Heilbronn's triangle problem and connections to projection theory

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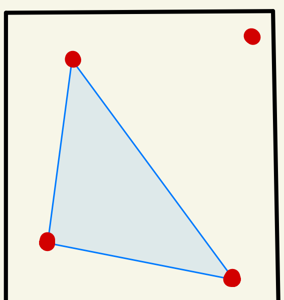
In any set of n points, there is a triangle of area $\leq \Delta(n)$

(and $\Delta(n)$ is the smallest number making this true)

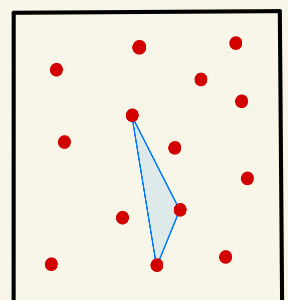
Example:



$\Delta(4) = \frac{1}{2}$

For any set of 4 pts there is a triangle with area $\leq \frac{1}{2}$

Problem: prove asymptotic upper bounds for $\Delta(n)$ as $n \to \infty$



Find small triangles!

Lower Bounds: Finding sets with no small triangles

- Erdős: $\Delta(n) \gtrsim \frac{1}{n^2}$

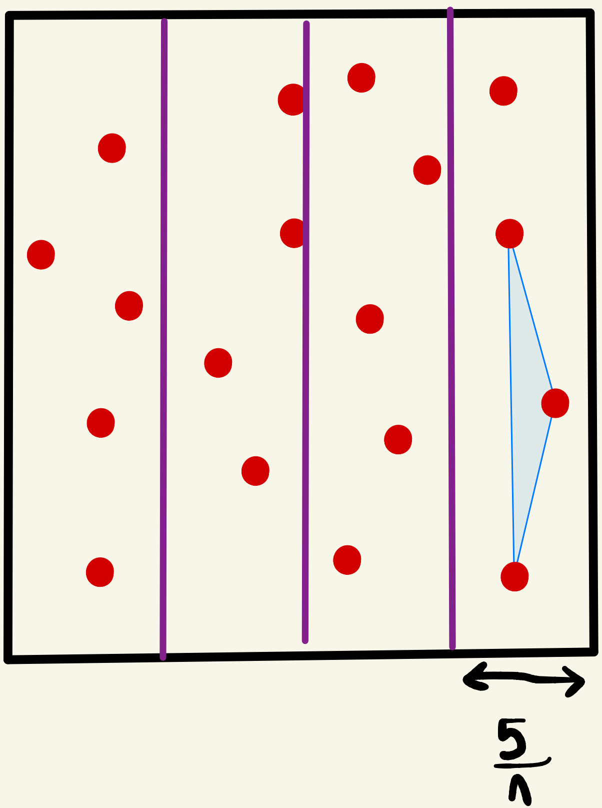
  - Explicit algebraic construction

- Komlós, Pintz, Szemerédi: $\Delta(n) \gtrsim \frac{\log n}{n^2}$

  - Semi-random method
  - Independence in hypergraph
  - Significant combinatorial resut

Now back to upper bounds

Observation 1: Trivial bound, $\Delta(n) \le \frac{5}{n}$

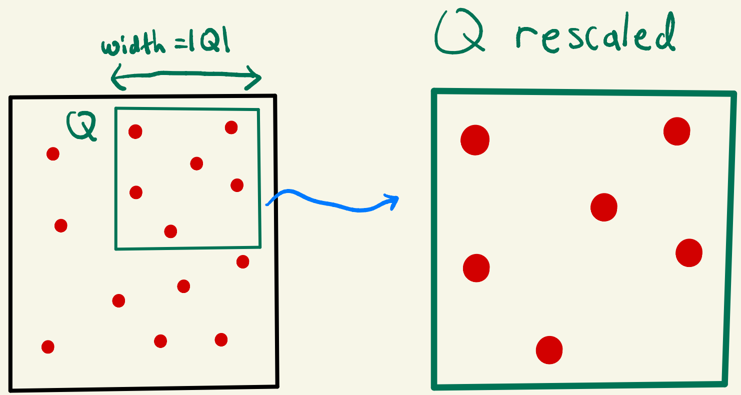


Each strip has area $\frac{5}{n}$

By the pigeonhole principle some strip has ≥ 5 points

First problem: prove $\Delta = o\left(\frac{1}{n}\right)$

Observation 2: Scaling



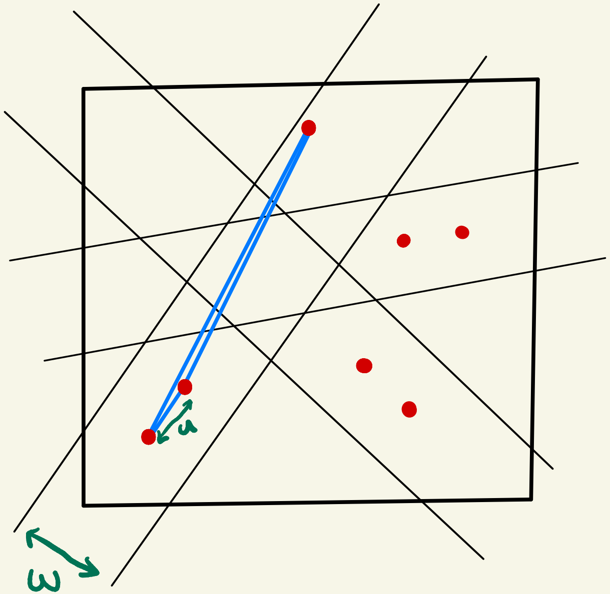
$$\Delta(P) \leq |Q|^2 \cdot \Delta(P \cap Q \text{ rescaled})$$

If points are concentrated in a subsquare, find a small triangle there

To improve on the trivial bound, we can assume P is well spaced

Otherwise, induct into a subsquare

Observation 3: Incidence setup



Area = Base × Height

$\Delta = u \times w$

Plan:
- select pairs at distance u
- form strips of width w
- find a third point in some strip

Base first, then height

Prior work

KPS = Komlós, Pintz, Szemerédi

- Trivial bound $\Delta \lesssim n^{-1}$
- Roth 1951 $\Delta \lesssim n^{-1}(\log\log n)^{-\frac{1}{2}}$ proof is based on density increment
- Schmidt 1972 $\Delta \lesssim n^{-1}(\log n)^{-\frac{1}{2}}$ proof considers pairs at many different scales

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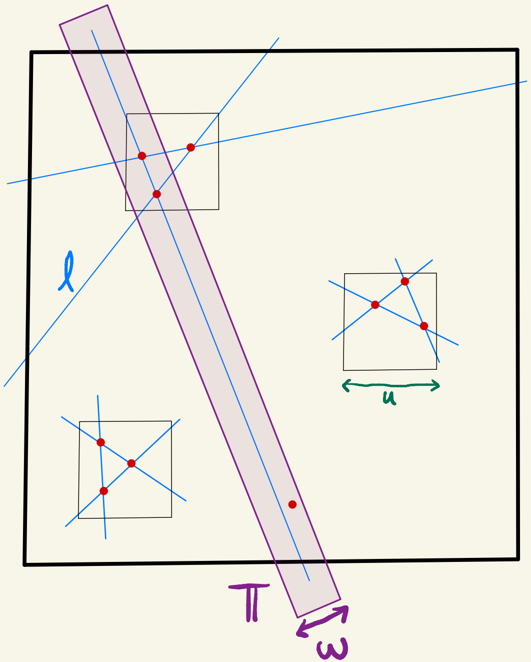
- Roth 1972-73 $\Delta \lesssim n^{-1.1}$ Proof has two steps: initial estimate and inductive step
- KPS 1981 $\Delta \lessapprox n^{-8/7}$ Proof extends Roth's approach to its natural limit
- Us 2023 $\Delta \lessapprox n^{-\frac{8}{7}-\frac{1}{2000}}$ We break the KPS barrier.

Roth's setup

$P \subset [0,1]^2$



$L = \{\text{Lines } \ell \text{ connecting pairs } x,y \in P \text{ with } |x-y| \le u\}$

$\mathbb{T} = \{\omega\text{-tube around } \ell : \ell \in L\}$ a set of tubes

Incidences:

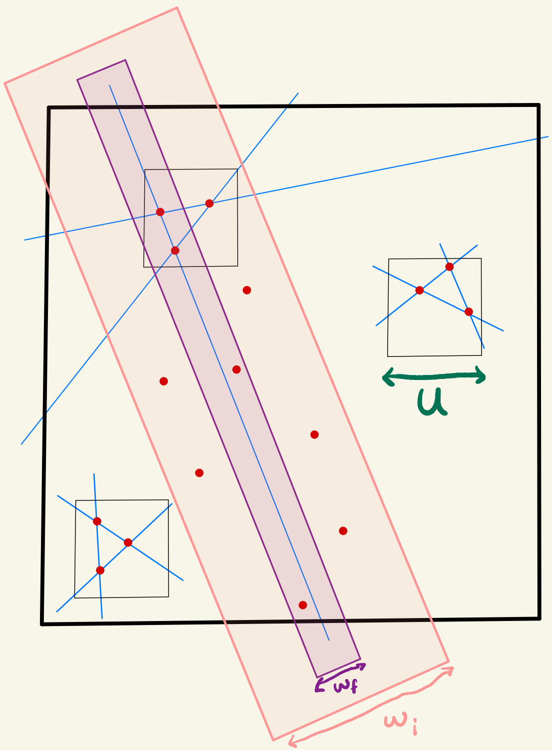
$$I(\omega; P, L) = \#\{(p,T) \in P \times \mathbb{T} : p \in T\}$$
$$= \#\{(p,\ell) \in P \times L : d(p,\ell) \le \omega/2\}$$

Goal: $I(\omega_f; P, L) > 2|L|$

Then a third point in some strip, and $\Delta \le u \cdot \omega_f$

E.g.: $u = n^{-1/3}$, $\omega_f = n^{-3/4}$, $\Delta = n^{-13/12}$

Roth's two steps



$$I(w; P, L) = \#\{(p,\ell) \in P \times L : d(p,\ell) \leq w/2\}$$

Pick $w_f \ll w_i$

- Initial Estimate: $I(w_i) \gtrsim w_i |P| \cdot |L|$
- Inductive step: $\left| \dfrac{I(w_i)}{w_i |P| \cdot |L|} - \dfrac{I(w_f)}{w_f |P| \cdot |L|} \right| \ll 1$

Normalized # of incidences doesn't change much as the scale varies

E.g. $u = n^{-\frac{1}{3}}$ $\quad w_i = n^{-0.1}$ $\quad w_f = n^{-3/4}$ $\quad \Delta = n^{-13/12}$

Main Theorem: $\Delta(n) \lesssim n^{-8/7-1/2000}$

comes from a 2017 paper of Guth, Solomon, and Wang

Topic in incidence geometry, projection theory, and Fourier analysis

1. Recast Roth's inductive step in terms of the high-low method
   - Basically the same as Roth/KPS. But a cool connection.

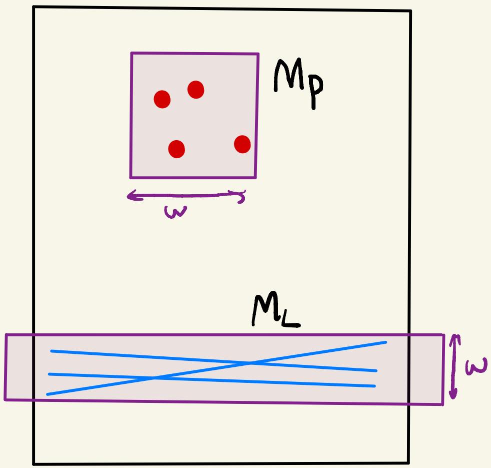
2. New approach to initial estimate using direction set estimates from projection theory
   - Very different from Roth/KPS
   - Marstrand 1954 direction set estimate recovers $\Delta \lesssim n^{-8/7}$
   - Main ingredient: Use Orponen, Shmerkin, and Wang 2023 direction set estimate in place of Marstrand

This is the culmination of a line of work based on Bourgain's discretized sum-product theorem

- Roth's 3 papers and KPS use the same initial estimate and refine inductive step
- We improve initial estimate

Inductive step and the high low method

Idea: If P and L are not too concentrated, then $\frac{I(w)}{w \cdot |P| \cdot |L|} \sim$ const. as w varies



$M_P(w \times w) = \max |P \cap Q|$, Q a $w \times w$ square

$M_L(w \times 1) = \max |L \cap T|$, T a w-tube

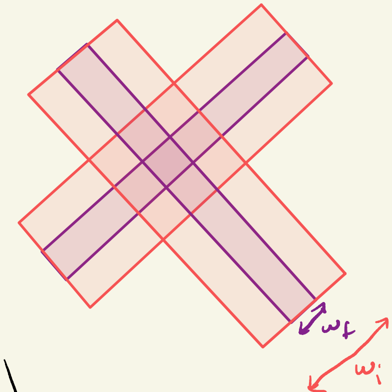
$w_f < w_i$ Scales

Thm: $$\left| \frac{I(w_i)}{w_i |P||L|} - \frac{I(w_f)}{w_f |P||L|} \right| \lesssim \sqrt{\frac{M_P(w_i \times w_i)}{|P|} \frac{M_L(w_i \times 1)}{|L|} w^{-3}}$$

GSW: Upper bounds Roth: Lower bounds

High-Low proof

The proof uses orthogonality.



$$g = \sum_{p \in P} w_f^{-2} 1_{B(p, w_f)}, \qquad \Phi = \sum_{\ell \in L} \left(\frac{1}{w_i} 1_{T_{w_i}(\ell)} - \frac{1}{w_f} 1_{T_{w_f}(\ell)} \right)$$

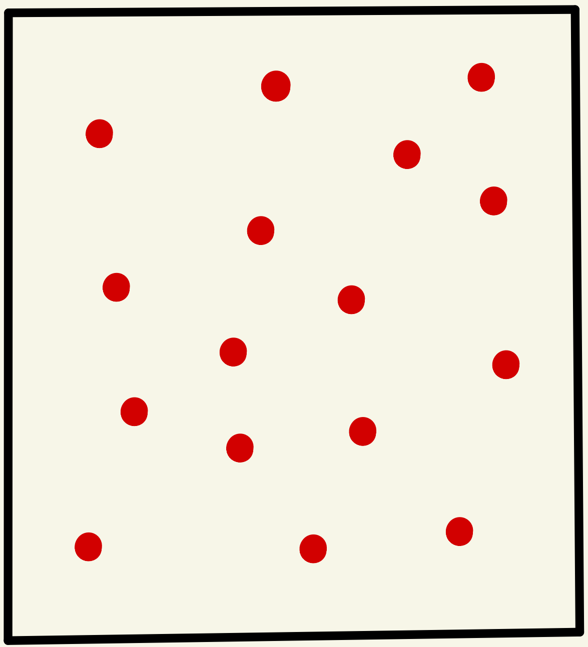
$$\left| \frac{I(w_i)}{w_i |P||L|} - \frac{I(w_f)}{w_f |P||L|} \right| = \frac{1}{|P||L|} |\langle g, \Phi \rangle| \le \frac{1}{|P||L|} \|g\|_2 \|\Phi\|_2 \lesssim \sqrt{\frac{M_P(w_i \times w_i)}{|P|} \frac{M_L(w_i \times 1)}{|L|} w^{-3}}$$

Estimate $\|\Phi\|_2$ using orthogonality between strips

Called high-low method because of Fourier analysis interpretation: Split into high & low frequencies

Taking stock of inductive step

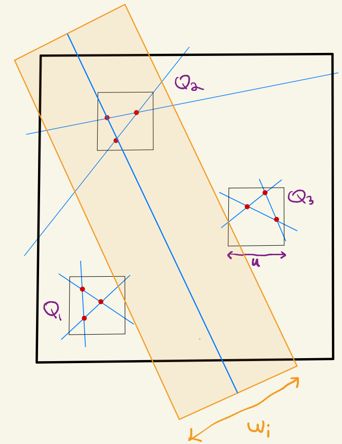
High-Low: If P and L are not too concentrated, then $\frac{I(\omega)}{\omega \cdot |P| \cdot |L|} \sim$ const. as ω varies



- If P is concentrated in a subsquare, induct and find a small triangle there
- If P is not concentrated, use high-low to find small triangles

Initial estimate intuition

Initial Estimate: $I(w_i) \gtrsim w_i |P| \cdot |L|$



- $L_Q = \{\text{lines } \ell \text{ connecting pairs } x,y \in P \cap Q\}$, $\quad L = \bigcup_Q L_Q$

- Direction set$_Q$ $= \left\{ \dfrac{x-y}{|x-y|} : x,y \in P \cap Q \right\}$
- $\text{Bad}_Q = \{\theta :$ w_i-tube in direction θ has $< \lambda w_i |P|$ points$\}$

  $\lambda \ll 1$

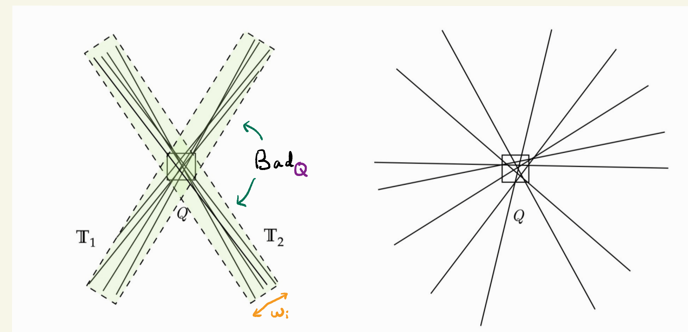
1. For most Q, $|\text{Bad}_Q| \leq \lambda$ (small)

   - What is probability a random pair (Q, θ) is bad?
     - Pick Q then θ: $\mathbb{E}_Q |\text{Bad}_Q|$
     - Pick θ then Q: $\leq \lambda$

2. Show direction set$_Q$ is spread out

   - Projection theory

3. Combine these:
   - $\#\{\ell \in L_Q : \theta(\ell) \in \text{Bad}_Q\} \leq \frac{1}{2} |L_Q|$
   - $|\mathbb{T}_{w_i}(\ell) \cap P| \gtrsim w_i |P|$ for most $\ell \in L_Q$
   - $I(w_i; P, L) = \sum_{\ell \in L} |\mathbb{T}_{w_i}(\ell) \cap P| \gtrsim w_i |P||L|$ ✓



lines are concentrated in Bad_Q

Direction set is spread out, so lines are not concentrated in Bad_Q

Direction set estimates from projection theory

Thm (Marstrand 1954): Let $X \subset [0,1]^2$ have Hausdorff dimension $s > 1$.

Then [dimension of direction set] $= 1$.

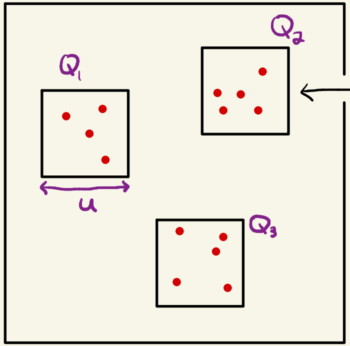
direction set $= \left\{ \frac{x-y}{|x-y|} : x, y \in X \right\}$

- Use discretized version to show line set is spread out

Defn: $P \subset [0,1]^2$ is s-regular above scale δ if

$$|P \cap Q| \leq C\,|Q|^s\,|P| \quad \text{for all squares } Q \text{ with } |Q| > \delta.$$

Frostman regular

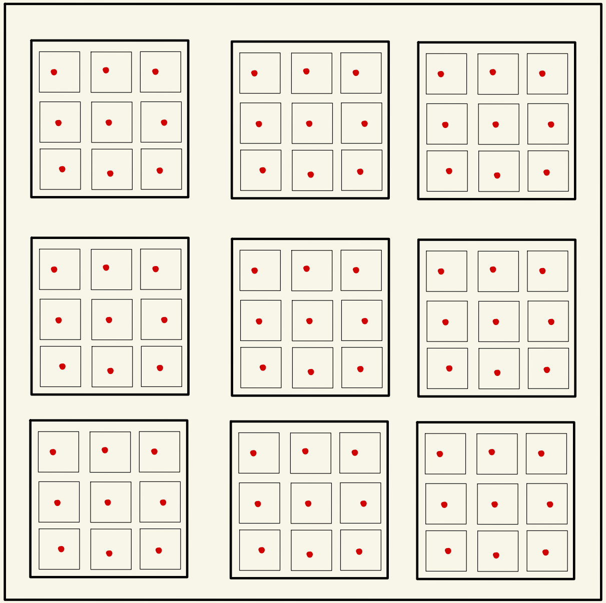


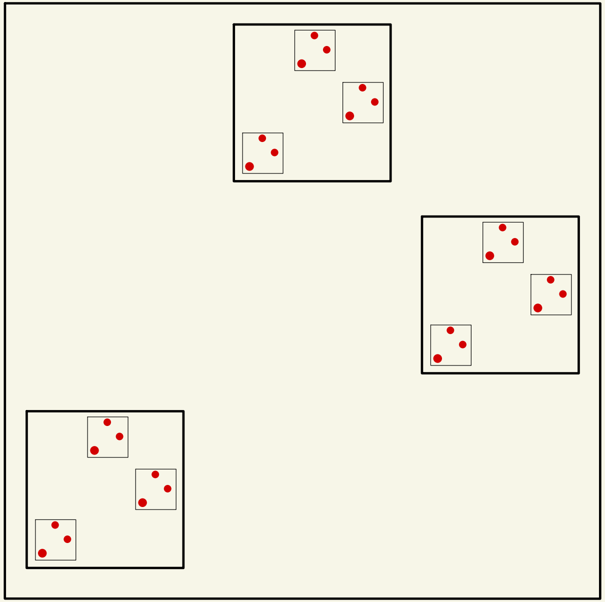
- If $P \cap Q$ is $s > 1$ regular, direction set$_Q$ is spread out

$$\#\{\ell \in L_Q : \theta(\ell) \in I\} \leq C\,|I|\,|L_Q| \quad \text{for all } I \subset S^1 \text{ with } |I| \geq \delta$$

- we get initial estimate

Examples of regular sets





- P is 2-regular
- Marstrand: direction set is 1-dim

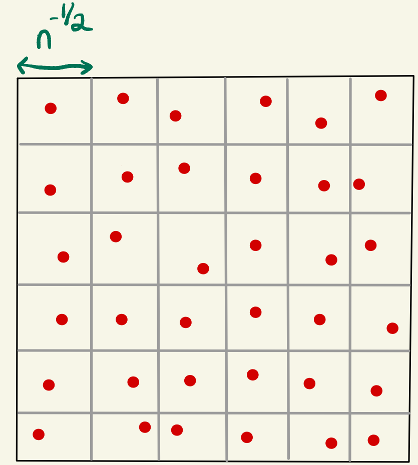
- P is 1-regular
- Marstrand gives no information on direction set

Taking stock of initial estimate

- First hard case (homogenous):

  P is 2-dim down to scale $\delta = n^{-1/2}$, then 1 pt

  $$\Delta \lessapprox n^{-7/6}$$

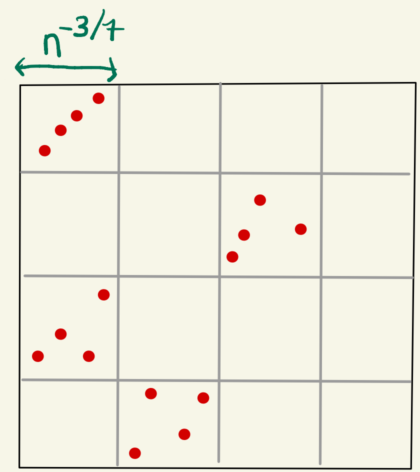


- Worst case scenario for Roth/KPS:

  P is 2-dim down to scale $n^{-3/7}$

  1-dim from scale $n^{-3/7} \to n^{-4/7}$

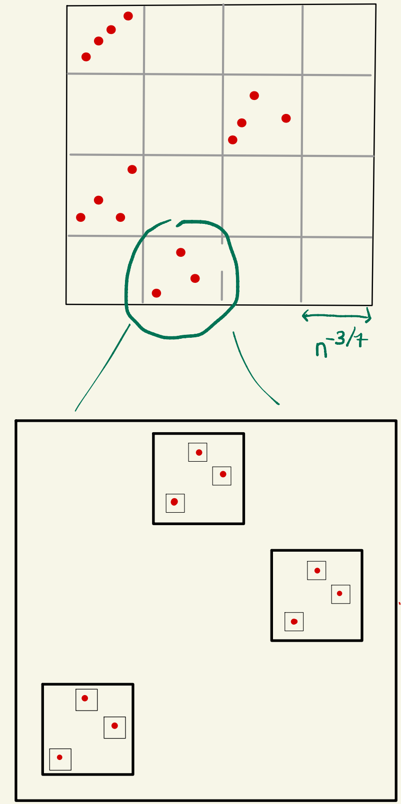
  $$\Delta \lessapprox n^{-8/7}$$ worse bound than homogenous!



Getting a better bound

Worst case scenario: P is 2-dim down to scale $n^{-3/7}$

1-dim from scale $n^{-3/7} \to n^{-4/7}$



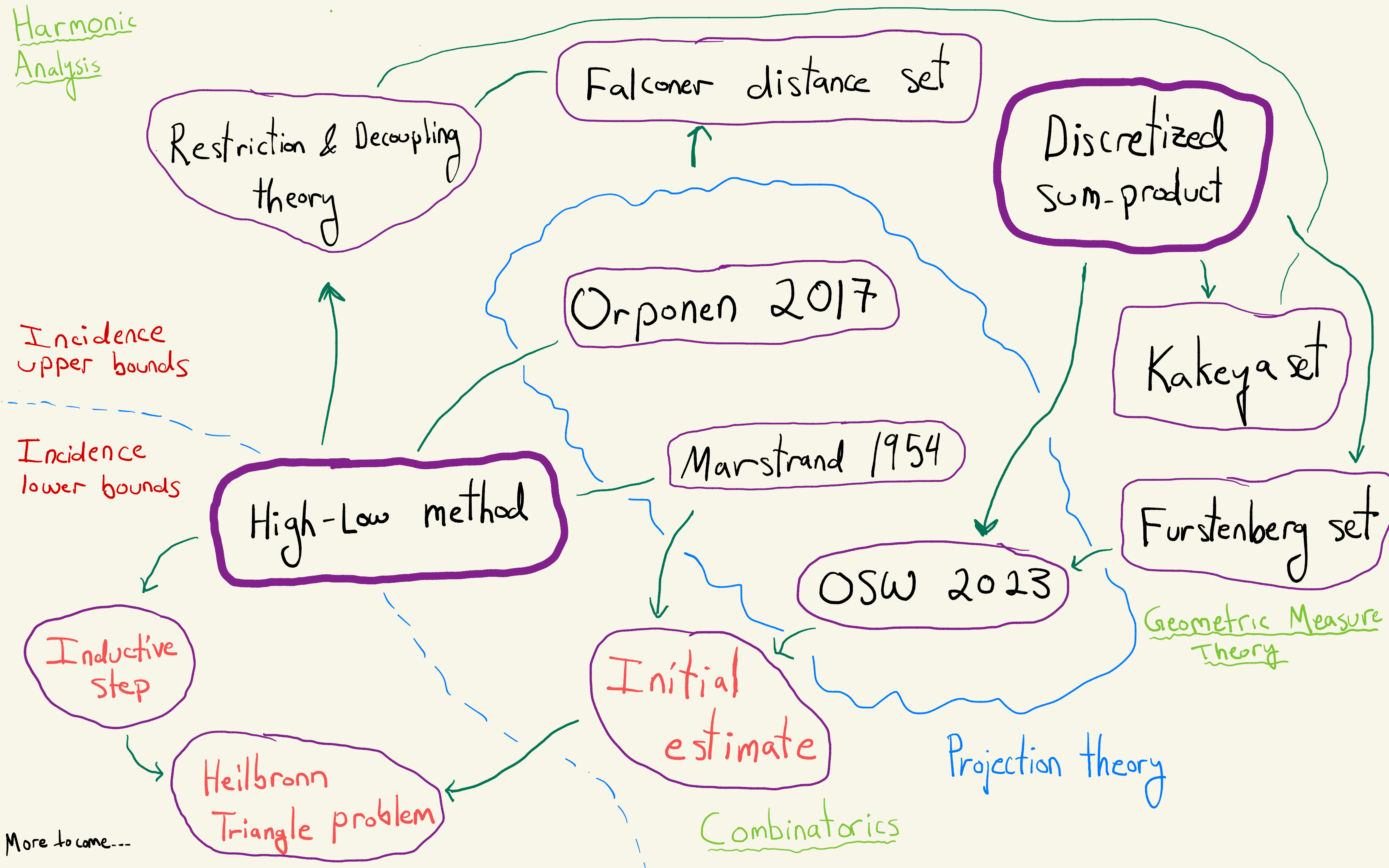
Thm (Orponen-Shmerkin-Wang): If

- $X \subset [0,1]^2$ is $0 < s \leq 1$ dimensional, and
- X is not contained in a line

Then [dimension of direction set] $\geq s$

- Gives initial estimate in worst case scenario
- Pf: discretized sum-product + proj. theory + bootstrapping
  I don't understand

- $P \cap Q$ is 1-regular
- Marstrand gives no information on direction set
- OSW: direction set is 1-dim


Harmonic Analysis
Restriction & Decoupling theory
Falconer distance set
Discretized sum-product
Orponen 2017
Kakeya set
Incidence upper bounds
Incidence lower bounds
Marstrand 1954
High-Low method
Furstenberg set
OSW 2023
Inductive step
Initial estimate
Geometric Measure Theory
Heilbronn Triangle problem
Projection theory
Combinatorics
More to come...