Heilbronn's triangle problem and connections to projection theory

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In any set of $n$ points, there is a triangle of area $\leq \Delta(n)$

(and $\Delta(n)$ is the smallest number making this true)

Example:

$$\Delta(4) = \frac{1}{2}$$

For any set of 4 pts, there is a triangle with area $\leq \frac{1}{2}$

Problem: prove asymptotic upper bounds for $\Delta(n)$ as $n \rightarrow \infty$

Find small triangles!
**Lower Bounds**: Finding sets with no small triangles

- Erdős: $\Delta(n) \geq \frac{1}{n^2}$
  - Explicit algebraic construction

- Komlós, Pintz, Szemerédi: $\Delta(n) \geq \frac{\log n}{n^2}$
  - Semi-random method
  - Independence in hypergraph
  - Significant combinatorial result

Now back to upper bounds
Observation 1: Trivial bound, $\Delta(n) \leq \frac{5}{n}$

Each strip has area $\frac{5}{n}$

By the pigeonhole principle some strip has $\geq 5$ points

First problem: prove $\Delta = o\left(\frac{1}{n}\right)$
Observation 2: Scaling

If points are concentrated in a subsquare, find a small triangle there.

\[ \triangle(P) \leq |Q|^2 \cdot \triangle(P \cap Q \text{ rescaled}) \]

To improve on the trivial bound, we can assume \( P \) is well spaced.

Otherwise, induct into a subsquare.
Observation 3: Incidence setup

\[ \text{Area} = \text{Base} \times \text{Height} \]

\[ \Delta = u \times w \]

Plan:
- select pairs at distance \( u \)
- form strips of width \( w \)
- find a third point in some strip

Base first, then height
Prior work

- Trivial bound: $\Delta \leq n^{-1}$

- Roth 1951: $\Delta \leq n^{-1} (\log \log n)^{\frac{1}{2}}$ proof is based on density increment

- Schmidt 1972: $\Delta \leq n^{-1} (\log n)^{\frac{1}{3}}$ proof considers pairs at many different scales

- Roth 1972-73: $\Delta \leq n^{-1.1}$ Proof has two steps: initial estimate and inductive step

- KPS 1981: $\Delta \leq n^{-8/7}$ Proof extends Roth's approach to its natural limit

- Us 2023: $\Delta \leq n^{-\frac{8}{7} - \frac{1}{2000}}$ We break the KPS barrier.
Roth's setup

\[ pc[0,1]^2 \]

\[ L = \{ \text{Lines } l \text{ connecting pairs } x, y \in \mathbb{P} \text{ with } |x - y| \leq u \} \]

\[ \Pi = \{ \omega \text{-tube around } l : l \in L \} \text{ a set of tubes} \]

Incidences:

\[ I(\omega; P, L) = \# \{ (p, T) \in P \times \Pi : p \in T \} \]

\[ = \# \{ (p, l) \in P \times L : d(p, l) \leq \omega/2 \} \]

Goal: \[ I(\omega^*_f; P, L) > 2|L| \]

Then a third point in some strip, and \( \Delta \leq u \cdot \omega^*_f \)

E.g.: \( u = n^{-1/3}, \omega^*_f = n^{-3/4}, \Delta = n^{-13/12} \)
Roth’s two steps

\[ I(\omega; P, L) = \# \{ (p, l) \in P \times L : d(p, l) \leq \omega/2 \} \]

Pick \( \omega_f < \omega \):

- Initial Estimate: \( I(\omega_i) \geq \omega_i |P| |L| \)

- Inductive step:
  \[ \left| \frac{I(\omega_i)}{\omega_i |P| |L|} - \frac{I(\omega_f)}{\omega_f |P| |L|} \right| \ll 1 \]

Normalized # of incidences doesn't change much as the scale varies

E.g. \( u = n^{3/5}, \omega_i = n^{0.1}, \omega_f = n^{-3/4}, \Delta = n^{-13/12} \)
Main Theorem: \( \Delta(n) \leq n^{\frac{-8}{7} - \frac{1}{2000}} \)

1. Recast Roth’s inductive step in terms of the high-low method
   - Basically the same as Roth/KPS. But a cool connection.

2. New approach to initial estimate using direction set estimates from projection theory
   - Very different from Roth/KPS
   - Marstrand 1954 direction set estimate recovers \( \Delta \leq n^{\frac{-8}{4}} \)
   - Main ingredient: Use Orponen, Shmerkin, and Wang 2023 direction set estimate in place of Marstrand

This is the culmination of a line of work based on Bourgain’s discretized sum-product theorem

- Roth’s 3 papers and KPS use the same initial estimate and refine inductive step
- We improve initial estimate
Inductive step and the high low method

Idea: If \( P \) and \( L \) are not too concentrated, then \( \frac{I(w)}{w \cdot |P| \cdot |L|} \sim \text{const.} \) as \( w \) varies

\[
M_p(w \times w) = \max \{P \cap Q, Q \text{ a } w \times w \text{ square} \}
\]

\[
M_L(w \times 1) = \max \{|L \cap T|, T \text{ a } w \text{-tube} \}
\]

\( w_f < w_i \) Scales

Thm: \[
\left| \frac{I(w_i)}{w_i \cdot |P| \cdot |L|} - \frac{I(w_f)}{w_f \cdot |P| \cdot |L|} \right| \lesssim \sqrt{\frac{M_p(w_i \times w_i)}{|P|} \frac{M_L(w_i \times 1)}{|L|} w^3}
\]

GSW: Upper bounds    Roth: Lower bounds
The proof uses orthogonality.

$$q = \sum_{p \in P} \omega^c_b \mathbf{B}(\omega^c_b, \omega^c_f), \quad \Phi = \sum_{\ell \in \mathcal{L}} \left( \frac{1}{\omega^c_\ell} \mathbf{1_{\omega^c_\ell}} - \frac{1}{\omega^c_f} \mathbf{1_{\omega^c_f}} \right)$$

$$\left| \frac{I(\omega^c_i)}{\omega^c_i} - \frac{I(\omega^c_f)}{\omega^c_f} \right| = \frac{1}{|\mathcal{P}|} \frac{1}{|\mathcal{L}|} \mathbf{g}^T \Phi \leq \frac{1}{|\mathcal{P}|} \frac{1}{|\mathcal{L}|} \| \mathbf{g} \|_2 \| \Phi \|_2 \leq \sqrt{\frac{M_p(\omega^c_k, \omega^c_i)}{|\mathcal{P}| \lambda}} \frac{M_r(\omega^c_i, x_i)}{|\mathcal{L}| \lambda^3}$$

Estimate $\| \Phi \|_2$ using orthogonality between strips.

Called high-low method because of Fourier analysis interpretation: Split into high & low frequencies.
Taking stock of inductive step

High-Low: If $P$ and $L$ are not too concentrated, then $\frac{I(w)}{w \cdot |P| \cdot |L|} \sim \text{const.}$ as $w$ varies

- If $P$ is concentrated in a subsquare, induct and find a small triangle there

- If $P$ is not concentrated, use high-low to find small triangles
Initial estimate intuition

- $L_Q = \{ \text{lines } l \text{ connecting pairs } x,y \in \mathcal{P} \}$, $L = \bigcup Q L_Q$

- Direction set $Q = \{ \frac{x-y}{|x-y|} : x,y \in \mathcal{P} \}$

- $Bad_Q = \{ \theta : \omega_i \text{-tube in direction } \theta \text{ has } < \lambda \omega_i |P| \text{ points} \}$

1. For most $Q$, $|Bad_Q| \leq \lambda$ (small)
   - What is probability a random pair $(Q, \theta)$ is bad?
     - Pick $Q$ then $\theta$: $|q Bad Q|

2. Show direction set $Q$ is spread out
   - Projection theory

3. Combine these:
   - $\# \{ l \in L_Q : \theta |n \in Bad_Q \} \leq \frac{1}{\Lambda} |L_Q|
   - $|T_{\omega_i}(e) n_p| \leq \omega_i |P|$ for most $l \in L_Q$
   - $I \{ \omega_i ; \beta, L \} = \sum_{x \in L} |T_{\omega_i}(x) n_p| \leq \omega_i |P| |L|$

Initial Estimate: $I(\omega_i) \geq \omega_i |P| |L|$

- $L_Q$ is the set of lines connecting pairs of points in $\mathcal{P}$.
- $L$ is the union of these sets over all $Q$.
- $Q$ is the direction set defined by lines connecting points.
- $Bad_Q$ contains directions for which the $\omega_i$-tube has too many points.

1. Most $Q$ have a small $|Bad_Q|$.
2. The direction set is spread out.
3. Combine these to show the initial estimate holds.
**Direction set estimates from projection theory**

**Thm (Marstrand 1954):** Let $X \subseteq [0,1]^2$ have Hausdorff dimension $s > 1$. Then $[\text{dimension of direction set}] = 1$.

- Use discretized version to show line set is spread out.

**Defn:** $P \subseteq [0,1]^2$ is $s$-regular above scale $S$ if $|P \cap Q| \leq C |Q|^s |P|$ for all squares $Q$ with $|Q| > S$.

Frostman regular.

- If $P \cap Q$ is $s > 1$ regular, direction set $\mathcal{Q}$ is spread out.

  \[
  \# \{ \ell \in \mathcal{L}_Q : \Theta(1) \in I \} \leq C |I| \|L_{Q}\| \quad \text{for all } I \in \mathcal{S} \text{ with } |I| > S
  \]

  - we get initial estimate
Examples of regular sets

- $P$ is 2-regular
- Marstrand: direction set is 1-dim

- $P$ is 1-regular
- Marstrand gives no information on direction set
Taking stock of initial estimate

- **First hard case (homogenous):**
  
  $P$ is 2-dim down to scale $s = n^{-1/2}$, then $Z_{pt}$
  
  $\Delta \leq n^{-7/6}$

- **Worst case scenario for Roth/KPS:**
  
  $P$ is 2-dim down to scale $n^{-3/4}$
  
  1-dim from scale $n^{-3/4}$ to $n^{-4/7}$
  
  $\Delta \leq n^{-8/7}$, worse bound than homogenous!
Getting a better bound

Worst case scenario: \( P \) is 2-dim down to scale \( n^{-3/4} \)

1-dim from scale \( n^{-3/4} \rightarrow n \)

Thm (Orponen-Shmerkin-Wang): If

- \( X \subseteq [0,1]^2 \) is 0<\( S \leq 1 \) dimensional, and
- \( X \) is not contained in a line

Then \( \text{[dimension of direction set]} = S \)

- Gives initial estimate in worst case scenario

- Pf: discretized sum-product + proj. theory + bootstrapping

Marstrand gives no information on direction set

OSW: direction set is 1-dim