text{Thm} \xrightarrow{\text{induction}} \text{Main result on } \mathbb{Q}.$$

The high-low inequality

$$I(w; P, L) = \#\{(p,\ell) \in P \times L : d(p,\ell) \le w\}$$

Thm (High-low)

$$\left| \frac{I(w)}{w|P||L|} - \frac{I(w/2)}{(w/2)|P||L|} \right| \lesssim$$

$$\left(\frac{\max_{Q \text{ a } w\times w \text{ square}} |P \cap Q|}{|P|} \frac{\max_{T \text{ a } 1\times w \text{ tube}} |L \cap T|}{|L|} w^{-3} \right)^{\frac{1}{2}}$$

Ex If P is α-Frostman

L is β-Frostman

$$\mathrm{Err} \lesssim w^{\frac{\alpha+\beta-3}{2}}$$

$\alpha+\beta>3$ in thm

Two steps and incidence lower bounds

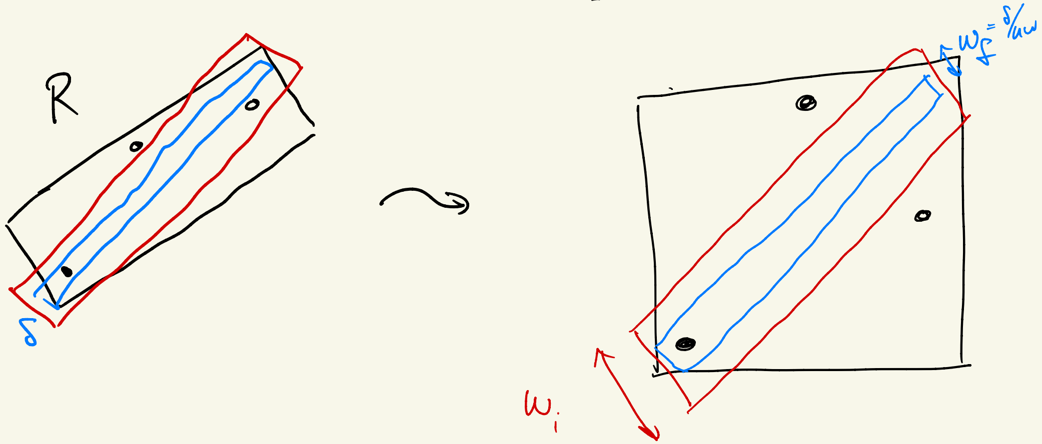
① Initial estimate $\dfrac{I(w_i)}{w_i |P||L|} \gtrsim 1$

for some large scale w_i

② Inductive step

$$\left| \frac{I(w_i)}{w_i |P||L|} - \frac{I(w_f)}{w_f |P||L|} \right| \ll 1$$

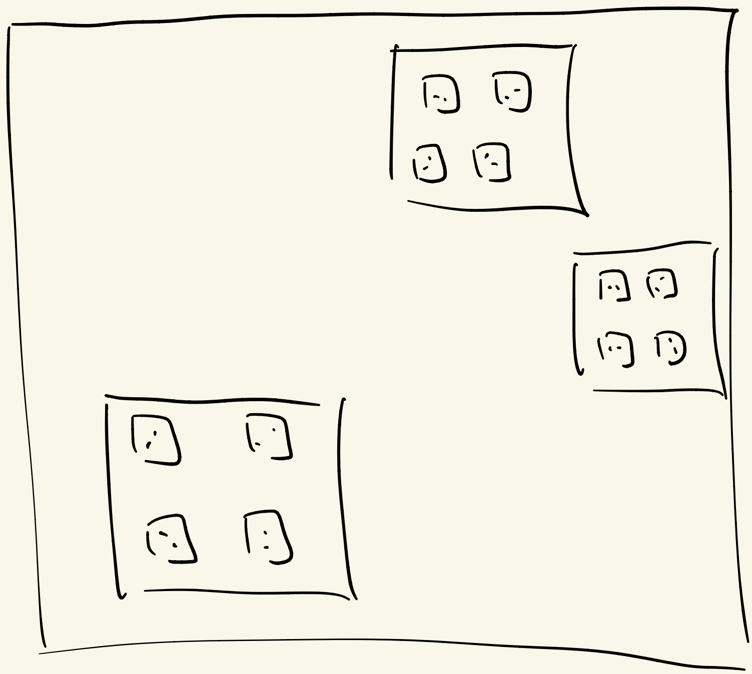
Goal: Find a rectangle R s.t. after blowing up into R, can run ① + ②.



- Not too hard to find some rectangles that satisfy 1 and some that satisfy 2
- <u>Hard</u> to satisfy both at once

Branching structure

Assume $P \subset [0,1]^2$ is built like a tree



Def P is uniform on a set of scales S if

for $u \in S$,

$$P \subset \bigcup_{Q \in \mathcal{Q}_u} Q$$

$\mathcal{Q}_u$ ← collection of $u \times u$ square

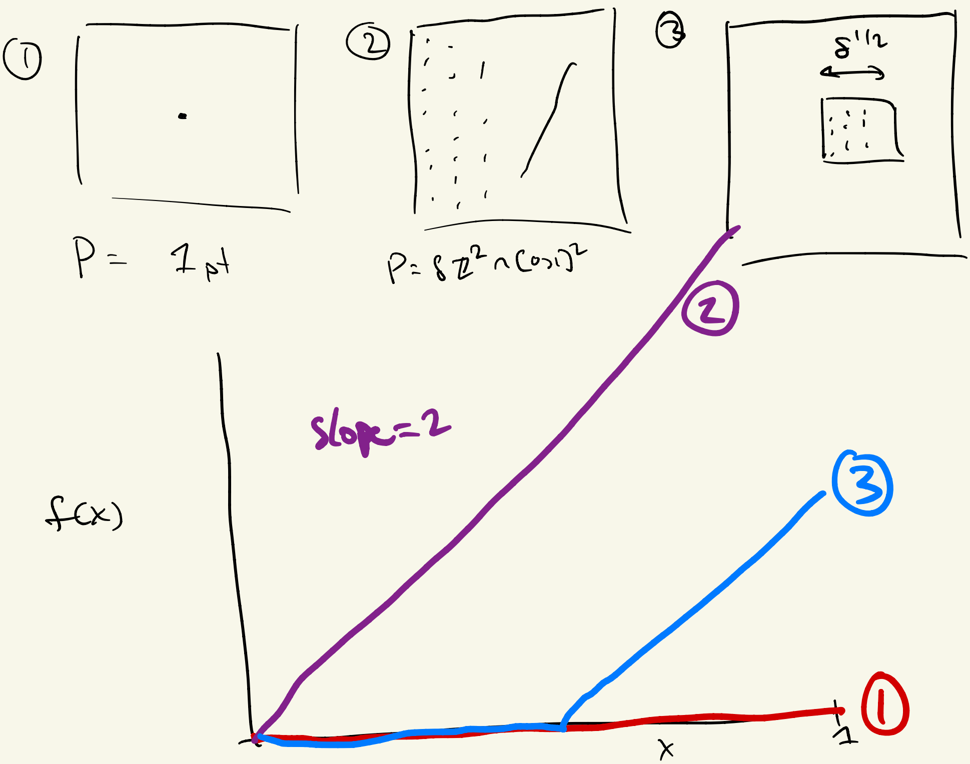
$$|P \cap Q| \sim \text{const.} \qquad \forall Q \in \mathcal{Q}_u$$

Def $f:[0,1]\to\mathbb{R}_{\geq 0}$ is the branching function

$$\delta^{-f(x)} = |\mathfrak{Q}_{\delta^x}|$$

covering number

Ex



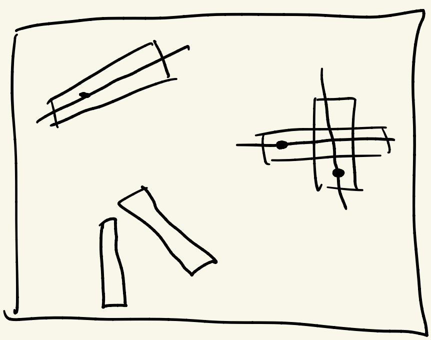
Fractal Geometry: # pts in a set $\longrightarrow$ branching fcn

$X = \{(p_j, \ell_j)\}_{j=1}^{n}$

<u>Def</u> X is uniform if $\forall\ u \times u \times w \in \mathcal{S}$,

$$X \subset \bigcup_{R \in \mathcal{R}_{u \times u \times w}} R$$

$$|X \cap R| = \text{const.}$$



<u>Def</u> $\delta^{-f(x,y)} = |\mathcal{R}_{\delta^x \times \delta^{x+y}}|$

Actually, f is a function of 3-variables

direction info

$$f(x,y,z)$$

Can phrase 2 steps in terms of f.

Problem now about Lip functions.

Q What functions $f(x,y,z)$ arise as the branching func?

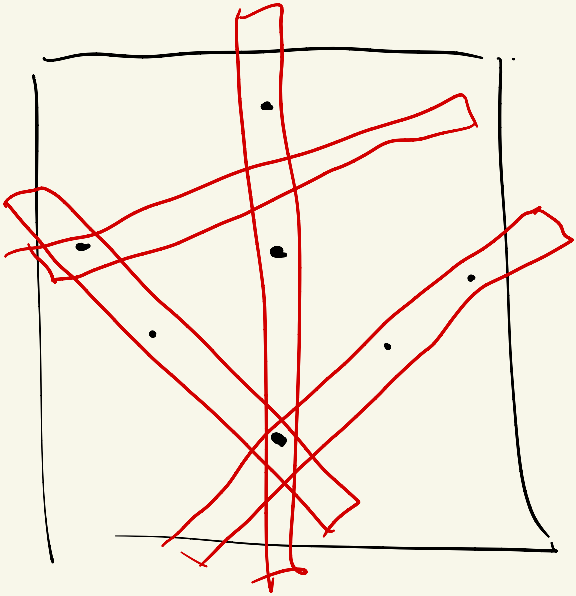
Def $\mathcal{L}$ = Limits of branching funcs as $\delta \to 0$.

Simple facts about rectangles $\to$ simple facts about $\mathcal{L}$

- Monotonic
- Lipschitz
- Submodular

Use these to prove thm

$\mathcal{L}$ also contains deeper info



Problem setup

$P \subset [0,1]^2$

$\mathbb{T}$ a set of δ- tubes

M- many tubes through each pt

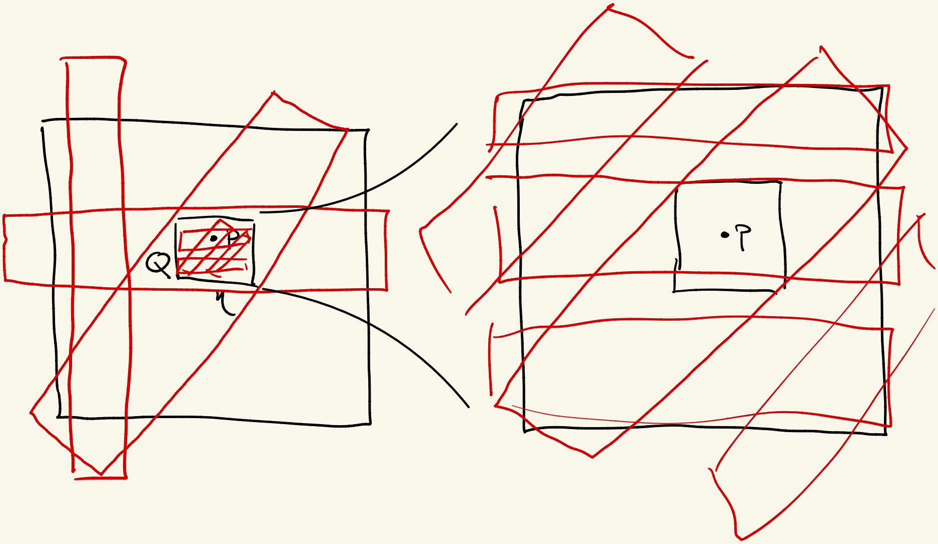
Q How big is $\mathbb{T}$?

This Q can be phrased in terms of $\mathcal{L}$, along w/ info about sizes at diff scales

Ex Ren-Wang Furstenberg set thm can be phrased as info about $\mathcal{L}$

General Q of 2D incidence geometry: What does $\mathcal{L}$ look like?

Initial estimate



- There are $\sim \eta^{-2}$-many tubes, each of which have the same # of lines
- Suppose $\eta^{-S} \ll \eta^{-1}$ many tubes go through Q.
  No initial estimate
- Blowup into Q.
- Direction stable: # slopes coming out of Q $\sim$ # slopes total

Problem: Can't run inductive step after blowing up (goal is hard)