

NONLINEAR WAVE EQUATIONS

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This topic proposal will deal with well-posedness questions for nonlinear wave equations of the form

$$\begin{aligned}\square u &= F \\ u[0] &= (f, g),\end{aligned}$$

where $\square := -\partial_t^2 + \Delta$ and $u[0] := (u, u_t)|_{t=0}$. The equation is semi-linear if F is a function only of u , (i.e. $F = F(u)$), and quasi-linear if F is also a function of the derivatives of u (i.e. $F = F(u, Du)$, where $D := (\partial_t, \nabla)$). The goal is to use energy methods to prove local well-posedness for quasi-linear equations with data $(f, g) \in H^s \times H^{s-1}$ for large enough s , and then to derive Strichartz estimates to deal with semi-linear problems with data $(f, g) \in \dot{H}^1 \times L^2$. We begin, however, by deriving various conservation laws for solutions of wave equations.

1. CONSERVATION LAWS

The general idea is that the various symmetries of the wave equation lead to certain conserved quantities. To see this, we start by observing that solutions to the homogeneous wave equation can be expressed as stationary points of a variational integral. Indeed, define \mathcal{L} by

$$\mathcal{L}(u) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^n} |\nabla u|^2 - |u_t|^2 dt dx.$$

Then, for every $\phi \in C_0^\infty$ we have, formally, that

$$\begin{aligned}\langle \mathcal{L}'(u), \phi \rangle &= \frac{d}{d\epsilon} \mathcal{L}(u + \epsilon\phi)|_{\epsilon=0} \\ &= \int (u_{tt} - \Delta u)\phi \\ &= - \int \square u \phi\end{aligned}$$

where $\mathcal{L}'(u)$ is the Fréchet derivative. Hence u is stationary for \mathcal{L} with respect to variations ϕ if and only if $\square u = 0$.

Noether's Theorem states that the invariance of a variational integral, \mathcal{L} , with respect to a 1-parameter family of diffeomorphisms implies a conservation law for any extreme value of \mathcal{L} . Since solutions, u , of the homogeneous wave equation arise as extreme values of \mathcal{L} , we can use Noether's Theorem to obtain conservation laws in the form of a divergence equation $\partial_j V^j = 0$.

To derive our first conservation law, observe that \mathcal{L} is invariant under time translation, i.e. $\mathcal{L}(u) = \mathcal{L}(G_\epsilon u)$ where $G_\epsilon u(t, x) = u(t + \epsilon, x)$. The vector field ∂_t generates the flow corresponding

to time translation. Therefore, by Noether's Theorem we are able to multiply the wave equation by u_t to obtain a divergence. Indeed, we have

$$\begin{aligned}
(1) \quad 0 &= u_t \square u \\
&= -u_{tt}u_t + \Delta uu_t \\
&= -\partial_t \left(\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2} \right) + \operatorname{div}(\nabla uu_t)
\end{aligned}$$

which is a divergence equation (observe that this is a divergence in space-time of the vector field give by $(-\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2}, \nabla uu_t)$). At time t , we define the energy $E(u(t)) := \|Du(t)\|_{L^2}^2 = \int_{\mathbb{R}^n} \left(\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2} \right) dx$. Integrating the above over \mathbb{R}^n and assuming, say, that $Du(t) \in L^2(\mathbb{R}^n)$, we obtain the conservation of energy

$$\partial_t E(u(t)) = 0$$

which implies that $E(u(t)) = E(u(s))$ for all s, t . We can localize this result to light cones to obtain more information. Let $K(z_0) := \{z = (t, x) \in \mathbb{R} \times \mathbb{R}^n : |x_0 - x| < t_0 - t\}$ be the backwards light cone based at $z_0 = (t_0, x_0)$, and let $K_s^t(z_0) := K(z_0) \cap ([s, t] \times \mathbb{R}^n)$ be the truncated cone. Integrating (1) over $K_s^t(z_0)$, we obtain

$$E(u; D(s; z_0)) = E(u; D(t; z_0)) + \operatorname{Flux}(u; M_s^t(z_0))$$

where

$$\begin{aligned}
D(s; z_0) &:= K(z_0) \cap (\{s\} \times \mathbb{R}^n) \\
E(u; D(s; z_0)) &:= \|Du(t)\|_{L^2(D(s; z_0))}^2 \\
M_s^t(z_0) &:= \{|x - x_0| = t_0 - t\} \cap ([s, t] \times \mathbb{R}^n) \\
\operatorname{Flux}(u; M_s^t(z_0)) &:= \int_{M_s^t(z_0)} \frac{1}{2} \left| \nabla u - u_t \frac{x - x_0}{|x - x_0|} \right|^2 d\sigma.
\end{aligned}$$

This tells us that the energy at time t is less than or equal to the energy at time s with the difference consisting of the Flux, which is what has escaped out the sides of the light cone.

To derive more conservation laws we observe that \mathcal{L} is invariant under the Poincaré group which consists of the isometries of Minkowski space, namely, translations and Lorentz transformations.

Invariance under spatial translation leads to the conservation of momentum. Spatial translation is generated by ∂_{x_k} so multiplying the wave equation by the u_k will produce a divergence equation.

$$\begin{aligned}
0 &= -u_{tt}u_k + \Delta uu_k \\
&= -\partial_t(u_t u_k) + \partial_k \left(\frac{|u_t|^2}{2} - \frac{|\nabla u|^2}{2} \right) + \operatorname{div}(\nabla uu_k).
\end{aligned}$$

Integrating this over \mathbb{R}^n and assuming $Du \in L^2(\mathbb{R}^n)$ gives that

$$\partial_t \int_{\mathbb{R}^n} u_t u_k dx = 0.$$

which shows that momentum is constant for all time. Similarly, elements of the subgroup of Lorentz transformations are generated by $\Gamma_{jk} := x_j \partial_k - x_k \partial_j$ or $\Gamma_j = x_j \partial_t + t \partial_j$. Proceeding as before we

obtain the conservation of angular momenta

$$\partial_t \int (x_k u_j - x_j u_k)(u_t) dx = 0$$

as well as

$$\partial_t \int x_k \left(\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2} \right) + t u_k u_t dx = 0.$$

2. ENERGY ESTIMATES

In the preceding section, symmetry allowed us to prove conservation of energy. More generally, we can use the Fourier transform to prove energy inequalities as well as an existence and uniqueness result for the linear wave equation. These results are necessary in order to address the issue of local well-posedness for our quasi and semi-linear problems.

Theorem 2.1. *Let $(f, g) \in H^s \times H^{s-1}$ and $h \in L^1([0, T], H^{s-1})$. Then the linear wave equation*

$$(2) \quad \begin{aligned} \square u &= h \\ u[0] &= (f, g) \end{aligned}$$

has a unique solution $u \in C([0, T]; H^s) \cap C([0, T]; H^{s-1})$ which satisfies, for each $0 \leq t \leq T$, the following energy estimates:

$$(3) \quad \|u(t, \cdot)\|_{H^s} + \|\partial_t u(t, \cdot)\|_{H^{s-1}} \leq C(1+t) \left(\|f\|_{H^s} + \|g\|_{H^{s-1}} + \int_0^t \|h(r, \cdot)\|_{H^{s-1}} dr \right)$$

Remark 2.2. The term $(1+t)$ in the above inequality is necessary because solutions, u , to the wave equation only satisfy $\|u(t)\|_{L^2} = O(t)$ as $t \rightarrow \infty$. Hence bounds for $\|u(t)\|_{H^s}$ will always need to include a term that depends on t . To remove this inconvenience we can estimate, instead, in \dot{H}^s where $s \geq 1$, giving us the following result.

Theorem 2.3. *Let $s \geq 1$. Let $(f, g) \in \dot{H}^s \times \dot{H}^{s-1}$ and suppose $h \in L^1([0, T]; \dot{H}^{s-1})$. Then solutions u to the linear wave equation*

$$\begin{aligned} \square u &= h \\ u[0] &= (f, g) \end{aligned}$$

satisfy

$$(4) \quad \|u(t, \cdot)\|_{\dot{H}^s} + \|\partial_t u(t, \cdot)\|_{\dot{H}^{s-1}} \leq C \left(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \int_0^t \|h(r, \cdot)\|_{\dot{H}^{s-1}} dr \right)$$

In the proofs of these theorems and in the following discussion we will also need the following standard energy inequality for solutions to the linear wave equation

Lemma 2.4. *Suppose $(f, g) \in \dot{H}^1 \times L^2$ and $h \in L^1(\mathbb{R}; L^2(\mathbb{R}^n))$. Then, if u solves the linear wave equation (2), we have*

$$\|Du\|_{L_t^\infty L_x^2} \leq \|u[0]\|_{\dot{H}^1 \times L^2} + \|h\|_{L_t^1 L_x^2}$$

Proof of Lemma 2.4. As in the proof of conservation of energy we multiply our equation (2) by u_t and integrate (assuming sufficient vanishing conditions on u), to obtain

$$\begin{aligned}\partial_t \|Du(t, \cdot)\|_{L^2}^2 &= 2 \int u_t(t, x) h(t, x) dx \\ &\leq 2 \|u_t(t, \cdot)\|_{L^2} \|h(t, \cdot)\|_{L^2} \\ &\leq 2 \|Du(t, \cdot)\|_{L^2} \|h(t, \cdot)\|_{L^2}\end{aligned}$$

When $\|Du(t, \cdot)\|_{L^2} \neq 0$ we can divide through by it and integrate from 0 to t to obtain

$$\|Du(t, \cdot)\|_{L^2} \leq \|Du(t, 0)\|_{L^2} + \int_0^t \|h(t, \cdot)\|_{L^2}.$$

Taking the supremum over all $0 \leq t \leq \infty$ finishes the proof. \square

With this result we can now prove Theorem 2.1.

Proof of Theorem 2.1. Assuming the energy inequality, we first prove the existence and uniqueness statement. Taking the Fourier transform in the space variables and solving the resulting ODE gives us the Fourier representation of the solution

$$u(t, \cdot) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \cos(t\sqrt{-\Delta}) f + \int_0^t \frac{\sin((t-r)\sqrt{-\Delta})}{\sqrt{-\Delta}} h(r, \cdot) dr.$$

This gives existence for $(f, g) \in \mathcal{S} \times \mathcal{S}$ and $h \in C^\infty([0, T]; \mathcal{S})$. For the general case, we can use an approximation argument, using (3) to remove the smoothness assumption. Uniqueness follows directly from (3) by observing that if u and v are two solutions, then their difference solves (2) with $(f, g) = (0, 0)$ and $h = 0$. To prove (3), we estimate each piece of the Fourier representation.

Letting $K_t(x) := \mathcal{F}^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\right)$, where \mathcal{F}^{-1} denotes the inverse Fourier transform, we have

$$u(t, \cdot) = K_t * g + K_t' * f + \int_0^t K_{t-r} * h(r, \cdot) dr.$$

Then,

$$\begin{aligned}\|K_t * g\|_{H^s} &= \|(1 - \Delta)^{\frac{s}{2}}(K_t * g)\|_{L^2} \\ &= \|K_t * (1 - \Delta)^{\frac{s}{2}} g\|_{L^2} \\ &= \|K_t * G\|_{L^2}\end{aligned}$$

where $G := (1 - \Delta)^{\frac{s}{2}} g$. Now, observe that by Plancherel

$$\begin{aligned}
\|K_t * G\|_{L^2}^2 &= \|\hat{K}_t \hat{G}\|_{L^2}^2 \\
&= \int \left| \frac{\sin t |\xi|}{|\xi|} \right|^2 |\hat{G}(\xi)|^2 d\xi \\
&= \int_{|\xi| \leq 1} \left| \frac{\sin t |\xi|}{|\xi|} \right|^2 |\hat{G}(\xi)|^2 d\xi + \int_{|\xi| > 1} \left| \frac{\sin t |\xi|}{|\xi|} \right|^2 |\hat{G}(\xi)|^2 d\xi \\
&\leq \int_{|\xi| \leq 1} t^2 |\hat{G}(\xi)|^2 d\xi + \int_{|\xi| > 1} |\xi|^{-2} |\hat{G}(\xi)|^2 d\xi \\
&\leq 2(1+t^2) \int (1+|\xi|^2)^{-1} |\hat{G}(\xi)|^2 d\xi \\
&= 2(1+t^2) \|g\|_{H^{s-1}}
\end{aligned}$$

with the first inequality following from the obvious bounds on $\left| \frac{\sin t |\xi|}{|\xi|} \right|^2$ and the second inequality following from the fact that $|\xi| \leq 1 \Rightarrow 1 \leq \frac{2}{1+|\xi|^2}$ and $|\xi| > 1 \Rightarrow \frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$. This proves that

$$\|K_t * g\|_{H^s} \leq C(1+t) \|g\|_{H^{s-1}}$$

Similarly, using Plancherel again, we have

$$\begin{aligned}
\|K'_t * f\|_{H^s}^2 &= \|K'_t * (1 - \Delta)^{\frac{s}{2}} f\|_{L^2}^2 \\
&= \|(\cos t |\xi|)(1 - |\xi|^2)^{\frac{s}{2}} f\|_{L^2}^2 \\
&\leq \|(1 - |\xi|^2)^{\frac{s}{2}} f\|_{L^2}^2 \\
&= \|f\|_{H^s}^2
\end{aligned}$$

In the same manner we estimate the inhomogeneous part:

$$\begin{aligned}
(5) \quad \left\| \int_0^t K_{t-r} * h(r, \cdot) dr \right\|_{H^s} &= \left\| (1 - \Delta)^{\frac{s}{2}} \int_0^t K_{t-r} * h(r, \cdot) dr \right\|_{L^2} \\
&\leq \int_0^t \|K_{t-r} * (1 - \Delta)^{\frac{s}{2}} h(r, \cdot)\|_{L^2} dr \\
&\leq C \int_0^t (1 + |t-r|) \|h(r, \cdot)\|_{H^{s-1}} dr \\
&\leq C(1+t) \int_0^t \|h(r, \cdot)\|_{H^{s-1}} dr
\end{aligned}$$

where the first inequality above is by Minkowski's inequality and the second by our previous argument that proved $\|K_t * (1 - \Delta)^{\frac{s}{2}} g\|_{L^2} \leq C(1+t)\|g\|_{H^{s-1}}$. Putting this all together we get that

$$\|u(t, \cdot)\|_{H^s} \leq C(1+t) \left(\|f\|_{H^s} + \|g\|_{H^{s-1}} + \int_0^t \|h(r, \cdot)\|_{H^{s-1}} dr \right).$$

To finish the proof of (3) we would like the same estimate for $\partial_t u(t, \cdot)$. To obtain this estimate, we will need the standard energy inequality from Lemma 2.4.

$$(6) \quad \begin{aligned} \|Du(t, \cdot)\|_{L^2} &\leq C \left(\|Du(0, \cdot)\|_{L^2} + \int_0^t \|h(r, \cdot)\|_{L^2} dr \right) \\ &\leq C \left(\|\nabla f\|_{L^2} + \|g\|_{L^2} + \int_0^t \|h(t, \cdot)\|_{L^2} dr \right) \end{aligned}$$

where $D = (\partial_t, \nabla)$.

Now, observe that $(1 - \Delta)^{\frac{s-1}{2}}$ commutes with the \square operator, i.e. for any function u , $(1 - \Delta)^{\frac{s-1}{2}} \square u = \square \left((1 - \Delta)^{\frac{s-1}{2}} u \right)$. Hence if u solves

$$\begin{aligned} \square u &= h \\ u[0] &= (f, g), \end{aligned}$$

then $w := (1 - \Delta)^{\frac{s-1}{2}} u$ solves

$$\begin{aligned} \square w &= (1 - \Delta)^{\frac{s-1}{2}} h \\ w[0] &= \left((1 - \Delta)^{\frac{s-1}{2}} f, (1 - \Delta)^{\frac{s-1}{2}} g \right) \end{aligned}$$

Applying (6) to w gives us that

$$\begin{aligned} \|\partial_t u(t, \cdot)\|_{H^{s-1}} &= \|\partial_t \left((1 - \Delta)^{\frac{s-1}{2}} u \right)\|_{L^2} \\ &\leq C \left(\left\| (1 - \Delta)^{\frac{s-1}{2}} \nabla f \right\|_{L^2} + \left\| (1 - \Delta)^{\frac{s-1}{2}} g \right\|_{L^2} + \int_0^t \left\| (1 - \Delta)^{\frac{s-1}{2}} h(t, \cdot) \right\|_{L^2} \right) \\ &\leq C \left(\|f\|_{H^s} + \|g\|_{H^{s-1}} + \int_0^t \|h(t, \cdot)\|_{H^{s-1}} \right) \end{aligned}$$

This, combined with our estimate for u gives us

$$\|u(t, \cdot)\|_{H^s} + \|\partial_t u(t, \cdot)\|_{H^{s-1}} \leq C(1+t) \left(\|f\|_{H^s} + \|g\|_{H^{s-1}} + \int_0^t \|h(r, \cdot)\|_{H^{s-1}} dr \right)$$

as desired. □

Proof of Theorem 2.3. To prove (4) we again estimate each piece of the Fourier representation

$$u(t, \cdot) = K_t * g + K'_t * f + \int_0^t K_{t-r} * h(r, \cdot) dr.$$

where $K_t(x) := \mathcal{F}^{-1} \left(\frac{\sin(t|\xi|)}{|\xi|} \right)$, and \mathcal{F}^{-1} denotes the inverse Fourier transform. Using the homogeneous Sobolev norms we now have

$$\|K_t * g\|_{\dot{H}^1} = \|\nabla K_t * g\|_{L^2}.$$

We compute ∇K_t .

$$\begin{aligned}\partial_{x_j} K_t(x) &= \partial_{x_j} \int e^{ix \cdot \xi} \frac{\sin(t|\xi|)}{|\xi|} d\xi \\ &= \int \frac{i\xi_j e^{ix \cdot \xi} \sin(t|\xi|)}{|\xi|} d\xi.\end{aligned}$$

Hence

$$\begin{aligned}\|\partial_{x_j} K_t * g\|_{L^2}^2 &= \|\widehat{\partial_{x_j} K_t \hat{g}}\|_{L^2}^2 \\ &= \int \left| \frac{i\xi_j \sin(t|\xi|)}{|\xi|} \right|^2 |\hat{g}(\xi)|^2 d\xi \\ &\leq C \|g\|_{L^2}\end{aligned}$$

where the constant C is independent of t . The above estimate implies that

$$\begin{aligned}\|K_t * g\|_{\dot{H}^s} &= \|\nabla K_t * g\|_{\dot{H}^{s-1}} \\ &= \|\nabla K_t * (-\Delta)^{\frac{s-1}{2}} g\|_{L^2} \\ &\leq \|(-\Delta)^{\frac{s-1}{2}} g\|_{L^2} \\ &= C \|g\|_{\dot{H}^{s-1}}.\end{aligned}$$

The rest of the proof now follows by proceeding in exactly the same manner as in the proof of Theorem 2.1 except with \dot{H}^s norms instead of H^s norms. \square

3. LOCAL WELL POSEDNESS FOR QUASI-LINEAR EQUATIONS

With the energy estimates in hand we can address the question of well-posedness for quasi-linear equations with data in $H^s \times H^{s-1}$. We will be considering equations of the form

$$\begin{aligned}\square u &= F(u, Du) \\ u[0] &= (f, g)\end{aligned}$$

where $F \in C^\infty$, $F(0) = 0$, and $\nabla F(0) = 0$.

Remark 3.1. We cannot hope to prove global well-posedness for such a wide class of nonlinearities F . The following is an easy example of an equation whose solution blows up in finite time:

$$(7) \quad \begin{aligned}\square u &= (\partial_t u)^2 \\ u[0] &= (0, g)\end{aligned}$$

where $g \in C_0^\infty(\mathbb{R}^n)$. To see this, suppose $g = c$ is constant. Then u must be a function only of time, and our problem reduces to solving $u_{tt}(t) = (u_t(t))^2$. The solution to this ODE is given by $u(t) = -\log(1 - ct)$ which blows up as $t \rightarrow \frac{1}{c}$. To reach the same conclusion with $g \in C_0^\infty$, one can show that solutions to (7) are unique on backwards light cones. Then solve (7) on a solid backward light cone with height $\frac{1}{c}$ and base the ball of radius $\frac{1}{c}$. Then take $g = c$ on the ball of radius $R > \frac{1}{c}$ and $\text{supp}(g) \in B(0, 2R)$.

We can, however, prove local well-posedness.

Theorem 3.2. Let $(f, g) \in H^s \times H^{s-1}$ for $s > \frac{n}{2} + 1$. Let $F : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ satisfy $F \in C^\infty$, $F(0) = 0$, and $\nabla F(0) = 0$. Then, there exists a $T > 0$ such that

$$(8) \quad \begin{aligned} \square u &= F(u, Du) \\ u[0] &= (f, g) \end{aligned}$$

is well-posed in the space $X_T := C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ with the norm on X_T defined by

$$\|u\|_{X_T} := \sup_{0 \leq t < T} (\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}).$$

The proof will require the following results:

Lemma 3.3 (Gronwall's Lemma). Suppose $\alpha, \beta \geq 0$ are constants and $G \geq 0$ is a continuous function on $[0, T]$ such that

$$(9) \quad G(t) \leq \alpha + \beta \int_0^t G(r) dr$$

for every $0 \leq t \leq T$. Then

$$G(t) \leq \alpha e^{\beta t}$$

for every $0 \leq t \leq T$.

Proof of Lemma 3.3. Define $R(t) = \int_0^t G(r) dr$. Then

$$R'(t) = G(t) \leq \alpha + \beta R(t).$$

Multiplying both sides by $e^{-\beta t}$ gives that

$$\frac{d}{dt} (R(t)e^{-\beta t}) \leq \alpha e^{-\beta t}.$$

Integrate from 0 to t and multiply both sides by $e^{\beta t}$ to get

$$R(t) \leq -\frac{\alpha}{\beta}(1 + e^{\beta t})$$

Plugging this into (9) finishes the proof. □

We will also need the fact that for $s > \frac{n}{2}$, H^s is an algebra.

Lemma 3.4. Suppose $f, g \in H^s$ for $s > \frac{n}{2}$. Then

$$\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}$$

Proof of Lemma 3.4. Let P_k denote the Littlewood Paley projection onto the k th dyadic shell, i.e. $P_k f := \mathcal{F}^{-1}(\psi \hat{f}) = \hat{\psi} * f$ and the $\{\psi_j\}$'s form a dyadic partition of unity, i.e. $\sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1$ for every $\xi \in \mathbb{R}^n$, $\psi_j \in C_0^\infty(\{2^{j-1} \leq |\xi| \leq 2^{j+1}\})$ and $\psi_j(\xi) = \psi_1(2^{-(j-1)}\xi)$. In this proof we will also assume that $\psi_0 := \sum_{j \leq 0} \psi_j$. We will assume the following two facts which follow from the Littlewood Paley Theorem.

- (a) Suppose there is $C > 0$ such that a sequence $\{f_j\}$ satisfies $\text{supp}(\hat{f}_j) \subset \{C^{-1}2^j \leq |\xi| \leq C2^j\}$. Then if $f = \sum f_j$ we have

$$\|f\|_{H^s}^2 \leq C' \sum_{j \geq 0} 2^{2js} \|f_j\|_{L^2}^2$$

If we further assume that $f_j := P_j f$, then the converse is true as well, i.e.

$$\sum_{j \geq 0} 2^{2js} \|f_j\|_{L^2}^2 \leq C' \|f\|_{H^s}^2.$$

This holds for any $s \in \mathbb{R}$.

- (b) If $s > 0$ and if there is a $C > 0$ such that the sequence $\{f_j\}$ satisfies $\text{supp}(\hat{f}_j) \subset \{|\xi| \leq R2^j\}$ then

$$\|f\|_{H^s}^2 \leq C \sum_{j \geq 0} 2^{2sj} \|f_j\|_{L^2}^2$$

If $R = 2^k$ then the constant C above is of the form $C = C'2^{2sk}$.

Now, observe that we can write

$$\begin{aligned} fg &= \sum_{j,k \geq 0} P_j f P_k g \\ &= \sum_{k \leq j-10} P_j f P_k g + \sum_{j \leq k-10} P_j f P_k g + \sum_{|j-k| < 10} P_j f P_k g \\ &=: \text{I} + \text{II} + \text{III} \end{aligned}$$

Observe that for $k \leq j-10$ we have that $\text{supp}(\widehat{P_j f P_k g}) \subset \{C^{-1}2^j \leq |\xi| \leq C2^j\}$. For $j \leq k-10$ we have $\text{supp}(\widehat{P_j f P_k g}) \subset \{C^{-1}2^k \leq |\xi| \leq C2^k\}$. And for $|j-k| < 10$ we have $\text{supp}(\widehat{P_j f P_k g}) \subset \{|\xi| \leq C2^j\}$. Then, using (a) on the first two terms and (b) on the third term, we have that

$$\|fg\|_{H^s} \leq \|\text{I}\|_{H^s} + \|\text{II}\|_{H^s} + \|\text{III}\|_{H^s}$$

$$\lesssim \left(\sum_{j \geq 0} 2^{2sj} \|P_j f\|_{L^2} \sum_{k \leq j-10} \|P_k g\|_{L^2} \right)^{\frac{1}{2}} + \left(\sum_{k \geq 0} 2^{2sk} \|P_k\|_{L^2} \sum_{j \leq k-10} \|P_j f\|_{L^2} \right)^{\frac{1}{2}} + \left(\sum_{j \geq 0} 2^{2sj} \|P_j f\|_{L^2} \sum_{|j-k| \leq 10} \|P_k g\|_{L^2} \right)^{\frac{1}{2}}$$

$$=: A + B + D$$

We start by estimating A . Since $\|\sum_{k \leq j-10} P_k g\|_{L^\infty} \leq C\|g\|_{L^\infty}$ we have that

$$\begin{aligned} A &\leq C\|g\|_{L^\infty} \left(\sum_{j \geq 0} 2^{2sj} \|P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq C\|g\|_{L^\infty} \|f\|_{H^s} \end{aligned}$$

Since A and B are completely symmetric, we also have

$$B \leq C\|f\|_{L^\infty} \|g\|_{H^s}$$

For D we also have that $\|\sum_{|j-k| < 10} P_k g\|_{L^\infty} \leq C\|g\|_{L^\infty}$. Hence

$$\begin{aligned} D &\leq D\|g\|_{L^\infty} \left(\sum_{j \geq 0} 2^{2sj} \|P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq D\|g\|_{L^\infty} \|f\|_{H^s} \end{aligned}$$

□

We will also need Moser's inequality.

Lemma 3.5 (Moser's Inequality). *Let $F : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy $F \in C^\infty$ and $F(0) = 0$. Then for every $s \geq 0$, there exists a continuous function γ such that*

$$\|F(f)\|_{H^s} \leq \gamma(\|f\|_\infty)\|f\|_{H^s}$$

for every $f \in H^s \cap L^\infty$.

To prove Moser's inequality we will need the following lemma due to Bernstein

Lemma 3.6 (Bernstein's Lemma). *Suppose $f \in L^p$ with $1 \leq p \leq \infty$ satisfies $\text{supp}(\hat{f}) \subset \{|\xi| \leq R\}$. Then*

$$(10) \quad \|\partial^\alpha f\|_{L^p} \leq C_\alpha R^{|\alpha|} \|f\|_{L^p}$$

If, in addition, we have $\text{supp}(\hat{f}) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, then

$$(11) \quad C_\alpha^{-1} 2^{jk} \|f\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \leq C_\alpha 2^{jk} \|f\|_{L^p}$$

Proof of Lemma 3.6. Suppose $f \in \mathcal{S}$ is such that $\text{supp}(\hat{f}) \subset \{|\xi| \leq R\}$. Let $\chi \in C_0^\infty$ satisfy $\chi(\xi) = 1$ if $\xi \in \{|\xi| \leq 1\}$ and $\text{supp}(\chi) \subset \{|\xi| \leq 2\}$. Then define $\chi_R(\xi) = \chi(R^{-1}\xi)$. Then we have $\hat{f}(\xi) = \chi_R(\xi)\hat{f}(\xi)$ and hence $f(x) = \widehat{\chi_R} * f(x) = R^n \hat{\chi}(R\cdot) * f(x)$. Then by Young's inequality we have

$$\begin{aligned} \|\partial^\alpha f\|_{L^p} &\leq \|R^n \partial^\alpha \hat{\chi}(R\cdot)\|_{L^1} \|f\|_{L^p} \\ &= C_\alpha R^{|\alpha|} \|f\|_{L^p} \end{aligned}$$

This proves (10). We will prove (11) in the case $n = 1$ and $|\alpha| = 1$. Suppose $\text{supp}(\hat{f}) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Then f' also satisfies $\text{supp}(\hat{f}') \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Then, letting $\{\psi_j\}$ be a dyadic partition of unity as in the proof of Lemma 3.4, we have

$$i\xi \hat{f}(\xi) = \widehat{f'}(\xi) = \psi_j(\xi) \widehat{f'}(\xi)$$

which implies that for $\xi \neq 0$ we have

$$\widehat{f}(\xi) = -i \frac{\psi_j(\xi)}{\xi} \widehat{f'}(\xi)$$

Hence, we can write

$$f(x) = G * f'(x)$$

where $G = \mathcal{F}^{-1} \left(-i \frac{\psi_j(\xi)}{\xi} \right)$. Therefore, by Young's inequality, it suffices to show that $\|G\|_{L^1} \leq C 2^{-j}$.

$$\begin{aligned} G(x) &= \int_{2^{j-1}}^{2^{j+1}} \frac{\psi_j(\xi)}{\xi} e^{ix\xi} d\xi - \int_{2^{j-1}}^{2^{j+1}} \frac{\psi_j(\xi)}{\xi} e^{-ix\xi} d\xi \\ &= \int_{2^{-1}}^2 \frac{\psi_0(\xi)}{\xi} e^{i2^j x \xi} d\xi - \int_{2^{-1}}^2 \frac{\psi_j(\xi)}{\xi} e^{-i2^j x \xi} d\xi \end{aligned}$$

For $|x| \leq 2^{-j}$ we will use the trivial estimate $|G(x)| \leq C$. To derive a useful estimate for $|x| > 2^{-j}$ we can integrate the above expression for G by parts twice to get

$$G(x) = \int_{2^{-1}}^2 \frac{d^2}{d\xi^2} \left(\frac{\psi_0(\xi)}{\xi} \right) \frac{e^{i2^j x \xi}}{2^{2j} x^2} d\xi - \int_{2^{-1}}^2 \frac{d^2}{d\xi^2} \left(\frac{\psi_0(\xi)}{\xi} \right) \frac{e^{-i2^j x \xi}}{2^{2j} x^2} d\xi$$

which implies that $|G(x)| \leq C2^{-2j} |x|^{-2}$. Then

$$\begin{aligned} \int |G(x)| dx &= \int_{|x| \leq 2^{-j}} |G(x)| dx + \int_{|x| > 2^{-j}} |G(x)| dx \\ &\leq \int_{|x| \leq 2^{-j}} C dx + \int_{|x| > 2^{-j}} C2^{-2j} |x|^{-2} dx \\ &\leq C2^{-j} \end{aligned}$$

This proves the left inequality in (11). The right inequality follows from the proof of (10). \square

Proof of Lemma 3.5. As in the proof of Lemma 3.4, let P_j denote the j th Littlewood Paley projection. Also define $S_N f := \sum_{j=-\infty}^N P_j f$. We claim that it suffices to show that

$$(12) \quad \|F(S_N f) - F(S_M f)\|_{H^s} \leq \gamma(\|f\|_{L^\infty}) \|S_N f - S_M f\|_{H^s}$$

for every N, M . To see that this suffices, observe that if (12) holds, then, since $S_j f \rightarrow f$ in H^s , we have that $F(S_j f)$ is Cauchy in H^s and

$$\|F(S_j f)\|_{H^s} \leq \gamma(\|f\|_{L^\infty}) \|f\|_{H^s}$$

It also follows easily from the convergence in H^s of $S_j f$ that $F(S_j f) \rightarrow F(f)$ is H^s , which, combined with the above inequality implies

$$\|F(f)\|_{H^s} \leq \gamma(\|f\|_{L^\infty}) \|f\|_{H^s}$$

To prove (12) observe that

$$\begin{aligned} F(S_j f) - F(S_{j-1} f) &= \int_0^1 \frac{d}{d\lambda} F(S_{j-1} f + \lambda(S_j f - S_{j-1} f)) d\lambda \\ &= \int_0^1 F'(S_{j-1} f + \lambda P_j f) P_j f d\lambda \\ &= m_j P_j f \end{aligned}$$

where $m_j := \int_0^1 F'(S_{j-1} f + \lambda P_j f) d\lambda$. Then we can write

$$\begin{aligned} F(S_N f) - F(S_M f) &= \sum_{j=M+1}^N F(S_j f) - F(S_{j-1} f) \\ &= \sum_{j=M+1}^N m_j P_j f \end{aligned}$$

Now, we also break up m_j into dyadic pieces and write $m_j = S_j m_j + \sum_{k=1}^{\infty} P_{j+k} m_j$. Hence,

$$F(S_N f) - F(S_M f) = A + \sum_{k=1}^{\infty} B_k$$

where $A := \sum_{j=M+1}^N S_j m_j P_j f$ and $B_k := \sum_{j=M+1}^N P_{j+k} m_j P_j f$.

We claim that for every j, k , and for every $L \in \mathbb{N}$, we have

$$(13) \quad \|P_{j+k} m_j\|_{L^\infty} \leq 2^{-Lk} \gamma_L(\|f\|_{L^\infty})$$

To prove the claim we observe that $\partial^\alpha m_j$ consists of terms of the form

$$\int_0^1 F^{l+1}(g_\lambda) \partial^{\alpha_1} g_\lambda \dots \partial^{\alpha_l} g_\lambda d\lambda$$

where $g_\lambda := S_{j-1}f + \lambda P_j f$, $\sum \alpha_i = |\alpha|$ and $l \leq |\alpha|$. Now, by Lemma 3.6, (since each $\widehat{g_\lambda}$ is supported in $\{|\xi| \leq C2^j\}$), we have

$$\|\partial^\alpha m_j\|_{L^\infty} \leq 2^{j|\alpha|} \gamma_\alpha(\|f\|_{L^\infty})$$

where γ_α is a continuous function that depends on α and derivatives of F' up to order α . Applying Bernstein's Lemma again, (this time since $\widehat{P_{j+k}m_j}$ is supported in $\{C^{-1}2^{j+k} \leq |\xi| \leq C2^{j+k}\}$), followed by the above estimates we get

$$\begin{aligned} \|P_{j+k}m_j\|_{L^\infty} &\leq C_L 2^{-L(j+k)} \sup_{|\alpha|=L} \|\partial^\alpha P_{j+k}m_j\|_{L^\infty} \\ &\leq C_L 2^{-L(j+k)} \sup_{|\alpha|=L} \|\partial^\alpha m_j\|_{L^\infty} \|\widehat{\psi_{j+k}}\|_{L^1} \\ &\leq C_L 2^{-Lk} \gamma_L(\|f\|_{L^\infty}) \end{aligned}$$

proving the claim. We can now estimate A and B_k . Using fact (b) from the proof of Lemma 3.4, our above estimates for $\|P_{j+k}m - J\|_{L^\infty}$, and fact (a) from Lemma 3.4 we have

$$\begin{aligned} \|A\|_{H^s} &= \left\| \sum_{j=M+1}^N S_j m_j P_j f \right\|_{H^s} \leq C \left(\sum_{j=M+1}^N 2^{2sj} \|S_j m_j P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{j=M+1}^N 2^{2sj} \|S_j m_j\|_{L^\infty}^2 \|P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq C \gamma_0(\|f\|_{L^\infty}) \left(\sum_{j=M+1}^N 2^{2sj} \|P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq C \gamma_0(\|f\|_{L^\infty}) \left\| \sum_{j=M+1}^N P_j f \right\|_{H^s} \\ &= C \gamma_0(\|f\|_{L^\infty}) \|S_N f - S_M f\|_{H^s}. \end{aligned}$$

Similarly, except this time using the fact that when we apply (b) from the proof of Lemma 3.4 we have $R = 2^k$, we have

$$\begin{aligned} \|B_k\|_{H^s} &= \left\| \sum_{j=M+1}^N P_{j+k} m_j P_j f \right\|_{H^s} \leq 2^{sk} \left(\sum_{j=M+1}^N 2^{2sj} \|P_{j+k} m_j P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq 2^{sk} \gamma_L(\|f\|_{L^\infty}) 2^{-Lk} \|S_N f - S_M f\|_{H^s}. \end{aligned}$$

Choosing $L > s$ we then sum over k to get

$$\sum_{k=1}^{\infty} \|B_k\|_{H^s} \leq C \gamma_L(\|f\|_{L^\infty}) \|S_N f - S_M f\|_{H^s}.$$

This proves (12). □

We can now prove Theorem 3.2.

Proof of Theorem 3.2. We will use an iterative process to prove existence. We define a sequence in X_T by letting $u_{-1} = 0$ and, for every j , defining u_j to be the solution to the linear wave equation

$$\begin{aligned}\square u_j &= F(u_{j-1}, Du_{j-1}) \\ u_j[0] &= (f, g)\end{aligned}$$

We can show by induction that $\{u_j\}$ is a well defined sequence in X_T . Clearly, $u_{-1} \in X_T$. Assume $u_{j-1} \in X_T$. By Moser's inequality we have that

$$\|F(u_{j-1}, Du_{j-1})\|_{H^{s-1}} \leq \gamma(\|(u_{j-1}, Du_{j-1})\|_{L^\infty}) \|(u_{j-1}, Du_{j-1})\|_{H^{s-1}}$$

Since $s > \frac{n}{2} + 1$ we have, by Sobolev Embedding, that

$$\begin{aligned}\|(u_{j-1}, Du_{j-1})\|_{L^\infty} &\leq C\|(u_{j-1}, Du_{j-1})\|_{H^{s-1}} \\ &\leq C\|u_{j-1}\|_{H^s} \\ &\leq C\|u_{j-1}\|_{X_T}\end{aligned}$$

Hence $\gamma(\|(u_{j-1}, Du_{j-1})\|_{L^\infty}) \leq \sup_{0 \leq r \leq CR} |\gamma(r)|$. Which tells us that $F(u_{j-1}, Du_{j-1}) \in H^{s-1}$. Therefore, by Theorem 2.1, we have that $\{u_j\}$ is a well defined sequence in X_T and each u_j satisfies the energy inequality (3). We now claim that the u_j 's form a bounded sequence in X_T . More precisely, we claim that there exists $T > 0$ such that $\|u_j\|_{X_T} \leq 2CE_s =: R$ where $E_s := \|f\|_{H^s} + \|g\|_{H^{s-1}}$, and C is the constant arising in the energy estimates (3).

We prove this by induction. We have that $\|u_{-1}\|_{X_T} \leq R$. Assume that $\|u_{j-1}\|_{X_T} \leq R$. Then using the energy estimates (3) and assuming right away that $T < 1$, we have

$$(14) \quad \|u_j(t)\|_{H^s} + \|\partial_t u_j(t)\|_{H^{s-1}} \leq C \left(E_s + \int_0^t \|F(u_{j-1}, Du_{j-1})\|_{H^{s-1}} dr \right)$$

By Moser's Inequality we have that

$$\|F(u_{j-1}, Du_{j-1})\|_{H^{s-1}} \leq \gamma(\|(u_{j-1}, Du_{j-1})\|_{L^\infty}) \|(u_{j-1}, Du_{j-1})\|_{H^{s-1}}.$$

And, again by Sobolev Embedding,

$$\begin{aligned}\|(u_{j-1}, Du_{j-1})\|_{L^\infty} &\leq C\|(u_{j-1}, Du_{j-1})\|_{H^{s-1}} \\ &\leq C\|u_{j-1}\|_{H^s} \\ &\leq C\|u_{j-1}\|_{X_T}.\end{aligned}$$

Now, since by our inductive hypothesis $0 \leq \|u_{j-1}\|_{X_T} \leq R$ and since γ is continuous, we have

$$\begin{aligned}\|F(u_{j-1}, Du_{j-1})\|_{H^{s-1}} &\leq A\|(u_{j-1}, Du_{j-1})\|_{H^{s-1}} \\ &\leq A_1\|u_{j-1}\|_{X_T}\end{aligned}$$

where the above constant $A := \max_{0 \leq y \leq CR} |\gamma(y)|$. Using this estimate in (14) and taking the supremum over $t \in [0, T]$ gives

$$\begin{aligned}\|u_j\|_{X_T} &\leq C(E_s + TA_1\|u_{j-1}\|_{X_T}) \\ &\leq C(E_s + TA_1R).\end{aligned}$$

Letting $T \leq \frac{1}{2A_1}$ ensures that $\|u_j\|_{X_T} \leq R$.

We can now show that the u_j 's satisfy

$$\|u_{j+1} - u_j\|_{X_T} \leq \frac{1}{2}\|u_j - u_{j-1}\|_{X_T}$$

and are therefore a Cauchy sequence in X_T . Observe that the difference $w := u_{j+1} - u_j$ solves the linear wave equation

$$\begin{aligned}\square w &= F(u_j, Du_j) - F(u_{j-1}, Du_{j-1}) \\ w[0] &= 0\end{aligned}$$

Therefore, by our energy estimates (3) we have

(15)

$$\|u_{j+1}(t) - u_j(t)\|_{H^s} + \|\partial_t u_{j+1}(t) - \partial_t u_j(t)\|_{H^{s-1}} \leq C \int_0^t \|F(u_j, Du_j) - F(u_{j-1}, Du_{j-1})\|_{H^{s-1}} dr$$

We can estimate the term inside the integral as follows. Let $U_j := (u_j, Du_j)$. Then,

$$\begin{aligned}|F(U_j) - F(U_{j-1})| &\leq \int_0^1 \left| \frac{d}{d\lambda} F(U_{j-1} + \lambda(U_j - U_{j-1})) \right| d\lambda \\ &= |U_j - U_{j-1}| \int_0^1 |\nabla F(U_{j-1} + \lambda(U_j - U_{j-1}))| d\lambda\end{aligned}$$

Taking the H^s norm of both sides and using the fact that H^{s-1} is an algebra (Lemma 3.4) for $s > \frac{n}{2} + 1$ gives us that

$$\|F(U_j) - F(U_{j-1})\|_{H^{s-1}} \leq \|U_j - U_{j-1}\|_{H^{s-1}} \int_0^1 \|\nabla F(U_{j-1} + \lambda(U_j - U_{j-1}))\|_{H^{s-1}} d\lambda.$$

And by Moser's Inequality again the above is

$$\leq \|U_j - U_{j-1}\|_{H^{s-1}} \gamma(\|U_{j-1} + \lambda(U_j - U_{j-1})\|_{L^\infty}) \|U_{j-1} + \lambda(U_j - U_{j-1})\|_{H^{s-1}}.$$

By Sobolev Embedding, the continuity of γ , and the fact that $\|u_j\|_{X_T} \leq R$ we have that $\gamma(\|U_{j-1} + \lambda(U_j - U_{j-1})\|_{L^\infty})$ is bounded by a constant depending on R and the nonlinearity F . And we have

$$\begin{aligned}\|U_{j-1} + \lambda(U_j - U_{j-1})\|_{H^{s-1}} &\leq C(\|U_{j-1}\|_{H^{s-1}} + \|U_j\|_{H^{s-1}}) \\ &\leq C(\|u_{j-1}\|_{H^s} + \|u_j\|_{H^s}) \\ &\leq 2CR\end{aligned}$$

Hence,

$$(16) \quad \|F(U_j) - F(U_{j-1})\|_{H^{s-1}} \leq CR\|u_j - u_{j-1}\|_{H^s} \leq CR\|u_j - u_{j-1}\|_{X_T}$$

Also we have $\|U_j - U_{j-1}\|_{H^{s-1}} \leq C\|u_j - u_{j-1}\|_{H^s}$. Hence, taking the supremum over $0 \leq t \leq T$ in (15) we get that

$$\|u_{j+1} - u_j\|_{X_T} \leq TCR\|u_j - u_{j-1}\|_{X_T}.$$

And we can choose $T > 0$ such that $TCR \leq \frac{1}{2}$. This shows that $\{u_j\}$ is Cauchy in X_T and hence there is a function $u \in X_T$ such that $u_j \rightarrow u$ in X_T . We would like to show that this u is our solution to (8). To see this observe that for every j we have

$$(17) \quad \begin{aligned}u_j(t, \cdot) &= \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \cos(t\sqrt{-\Delta}) f + \int_0^t \frac{\sin((t-r)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u_{j-1}, Du_{j-1}) dr. \\ &= K_t * g + K'_t * f + \int_0^t K_{t-r} * F(U_{j-1})\end{aligned}$$

where $K_t := \mathcal{F}^{-1}\left(\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\right)$ and $U_j := (u_j, Du_j)$. By our above work, the left hand side of (17) converges in X_T to u . To deal with convergence of the right hand side we first show that $F(U_j) \rightarrow F(U)$ in H^{s-1} . Indeed, we have

$$|F(U_j) - F(U)| \leq |U_j - U| \int_0^1 |\nabla F(U + \lambda(U_j - U))| d\lambda$$

which implies, as in (16), that

$$\begin{aligned} \|F(U_j) - F(U)\|_{H^{s-1}} &\leq CR \|u_j - u\|_{H^s} \\ &\leq CR \|u_j - u\|_{X_T} \rightarrow 0. \end{aligned}$$

Hence $F(U_j) \rightarrow F(U)$ in H^{s-1} .

We would like to show that $\int_0^t K_{t-r} * F(U_{j-1}) \rightarrow \int_0^t K_{t-r} * F(U)$ in X_T . Using the same estimates as in (5) we get

$$\begin{aligned} \left\| \int_0^t K_{t-r} * (F(U_{j-1}) - F(U)) \right\|_{H^s} &\leq \int_0^t \|K_{t-r} * (F(U_{j-1}) - F(U))\|_{H^s} \\ &\leq C \int_0^t (1 + |t-r|) \|F(U_{j-1}) - F(U)\|_{H^{s-1}} \\ &\leq C \int_0^t \|F(U_{j-1}) - F(U)\|_{H^{s-1}} \\ &\leq CR \int_0^t \|u_j - u\|_{X_T} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \left\| \partial_t \left(\int_0^t K_{t-r} * (F(U_{j-1}) - F(U)) \right) \right\|_{H^{s-1}} &= \left\| \int_0^t K'_{t-r} * (F(U_{j-1}) - F(U)) \right\|_{H^{s-1}} \\ &\leq \int_0^t \|K'_{t-r} * (F(U_{j-1}) - F(U))\|_{H^{s-1}} \\ &\leq \int_0^t \|F(U_{j-1}) - F(U)\|_{H^{s-1}} \\ &\leq \int_0^t \|u_j - u\|_{X_T} \end{aligned}$$

Putting together the last two estimates and taking the supremum over $0 \leq t \leq T$, we have

$$\left\| \int_0^t K_{t-r} * (F(U_{j-1}) - F(U)) \right\|_{X_T} \leq CRT \|u_j - u\|_{X_T} \rightarrow 0$$

Therefore we can let $j \rightarrow \infty$ in (17) to get that

$$u(t, \cdot) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \cos(t\sqrt{-\Delta}) f + \int_0^t \frac{\sin((t-r)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u, Du) dr.$$

which implies that u is our desired solution.

We would now like to show that the solution $u \in X_T$ is unique. To see this, assume that there are two solutions $u, v \in X_T$. Then the difference $w := u - v$ solves the following equation

$$\begin{aligned}\square w &= F(u, Du) - F(v, Dv) \\ w[0] &= 0\end{aligned}$$

By the energy estimates (3), we have, for every $0 \leq t \leq T$, that

$$\|u - v\|_{H^s} + \|\partial_t u - \partial_t v\|_{H^{s-1}} \leq C \int_0^t \|F(U) - F(V)\|_{H^{s-1}}$$

where again we are using the notation $U := (u, Du)$. As we have seen above in (16), we have that

$$\begin{aligned}\|F(U) - F(V)\|_{H^{s-1}} &\leq CR\|u - v\|_{H^s} \\ &\leq CR(\|u - v\|_{H^s} + \|\partial_t(u - v)\|_{H^{s-1}})\end{aligned}$$

Hence

$$\|u - v\|_{H^s} + \|\partial_t(u - v)\|_{H^{s-1}} \leq CR \int_0^t (\|u - v\|_{H^s} + \|\partial_t(u - v)\|_{H^{s-1}})$$

which, by Gronwall's inequality, tells us that $\|u - v\|_{X_T} = 0$, proving uniqueness.

Finally, we show that solutions depend continuously on the initial data. Let $u[0] := (f_0, g_0) \in H^s \times H^{s-1}$ and let $T > 0$ be defined as above so that there exists a unique $u \in X_T$ such that u solves (8). We would like to show that there is a neighborhood $N \ni (f, g)$ in $H^s \times H^{s-1}$ and a continuous map $S : N \rightarrow X_T$ which associates to each $(f, g) \in N$ the unique $v \in X_T$ such that v solves our quasi-linear problem with initial data (f, g) .

To see this let $v[0] := (f, g) \in H^s \times H^{s-1}$ and let $T_1 > 0$ and $v \in X_{T_1}$ be the corresponding time and solution. Choosing T small enough above we can take $v[0]$ close enough in $H^s \times H^{s-1}$ to $u[0]$ so that we can take $T_1 = T$. Then, by the energy inequality (3), we have

$$\|u - v\|_{H^s} + \|\partial_t(u - v)\|_{H^{s-1}} \leq C \left(\|u[0] - v[0]\|_{H^s \times H^{s-1}} + \int_0^t \|F(u, Du) - F(v, Dv)\|_{H^{s-1}} \right).$$

Using the same methods as in the proofs of existence and uniqueness, the above inequality implies, by our choice of T , that

$$\|u - v\|_{X_T} \leq C\|u[0] - v[0]\| + \frac{1}{2}\|u - v\|_{X_T}$$

which proves that the solution map is, in fact, not only continuous, but Lipschitz.

Also, by differentiating the equation and using the same ideas we can prove that additional smoothness in the initial data persist for the full time of existence. In particular, if $f, g \in C_0^\infty(\mathbb{R}^n)$ then $u \in C^\infty([0, T] \times \mathbb{R}^n)$. \square

Remark 3.7. If our nonlinearity F is of the form $F = F(u)$ then the same argument works if we only require that $s > \frac{n}{2}$.

4. STRICHARTZ ESTIMATES

Having proven local well-posedness for quasi-linear equations with data in $H^s \times H^{s-1}$ for large s , we now address the same questions for the semi-linear equation in (1+3)-dimensions with data in energy space $\dot{H}^1 \times L^2$. Our model semi-linear problem is

$$\begin{aligned}\square u &= \pm |u|^{\alpha-1} u \\ u[0] &= (f, g) \in \dot{H}^1 \times L^2\end{aligned}$$

We can prove local existence and uniqueness for $1 \leq \alpha \leq 3$ by using only energy methods. The energy critical exponent for this problem, however, is $\alpha = 5$ so we would like to be able to prove local well-posedness for α up to and including the critical exponent. Strichartz estimates are the tool that will allow us to do so.

Definition 4.1. A pair of exponents (q, p) are said to be wave-admissible if $p, q \geq 2$ and

$$\frac{1}{q} + \frac{n-1}{2p} \leq \frac{n-1}{4}.$$

Theorem 4.2. *Let u be a solution to the wave equation*

$$(18) \quad \begin{aligned}\square u &= h \\ u[0] &= (f, g),\end{aligned}$$

where $(f, g) \in \dot{H}^\gamma \times \dot{H}^{\gamma-1}$ and $h \in L_t^{a'} L_x^{b'}$. Then, u satisfies the following a-priori estimates:

$$(19) \quad \|u\|_{L_t^q L_x^p} \leq C(\|u[0]\|_{\dot{H}^\gamma \times \dot{H}^{\gamma-1}} + \|h\|_{L_t^{a'} L_x^{b'}})$$

provided (q, p) and (a, b) are “wave-admissible” and (q, p) , (a, b) , and γ satisfy the scaling condition

$$(20) \quad \frac{1}{q} + \frac{n}{p} = \frac{1}{a'} + \frac{n}{b'} - 2 = \frac{n}{2} - \gamma.$$

Remark 4.3.

- The wave-admissibility condition is necessary because we can show that the case $f = 0$ and $h = 0$ in the above estimate is equivalent to the Stein-Tomas Restriction Theorem when $p = q$. And, the Knapp example shows that the condition that (p, p) be wave-admissible is, in fact, optimal in the case of the Restriction Theorem.
- Condition (20) is necessitated by scaling considerations.

To prove Theorem 4.2, we will need to derive pointwise estimates for solutions to the wave equation. Recall the Fourier representation of solutions to the homogeneous wave equation

$$u(\cdot, t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \cos(t\sqrt{-\Delta}) f.$$

If we assume that f and g have Fourier support in the dyadic shell $\{\frac{1}{2} \leq |\xi| \leq 2\}$, we can derive pointwise estimates by controlling the operator $e^{\pm it\sqrt{-\Delta}}$.

Lemma 4.4 (Pointwise Estimates). *Let $f \in \mathcal{S}$ satisfy $\text{supp}(\hat{f}) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. Then*

$$(21) \quad \left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_x^\infty} \leq C \langle t \rangle^{-\frac{(n-1)}{2}} \|f\|_{L^1}$$

$$(22) \quad \left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_x^2} \leq \|f\|_{L^2}$$

Proof. First we write

$$\begin{aligned} e^{\pm it\sqrt{-\Delta}} f &= \int \int e^{i(\pm t|\xi| + (x-y)\cdot\xi)} \chi(|\xi|) d\xi f(y) dy \\ &= K_t * f \end{aligned}$$

where $\chi \in C_0^\infty$ satisfies $\chi \equiv 1$ on $\{\frac{1}{2} \leq |\xi| \leq 2\}$ and $K_t(x) = \int e^{i(\pm t|\xi| + x\cdot\xi)} \chi(|\xi|) d\xi$. Then, we have $\|e^{\pm it\sqrt{-\Delta}} f\|_{L_x^\infty} \leq \|K_t\|_{L^\infty} \|f\|_{L^1}$. Hence, for the first inequality, it suffices to show that

$$\|K_t\|_{L^\infty} \leq C \langle t \rangle^{-\frac{(n-1)}{2}}$$

where

$$K_t(x) = \int e^{i(\pm t|\xi| + x\cdot\xi)} \chi(|\xi|) d\xi.$$

First we observe that $|K_t|(x) \leq C$ for all t , so we will only concern ourselves with t large. The phase function for K_t is $\phi(\xi) := \pm t|\xi| + x\cdot\xi$. We have $\nabla_\xi \phi(\xi) = \pm t \frac{\xi}{|\xi|} + x = 0$ only if $|x| = t$. This tells us that the critical points for ϕ lie on a line and, therefore, are not isolated. Hence we cannot apply stationary phase estimates directly. To avoid this problem we change to polar coordinates. Set $\xi = r\theta$, where $r = |\xi|$ and $\theta \in S^{n-1}$. Then we have

$$\begin{aligned} K_t(x) &= \int_{\frac{1}{2}}^2 e^{i\pm tr} \chi(r) r^{n-1} \int_{S^{n-1}} e^{irx\cdot\theta} d\sigma(\theta) dr \\ &= \int_{\frac{1}{2}}^2 e^{i\pm tr} \chi(r) r^{n-1} \hat{\sigma}(rx) dr \end{aligned}$$

Now, recall the following stationary phase type estimates for the surface measure of the sphere.

$$\hat{\sigma}(\eta) = e^{-i|\eta|} \omega_- (|\eta|) + e^{i|\eta|} \omega_+ (|\eta|)$$

where ω_\pm are functions that satisfy $|\partial^\alpha \omega_\pm(\eta)| \leq C |\eta|^{-\frac{(n-1)}{2} - |\alpha|}$ for each α . Plugging in these estimates for $\hat{\sigma}(rx)$, we get that

$$K_t(x) = \int_{\frac{1}{2}}^2 e^{i\pm tr} \chi(r) r^{n-1} (e^{-i|\eta|} \omega_- (|\eta|) + e^{i|\eta|} \omega_+ (|\eta|)) dr$$

Then for $|x|$ such that $C^{-1}t \leq |x| \leq Ct$ and t large, we have that

$$|K_t(x)| \leq Ct^{-\frac{(n-1)}{2}} \leq C' \langle t \rangle^{-\frac{(n-1)}{2}}$$

Now, for $|x|$ and t far apart we see that our original phase function ϕ is non-stationary so we expect to get any type of decay that we want. To be more precise, for $|x| \neq t$, K_t consists of terms of the form

$$A(x, t) := \int_{\frac{1}{2}}^2 e^{ir(\pm t \mp |x|)} \omega_\pm(rx) \chi(r) dr$$

Integrating by parts as many times as we like and using the decay estimates for $\partial^\alpha \omega_\pm$, we get that

$$|A| \leq C |x|^{-\frac{(n-1)}{2}} (\pm t \mp |x|)^{-N}$$

for any N . And since $|x|$ is bounded away from t we can make this as small as we want. this proves (21).

(22) follows from the following:

$$\begin{aligned}
\left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_x^2} &\leq \|K_t * f\|_{L^2} \\
&= \|\hat{K}_t \hat{f}\|_{L^2} \\
&\leq \|\hat{K}_t\|_{L^\infty} \|\hat{f}\|_{L^2} \\
&\leq C \|f\|_{L^2}
\end{aligned}$$

□

Interpolating between the two inequalities in Theorem 4.4 we immediately get

Lemma 4.5. *Let $f \in \mathcal{S}$ satisfy $\text{supp}(\hat{f}) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. Then, for $2 \leq p \leq \infty$ we have*

$$(23) \quad \left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_x^p} \leq C \langle t \rangle^{-\frac{(n-1)}{2}(\frac{1}{p'} - \frac{1}{p})} \|f\|_{L^{p'}}$$

Now, for the Strichartz estimates for solutions of the homogeneous wave equation we would like estimates of the form

$$(24) \quad \left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_t^q L_x^p} \leq C \|f\|_{L^2}$$

To obtain these we will use the standard TT^* argument which gets its name from the following lemma:

Lemma 4.6. *Let A be a Hilbert space and B a Banach space. Then the following are equivalent:*

- (i) $T : A \rightarrow B$ is bounded
- (ii) $T^* : B^* \rightarrow A$ is bounded
- (iii) $TT^* : B^* \rightarrow B$ is bounded

We will also need the following version of the Hardy-Littlewood-Sobolev inequality

Lemma 4.7. *Let $I_r(F) = \int_{-\infty}^{\infty} \langle t-s \rangle^{-\frac{1}{r}} F(s) ds$ and suppose $1 < q' < q < \infty$. Then, if $r > 1$ and $\frac{1}{r} \geq 1 - (\frac{1}{q'} - \frac{1}{q})$ we have*

$$\|I_r(F)\|_{L^q} \leq C \|F\|_{L^{q'}}.$$

With this we can now prove the key lemma

Lemma 4.8. *Let $f \in \mathcal{S}$ satisfy $\text{supp}(\hat{f}) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$ and $g \in \mathcal{S}(\mathbb{R}^{n+1})$ have spacial Fourier support, $\text{supp}(\hat{g}(\cdot, t)) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. Then for (q, p) and (a, b) wave-admissible, we have*

- (i) $\left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_t^q L_x^p} \leq C \|f\|_{L^2}$
- (ii) $\left\| \int_{-\infty}^{\infty} e^{\mp is\sqrt{-\Delta}} g(\cdot, s) ds \right\|_{L_x^2} \leq C \|g\|_{L_t^{a'} L_x^{b'}}$
- (iii) $\left\| \int_{-\infty}^{\infty} e^{\pm i(t-s)\sqrt{-\Delta}} g(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq C \|g\|_{L_t^{a'} L_x^{b'}}$

Proof. Define $T(f) = e^{\pm it\sqrt{-\Delta}} f$. A simple computation shows that then T^* and TT^* are given by $T^*(g) = \int_{-\infty}^{\infty} e^{\mp is\sqrt{-\Delta}} g(\cdot, s) ds$, and $TT^*(g) = \int_{-\infty}^{\infty} e^{\pm i(t-s)\sqrt{-\Delta}} g(\cdot, s) ds$ respectively.

We now start by proving that (iii) holds with $(a', b') = (q', p')$. By Minkowski's integral inequality, (since $p \geq 2 \geq 1$), and the $L^{p'} \rightarrow L^p$ estimates from Lemma 4.5, we have

$$\begin{aligned} \|TT^*g(\cdot, t)\|_{L_x^p} &\leq \int \|e^{\pm i(t-s)\sqrt{-\Delta}}g(\cdot, s)\|_{L_x^p} ds \\ &\leq C \int \langle t-s \rangle^{-\frac{n-1}{2}(\frac{1}{p'} - \frac{1}{p})} \|g(\cdot, s)\|_{L_x^{p'}} ds \end{aligned}$$

Now, we can apply the Hardy-Littlewood-Sobolev inequality to get

$$(25) \quad \|TT^*g\|_{L_t^q L_x^p} \leq C \|g\|_{L_t^{q'} L_x^{p'}}$$

as long as we have $\frac{n-1}{2}(\frac{1}{p'} - \frac{1}{p}) \geq 1 - (\frac{1}{q'} - \frac{1}{q})$ and $1 < q' < q < \infty$. But this is exactly satisfied by the the wave-admissibility condition except for the case $q = 2$.¹ Now, we apply Lemma 4.6 to conclude that both (i) and (ii) hold for all wave-admissible pairs (q, p) and (a, b) (Here we actually use Lemma 4.6 twice; once with $A = L_x^2$, $B = L_t^q L_x^p$ and then with $A = L_x^2$, $B = L_t^a L_x^b$). Finally we conclude that (iii) is true by setting $f = \int_{-\infty}^{\infty} e^{\mp is\sqrt{-\Delta}}g(\cdot, s) ds$ in (i) and then applying (ii) to the right hand side. \square

Remark 4.9. We have now proved Strichartz estimates for the homogeneous wave equation in the special case where f and g both have Fourier support in $\{\frac{1}{2} \leq |\xi| \leq 2\}$. To deal with the inhomogeneous equation, with h having spatial Fourier support in $\{\frac{1}{2} \leq |\xi| \leq 2\}$, we have to control the operator $h \mapsto \int_0^t e^{\pm i(t-s)\sqrt{-\Delta}}h(\cdot, s) ds$.

Lemma 4.10. *Suppose $h \in \mathcal{S}(\mathbb{R}^{n+1})$ is such that the the spatial Fovier support of h satisfies $\text{supp}(\hat{h}(\cdot, t)) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. Then, for all wave admissible (q, p) and (a, b) we have*

$$(26) \quad \left\| \int_0^t e^{\pm i(t-s)\sqrt{-\Delta}}h(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq C \|h\|_{L_t^{a'} L_x^{b'}}$$

To prove Lemma 4.10 we observe that by Lemma 4.8 part (iii) we have

$$(27) \quad \left\| \int_{-\infty}^{\infty} e^{\pm i(t-s)\sqrt{-\Delta}}h(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq C \|h\|_{L_t^{a'} L_x^{b'}}$$

for all wave-admissible (q, p) and (a, b) . Then Lemma 4.10 follows from the following lemma of Christ and Kiselev.

Lemma 4.11 (Christ-Kiselev Lemma). *Suppose X and Y are Banach spaces and suppose $K(t, s)$ is a continuous function taking values in the $B(X, Y)$, the space of bounded linear mappings from Y to X . Suppose $-\infty \leq a < b \leq \infty$ and set*

$$\begin{aligned} T(f)(t) &= \int_a^b K(t, s)f(s) ds. \\ W(f)(t) &= \int_a^t K(t, s)f(s) ds. \end{aligned}$$

And suppose

$$\|T(f)\|_{L^q((a,b);X)} \leq C \|f\|_{L^r((a,b);Y)}.$$

¹the endpoint case $q = 2$ requires a different argument that I will not give here.

Then, if $1 \leq r < q \leq \infty$, we have

$$\|W(f)\|_{L^q((a,b);X)} \leq C_{pr} \|f\|_{L^r((a,b);Y)}.$$

Proof of Lemma 4.11. We prove the case $q < \infty$. Normalize f so that $\|f\|_{L^r((a,b);Y)} = 1$. We can assume, without loss of generality, that f is continuous and if

$$F(t) := \int_a^t \|f(s)\|_Y^r ds$$

then $F : (a, b) \rightarrow (0, 1)$ is monotone and therefore a bijection. Hence, if $I = (c, d) \in (0, 1)$ is an interval, then $F^{-1}(I) = (F^{-1}(c), F^{-1}(d))$ is also an interval and

$$\begin{aligned} \|\chi_{F^{-1}(I)}(s)f(s)\|_{L^r((a,b);Y)}^r &= \int_{F^{-1}(c)}^{F^{-1}(d)} \|f(s)\|_Y^r ds \\ &= F(F^{-1}(d)) - F(F^{-1}(c)) \\ &= d - c \\ &= |I| \end{aligned}$$

Now, consider the set of all dyadic subintervals of $(0, 1)$,

$$\{(k-1)2^{-j}, k2^{-j} : 1 \leq k \leq 2^j, j = 1, 2, 3, \dots\}.$$

We define a relation, \sim , on dyadic subintervals by setting $I \sim J$ if $|I| = |J|$ and I and J are not adjacent but have adjacent parent intervals. Then for a fixed J there are at most 3 intervals I such that $I \sim J$. Now let W denote the Whitney decomposition of the cube $(0, 1) \times (0, 1)$, (where the size of each subcube is proportional to the distance to the diagonal), and let $\pi_1(W)$ and $\pi_2(W)$ denote the projections of this decomposition onto the x and y axes, respectively. With these projections, we can see that for almost every $(x, y) \in (0, 1) \times (0, 1)$ with $x < y$, there is a unique pair of dyadic subintervals $I \ni x$, $J \ni y$ such that $I \sim J$. Now, setting $x = F(s)$ and $y = F(t)$, we have that (a.e)

$$\begin{aligned} \chi_{\{(s,t) \in (a,b) \times (a,b) : s < t\}}(s, t) &= \chi_{\{(x,y) \in (0,1) \times (0,1) : x < y\}}(x, y) \\ &= \sum_{I \sim J} \chi_I(x) \chi_J(y) \\ &= \sum_{I \sim J} \chi_{F^{-1}(I)}(s) \chi_{F^{-1}(J)}(t). \end{aligned}$$

Then we have

$$\begin{aligned} W(f) &= \int_a^b \chi_{\{s < t\}} K(t, s) f(s) ds \\ &= \int_a^b \sum_{I \sim J} \chi_{F^{-1}(I)}(s) \chi_{F^{-1}(J)}(t) K(t, s) f(s) ds \\ &= \sum_{I \sim J} \chi_{F^{-1}(J)}(t) T(\chi_{F^{-1}(I)} f) \end{aligned}$$

Therefore, since there are only finitely many I with $I \sim J$ and since the J with $|J| = 2^{-j}$ are disjoint, we have

$$\begin{aligned}
\|W(f)\|_{L^q((a,b);X)} &\leq \sum_{j=1}^{\infty} \left\| \sum_{I \sim J, |I|=2^{-j}} \chi_{F^{-1}(J)}(t) T(\chi_{F^{-1}(I)} f) \right\|_{L^q((a,b);X)} \\
&\leq \sum_{j=1}^{\infty} C \left(\sum_{|I|=2^{-j}} \|T(\chi_{F^{-1}(I)} f)\|_{L^q((a,b);X)}^q \right)^{\frac{1}{q}} \\
&\leq \sum_{j=1}^{\infty} C \left(\sum_{|I|=2^{-j}} \|\chi_{F^{-1}(I)} f\|_{L^r((a,b);Y)}^q \right)^{\frac{1}{q}} \\
&= \sum_{j=1}^{\infty} C \left(\sum_{|I|=2^{-j}} |I|^{\frac{q}{r}} \right)^{\frac{1}{q}} \\
&= \sum_{j=1}^{\infty} C 2^{-j(\frac{1}{r} - \frac{1}{q})} \\
&\leq C
\end{aligned}$$

since we are assuming $q > r$. And we are done since we assumed $\|f\|_{L^r((a,b);Y)} = 1$. □

This takes care of Lemma 4.10 for all wave-admissible (q, p) and (a, b) except for the case $q = a' = 2$. This is, again, an endpoint that will not be dealt with here. We are now ready to prove the non-endpoint Strichartz estimates.

Proof of Theorem 4.2. The outline for the proof will be as follows

- (a) show the theorem holds in the case when f, g and h all have Fourier support in the dyadic shell $\{\frac{1}{2} \leq |\xi| \leq 2\}$
- (b) use the scaling condition (8) to show that these frequency 0 estimates imply the same estimates for all f, g and h with Fourier support in the dyadic shell $\{2^{j-1} \leq |\xi| \leq 2^j\}$ independently of j .
- (c) Observe that step (b) is enough by the Littlewood-Paley theorem.

We begin by proving (a).

Suppose f, g and h all have Fourier support in the dyadic shell $\{\frac{1}{2} \leq |\xi| \leq 2\}$. Since $\hat{f}(\xi) = 0$ if $\xi \notin \{\frac{1}{2} \leq |\xi| \leq 2\}$, we have by Plancherel that

$$C^{-1} \|f\|_{L^2} \leq \left(\int |\xi|^{2\gamma} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \|f\|_{\dot{H}^\gamma} \leq C \|f\|_{L^2}$$

for any γ . Hence,

$$\left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_t^q L_x^p} \leq C \|f\|_{\dot{H}^\gamma}$$

By the same reasoning we have

$$\left\| \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} g \right\|_{L_t^q L_x^p} \leq C \left\| e^{\pm it\sqrt{-\Delta}} g \right\|_{L_t^q L_x^p} \leq C \|g\|_{\dot{H}^{\gamma-1}}$$

where the first inequality comes from the fact that an extra $|\xi|^{-1}$ term does not matter in the pointwise estimates, (Lemma 4.4), since, in this case, $\frac{1}{2} \leq |\xi|^{-1} \leq 2$. And finally we have, by Lemma 4.10 and the same reasoning as above, that

$$(28) \quad \left\| \int_0^t \frac{e^{\pm(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} h(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq \left\| \int_0^t e^{\pm(t-s)\sqrt{-\Delta}} h(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq \|h\|_{L_t^{q'} L_x^{p'}}$$

Putting these last three estimates together proves Theorem 4.2 in the special case when f , g and h have Fourier support in $\{\frac{1}{2} \leq |\xi| \leq 2\}$.

Now let $\{\psi_j\}$ be a dyadic partition of unity, i.e. $\sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1$ for every $\xi \in \mathbb{R}^n$, $\psi_j \in C_0^\infty(\{2^{j-1} \leq |\xi| \leq 2^{j+1}\})$ and $\psi_j(\xi) = \psi_0(2^{-j}\xi)$. Then we have,

$$\begin{aligned} e^{\pm it\sqrt{-\Delta}} f &= \sum_{j \in \mathbb{Z}} \int e^{i(\pm t|\xi| + x \cdot \xi)} \psi_j(\xi) \hat{f}(\xi) d\xi \\ &=: \sum_{j \in \mathbb{Z}} K_j(f) \end{aligned}$$

And for each j we have,

$$\begin{aligned} K_j(f)(x, t) &:= \int e^{i(\pm t|\xi| + x \cdot \xi)} \psi_0(2^{-j}\xi) \hat{f}(\xi) d\xi \\ &= \int e^{i(\pm 2^j t|\xi| + 2^j x \cdot \xi)} \psi_0(\xi) 2^{nj} \hat{f}(2^j \xi) d\xi \\ &= \int e^{i(\pm 2^j t|\xi| + 2^j x \cdot \xi)} \psi_0(\xi) \hat{f}_{2^{-j}}(\xi) d\xi \\ &= K_0(f_{2^{-j}})(2^j x, 2^j t) \end{aligned}$$

where $f_{2^{-j}}(x) = f(2^{-j}x)$.

Therefore, by the above, by (a), and by the scaling condition, (20), we have

$$\begin{aligned} \|K_j(f)\|_{L_t^q L_x^p} &= 2^{j(-\frac{1}{q} - \frac{n}{p})} \|K_0(f_{2^{-j}})\|_{L_t^q L_x^p} \\ &\leq C 2^{j(-\frac{1}{q} - \frac{n}{p})} \|f_{2^{-j}}\|_{\dot{H}^\gamma} \\ &= C 2^{j(-\frac{1}{q} - \frac{n}{p} + \frac{n}{2} - \gamma)} \|f\|_{\dot{H}^\gamma} \\ &= C \|f\|_{\dot{H}^\gamma}, \end{aligned}$$

or, equivalently, if we assume above that the Fourier support of f is contained in the dyadic shell $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, we have

$$\left\| e^{\pm it\sqrt{-\Delta}} (\hat{\psi}_j * f) \right\|_{L_t^q L_x^p} \leq C \|\hat{\psi}_j * f\|_{\dot{H}^\gamma}$$

Using similar arguments we also get

$$\left\| \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (\hat{\psi}_j * g) \right\|_{L_t^q L_x^p} \leq C \|\hat{\psi}_j * g\|_{\dot{H}^{\gamma-1}}$$

and,

$$\left\| \int_0^t \frac{e^{\pm(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} (\hat{\psi}_j * h)(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq \|\hat{\psi}_j * h\|_{L_t^{q'} L_x^{p'}}$$

for every j , proving (b).

The final step is to go from the estimates on dyadic blocks to the general case. The key ingredient will be the Littlewood-Paley theorem. First, observe that we can write

$$\begin{aligned} e^{\pm it\sqrt{-\Delta}} (\hat{\psi}_j * f) &= \hat{\psi}_j * e^{\pm it\sqrt{-\Delta}} f \\ &=: F_j \end{aligned}$$

Then, observe that we have,

$$\begin{aligned} \left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_x^p}^2 &= \left\| \sum F_j(\cdot, t) \right\|_{L_x^p}^2 \\ &\leq \left\| \left(\sum |F_j(\cdot, t)|^2 \right)^{\frac{1}{2}} \right\|_{L_x^p}^2 \\ &\leq \sum_j \|F_j(\cdot, t)\|_{L_x^p}^2 \end{aligned}$$

where the first inequality is by Littlewood-Paley and the second is by Minkowski's integral inequality (since $\frac{p}{2} \geq 1$). Hence, by the above, and by another application of Minkowski (this time since $\frac{q}{2} \geq 1$) we have

$$\begin{aligned} \left\| e^{\pm it\sqrt{-\Delta}} f \right\|_{L_t^q L_x^p}^2 &= \left(\int (\|e^{\pm it\sqrt{-\Delta}} f\|_{L_x^p}^2)^{\frac{q}{2}} dt \right)^{\frac{2}{q}} \\ &\leq \left(\int \left(\sum_j \|F_j(\cdot, t)\|_{L_x^p}^2 \right)^{\frac{q}{2}} dt \right)^{\frac{2}{q}} \\ &\leq \sum_j \|F_j(\cdot, t)\|_{L_t^q L_x^p}^2 \end{aligned}$$

Now we will apply our estimates from (b) along with another application of the Littlewood-Paley theorem to get

$$\begin{aligned}
\left(\sum_j \|F_j(\cdot, t)\|_{L_t^q L_x^p}^2 \right)^{\frac{1}{2}} &\leq C \left(\sum_j (\|\hat{\psi}_j * f\|_{\dot{H}^\gamma}^2) \right)^{\frac{1}{2}} \\
&= C \left(\sum_j (\|(-\Delta)^{\frac{\gamma}{2}}(\hat{\psi}_j * f)\|_{L_x^2}^2) \right)^{\frac{1}{2}} \\
&= C \left(\sum_j \|\hat{\psi}_j * (-\Delta)^{\frac{\gamma}{2}} f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\
&\leq C \left\| \left(\sum_j |\hat{\psi}_j * (-\Delta)^{\frac{\gamma}{2}} f|^2 \right)^{\frac{1}{2}} \right\|_{L_x^2} \\
&\leq C \left\| (-\Delta)^{\frac{\gamma}{2}} f \right\|_{L_x^2} \\
&= C \|f\|_{\dot{H}^\gamma}
\end{aligned}$$

where the second to last inequality above is Fubini's Theorem and the last inequality is by Littlewood-Paley.

Using similar arguments we get

$$\left\| \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} g \right\|_{L_t^q L_x^p} \leq C \|g\|_{\dot{H}^{\gamma-1}}$$

and,

$$(29) \quad \left\| \int_0^t \frac{e^{\pm(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} h(\cdot, s) ds \right\|_{L_t^q L_x^p} \leq \|h\|_{L_t^{q'} L_x^{p'}}$$

completing the proof of Theorem 4.2 □

5. WELL-POSEDNESS FOR SEMILINEAR EQUATIONS

We are now ready to address our model semilinear equation. We will start with the subcritical case and we assume, from here out, that $n = 3$. We will also only deal with the cases where the exponent, α , satisfies $3 < \alpha \leq 5$. For $1 \leq \alpha \leq 3$ we can obtain global existence and uniqueness results in the case of the defocusing equation $\square u = |u|^{\alpha-1} u$ by just using energy methods. In the case of the focusing equation with small data we can also obtain global existence results for $\alpha \leq 3$ by methods similar to those presented below, but only if we require $\alpha > 1 + \sqrt{2}$. A result of F. John shows that there is always blow-up in the case of the focusing equation with $1 < \alpha < 1 + \sqrt{2}$.

Theorem 5.1. *Let $(f, g) \in \dot{H}^1 \times L^2$ and suppose $3 < \alpha < 5$. Then, there exists a $T > 0$ for which the Cauchy problem*

$$\begin{aligned}
\square u &= \pm |u|^{\alpha-1} u \\
u[0] &= (f, g)
\end{aligned}$$

is well-posed in $X := C^0([0, T]; \dot{H}^1) \cap C^1([0, T]; L^2) \cap L^{\frac{2\alpha}{\alpha-3}}([0, T]; L^{2\alpha})$. Moreover, the time T is proportional to the size of the initial data, i.e. $T \sim E_0^{-\lambda}$, where $E_0 := \|u[0]\|_{\dot{H}^1 \times L^2}$, for some $\lambda > 0$.

Proof of Theorem 5.1. We will set up a contraction mapping argument. Let X be the space

$$X := \left\{ u \in C^0([0, T]; \dot{H}^1) \cap C^1([0, T]; L^2) \cap L^{\frac{2\alpha}{\alpha-3}}([0, T]; L^{2\alpha}) \right\}$$

with the norm

$$\|u\|_X := \|Du\|_{L_t^\infty L_x^2} + \|u\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}}$$

Define X_R to be the space

$$X_R = \{u \in X : \|u\|_X \leq R := 2CE_0\}$$

where C is the constant appearing in the Strichartz estimates. Now, define a map $L : X_R \ni v \mapsto u$, where u is the unique solution to the inhomogeneous linear wave equation

$$\begin{aligned} \square u &= \pm |v|^{\alpha-1} v \\ u[0] &= (f, g) \end{aligned}$$

First we would like to show that $L : X_R \rightarrow X_R$. Combining our Strichartz and energy estimates gives the following inequality

$$\begin{aligned} \|u\|_X &:= \|Du\|_{L_t^\infty L_x^2} + \|u\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}} \leq C(\|u[0]\|_{\dot{H}^1 \times L^2} + \| |v|^{\alpha-1} v \|_{L_t^1 L_x^2}) \\ &= C(E_0 + \| |v|^{\alpha-1} v \|_{L_t^1 L_x^2}) \end{aligned}$$

since for $3 < \alpha < 5$ the pairs $(\frac{2\alpha}{\alpha-3}, 2\alpha)$ and $(\infty, 2)$ are both wave-admissible and satisfy the scaling condition $\frac{\alpha-3}{2\alpha} + \frac{3}{2\alpha} = 1 + \frac{3}{2} - 2 = \frac{3}{2} - 1 = \frac{1}{2}$.

Now, by Hölder's inequality we have

$$\| |v|^{\alpha-1} v \|_{L_t^1 L_x^2} = \|v\|_{L_t^\alpha L_x^{2\alpha}}^\alpha \leq T^{\frac{5-\alpha}{2}} \|v\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}}^\alpha$$

Therefore, since $v \in X_R$, we have

$$\begin{aligned} \|u\|_X &\leq C(E_0 + T^{\frac{5-\alpha}{2}} \|v\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}}^\alpha) \\ &\leq C(E_0 + T^{\frac{5-\alpha}{2}} \|v\|_X^\alpha) \\ &\leq C(E_0 + T^{\frac{5-\alpha}{2}} R^\alpha) \end{aligned}$$

This shows that $L : X_R \rightarrow X_R$ as long as $T^{\frac{5-\alpha}{2}} R^\alpha \leq E_0$.

The next step is to show that L is a contraction mapping on X_R . First we observe that given v_1 and v_2 in X_R , $L(v_1) - L(v_2) = u_1 - u_2$ satisfies the linear wave equation

$$\begin{aligned} \square w &= |v_1|^{\alpha-1} v_1 - |v_2|^{\alpha-1} v_2 \\ w[0] &= (0, 0) \end{aligned}$$

Then, as before we combine our Strichartz and energy estimates to get

$$\|u_1 - u_2\|_X := \|D(u_1 - u_2)\|_{L_t^\infty L_x^2} + \|u_1 - u_2\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}} \leq \| |v_1|^{\alpha-1} v_1 - |v_2|^{\alpha-1} v_2 \|_{L_t^1 L_x^2}$$

Now, since $\left| |v_1|^{\alpha-1} v_1 - |v_2|^{\alpha-1} v_2 \right| \leq |v_1 - v_2| (|v_1|^{\alpha-1} + |v_2|^{\alpha-1})$ we have

$$\begin{aligned}
\| |v_1|^{\alpha-1} v_1 - |v_2|^{\alpha-1} v_2 \|_{L_t^1 L_x^2} &\leq \| |v_1 - v_2| (|v_1|^{\alpha-1} + |v_2|^{\alpha-1}) \|_{L_t^1 L_x^2} \\
&\leq \|v_1 - v_2\|_{L_t^\alpha L_x^{2\alpha}} \| (|v_1|^{\alpha-1} + |v_2|^{\alpha-1}) \|_{L_t^{\frac{\alpha}{\alpha-1}} L_x^{\frac{2\alpha}{\alpha-1}}} \\
&\leq \|v_1 - v_2\|_{L_t^\alpha L_x^{2\alpha}} \left(\|v_1\|_{L_t^\alpha L_x^{2\alpha}}^{\alpha-1} + \|v_2\|_{L_t^\alpha L_x^{2\alpha}}^{\alpha-1} \right) \\
&\leq T^{\frac{5-\alpha}{2}} \|v_1 - v_2\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}} \left(\|v_1\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}}^{\alpha-1} + \|v_2\|_{L_t^{\frac{2\alpha}{\alpha-3}} L_x^{2\alpha}}^{\alpha-1} \right) \\
&\leq T^{\frac{5-\alpha}{2}} \|v_1 - v_2\|_X \left(\|v_1\|_X^{\alpha-1} + \|v_2\|_X^{\alpha-1} \right)
\end{aligned}$$

Therefore, since $v_1, v_2 \in X_R$ we have

$$\|u_1 - u_2\|_X \leq 2R^{\alpha-1} T^{\frac{5-\alpha}{2}} \|v_1 - v_2\|_X$$

Then, for L to be a contraction on X_R we simply need $2R^{\alpha-1} T^{\frac{5-\alpha}{2}} \ll 1$. By imposing this condition on T we have, by the contraction mapping principle, that L has a unique fixed point. Hence, we have proven existence, uniqueness and that T can be chosen so that $T \sim E_0^{-\lambda}$. To show continuous dependence on the initial data we will again use our Strichartz estimates.

By our existence and uniqueness proof we can define a map $S : \dot{H}^1 \times L^2 \rightarrow X$ which takes $(f, g) \in \dot{H}^1 \times L^2$ to the corresponding solution $u \in X$ of our model equation. We would like to show that S is continuous.

Let $u[0] := (f_0, g_0) \in \dot{H}^1 \times L^2$ and $v[0] := (f_1, g_1) \in \dot{H}^1 \times L^2$ be two pairs of initial data. And let $S(u[0]) = u$ and $S(v[0]) = v$ be the corresponding solutions to our model problem. Also, let $E_0 := \|u[0]\|_{\dot{H}^1 \times L^2}$ and $E_1 := \|v[0]\|_{\dot{H}^1 \times L^2}$. As above our solutions u and v satisfy $\|u\|_X \leq R_0$ and $\|v\|_X \leq R_1$ where the R_i are determined by the E_i . Again, we combine our Strichartz and energy estimates and proceed as in the proof of the contraction mapping to get

$$\begin{aligned}
\|u - v\|_X &\leq \|u[0] - v[0]\|_{\dot{H}^1 \times L^2} + T^{\frac{5-\alpha}{2}} \|u - v\|_X \left(\|u\|_X^{\alpha-1} + \|v\|_X^{\alpha-1} \right) \\
&\leq \|u[0] - v[0]\|_{\dot{H}^1 \times L^2} + T^{\frac{5-\alpha}{2}} \|u - v\|_X \left(R_1^{\alpha-1} + R_2^{\alpha-1} \right)
\end{aligned}$$

Now if we assume that $\|u[0] - v[0]\|_{\dot{H}^1 \times L^2}$ is small, then we also have that $R_1 \sim R_2$. And if we choose T such that $2R_1^{\alpha-1} T^{\frac{5-\alpha}{2}} \leq \frac{1}{2}$ we get

$$\|u - v\|_X \leq \|u[0] - v[0]\|_{\dot{H}^1 \times L^2} + \frac{1}{2} \|u - v\|_X$$

which implies

$$\|u - v\|_X \leq C \|u[0] - v[0]\|_{\dot{H}^1 \times L^2}$$

proving that S is actually Lipschitz.

Finally we show that if we assume that our data is smoother, then that additional regularity persists in the solution. Assume that we have initial data (f, g) such that f , and g are compactly supported in some set Ω and are in, say, $H^2 \times H^1(\Omega)$. Since $\dot{H}^1 \times L^2(\Omega) \subset H^2 \times H^1(\Omega)$, we can, by the

above arguments, find a $T > 0$ and a solution $u \in C^0([0, T]; \dot{H}^1) \cap C^1([0, T]; L^2) \cap L^{\frac{2\alpha}{\alpha-3}}([0, T]; L^{2\alpha})$. We would like to show that $u(t)$ is in $H^2 \times H^1(\Omega_t)$ for each t where Ω_t is the appropriate domain with respect to the propagation speed.

To see this, observe that for each $j = 1, \dots, n$ we have that $w := \partial_j u$ solves the following equation:

$$\begin{aligned}\square w &= |u|^{\alpha-1} \partial_j u \\ w[0] &= (\partial_j f, \partial_j g)\end{aligned}$$

By the Strichartz and energy estimates we have

$$\begin{aligned}\|w\|_X &= \|\partial_j u\|_X \leq \|w[0]\|_{\dot{H}^1 \times L^2} + \||u|^{\alpha-1} \partial_j u\|_{L_t^1 L_x^2} \\ &\leq \|w[0]\|_{\dot{H}^1 \times L^2} + \|\partial_j u\|_{L_t^\alpha L_x^{2\alpha}} \|u^{\alpha-1}\|_{L_t^{\frac{\alpha}{\alpha-1}} L_x^{\frac{2\alpha}{\alpha-1}}} \\ &\leq \|w[0]\|_{\dot{H}^1 \times L^2} + T^{\frac{5-\alpha}{2}} \|u\|_{L_t^\alpha L_x^{2\alpha}}^{\alpha-1} \|\partial_j u\|_{L_t^\alpha L_x^{2\alpha}} \\ &\leq \|w[0]\|_{\dot{H}^1 \times L^2} + T^{\frac{5-\alpha}{2}} \|u\|_X^{\alpha-1} \|\partial_j u\|_X \\ &\leq \|w[0]\|_{\dot{H}^1 \times L^2} + T^{\frac{5-\alpha}{2}} R^{\alpha-1} \|\partial_j u\|_X\end{aligned}$$

And since we have chosen T such that $T^{\frac{5-\alpha}{2}} R^{\alpha-1} \ll 1$ we obtain $\|w\|_X \leq C \|w[0]\|_{\dot{H}^1 \times L^2}$. From this we can conclude that $u \in C^0([0, T]; H^2(\Omega_t)) \cap C^1([0, T]; H^1(\Omega_t)) \cap L^{\frac{2\alpha}{\alpha-3}}([0, T]; L^{2\alpha})$. Similar arguments show that higher regularity persists as well. \square

In the case of the defocusing equation, $\square u = |u|^{\alpha-1} u$, we can easily show global well-posedness.

Corollary 5.2 (Global Well-Posedness for the Defocusing Equation). *Let $(f, g) \in \dot{H}^1 \times L^2$ and suppose $3 < \alpha < 5$. Then the defocusing problem*

$$\begin{aligned}\square u &= |u|^{\alpha-1} u \\ u[0] &= (f, g)\end{aligned}$$

is globally well-posed in $X := C^0([0, \infty); \dot{H}^1) \cap C^1([0, \infty); L^2) \cap L^{\frac{2\alpha}{\alpha-3}}([0, \infty); L^{2\alpha})$.

Proof of Corollary 5.2. As in the proof of conservation of energy for the homogenous equation, we multiply the equation on both sides by u_t and integrate over \mathbb{R}^n .

$$\begin{aligned}0 &= \int -\square u u_t + |u|^{\alpha-1} u u_t \\ &= \int \partial_t \left(\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2} \right) - \operatorname{div}(\nabla u u_t) + \partial_t \left(\frac{|u|^{\alpha+1}}{\alpha+1} \right) \\ &= \partial_t \int \left(\frac{|u_t|^2}{2} + \frac{|\nabla u|^2}{2} + \frac{|u|^{\alpha+1}}{\alpha+1} \right) dx\end{aligned}$$

Integrating from 0 to t then gives that

$$\begin{aligned} \int \left(\frac{|u_t(t)|^2}{2} + \frac{|\nabla u(t)|^2}{2} + \frac{|u(t)|^{\alpha+1}}{\alpha+1} \right) dx &= \int \left(\frac{|u_t(0)|^2}{2} + \frac{|\nabla u(0)|^2}{2} + \frac{|u(0)|^{\alpha+1}}{\alpha+1} \right) dx \\ &= \|u[0]\|_{\dot{H}^1 \times L^2}^2 + \int \frac{|u(0)|^{\alpha+1}}{\alpha+1} dx \\ &:= \mathcal{E}_0 \end{aligned}$$

This implies that for any time t we have

$$(30) \quad \int \frac{|u_t(t)|^2}{2} + \frac{|\nabla u(t)|^2}{2} \leq \mathcal{E}_0$$

The previous theorem gives us existence and uniqueness up to a time T that depends only on the size of the initial data E_0 . Since (30) tells us that the energy remains bounded by \mathcal{E}_0 , we can apply Theorem 5.1 to solve the Cauchy problem again, this time starting at time T . This extends our solution by a time, T_1 , proportional to \mathcal{E}_0 . Since (30) holds for all t we can continue in this fashion each time extending our solution by a fixed time T_1 which is proportional to \mathcal{E}_0 . This proves global well-posedness. \square

In the subcritical case the proof of local well-posedness relied on the relationship between time and the radius, R , of the ball we restricted ourselves to in the space X . We were able to choose time small enough so that the ratio $R^{\alpha-1}T^{\frac{5-\alpha}{2}}$ was small, and this allowed for a contraction mapping argument. When $\alpha = 5$, the critical case, this option is no longer available. To deal with this problem, we instead use small time to make the solution to the corresponding homogeneous problem small. This will allow us to prove local well-posedness for the critical problem.

Theorem 5.3. *Let $(f, g) \in \dot{H}^1 \times L^2$. Then, there exists a $T > 0$ such that the energy-critical problem*

$$\begin{aligned} \square u &= u^5 \\ u[0] &= (f, g) \end{aligned}$$

is well-posed in $X_T := C^0([0, T]; \dot{H}^1) \cap C^1([0, T]; L^2) \cap L^5([0, T]; L^{10})$

Proof of Theorem 5.3. Let $S_0 u[0]$ denote the solution to the homogeneous wave equation

$$\begin{aligned} \square u &= 0 \\ u[0] &= (f, g) \end{aligned}$$

That is

$$S_0 u[0] = \cos(t\sqrt{-\Delta})f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g$$

By the Strichartz estimates we have that

$$\|S_0 u[0]\|_{L_t^5 L_x^{10}} \leq \|u[0]\|_{\dot{H}^1 \times L^2} \leq C$$

Therefore, by continuity of the integral, by choosing T small we can make $\|S_0 u[0]\|_{L^5([0, T]; L_x^{10})}$ as small as we want. With this in mind we set up iterative argument in the space $Y_T := L^5([0, T]; L_x^{10})$.

Set $u_1 = 0$ and define u_k so that

$$\begin{aligned}\square u_k &= u_{k-1}^5 \\ u_k[0] &= u[0] = (f, g)\end{aligned}$$

We show first that for small enough T , the sequence is bounded in Y_T . In particular, we show that there exists a $T > 0$ such that $\|u_k\|_{Y_T} \leq 2C\|S_0u[0]\|_{Y_T}$ for every k , where C is the constant arising in the Strichartz estimates, (Theorem 4.2). To see this observe that for every j, k we have, by Theorem 4.2, that

$$\begin{aligned}(31) \quad \|u_{k+1} - u_{j+1}\|_{Y_T} &\leq C\|u_k^5 - u_j^5\|_{L^1([0,T];L_x^2)} \\ &\leq C\|u_k - u_j\|_{L^5([0,T];L_x^{10})}\|u_k^4 + u_j^4\|_{L^{\frac{5}{4}}([0,T];L_x^{\frac{5}{2}})} \\ &\leq C\|u_k - u_j\|_{Y_T}(\|u_k\|_{Y_T}^4 + \|u_j\|_{Y_T}^4)\end{aligned}$$

Now, choose $T > 0$ small enough so that $2C\|S_0u[0]\|_{Y_T} \leq \frac{1}{4}$. Clearly $u_{-1} = 0$ satisfies $\|u_{-1}\|_{Y_T} \leq 2C\|S_0u[0]\|_{Y_T}$. Suppose, for induction, that $\|u_k\|_{Y_T} \leq 2C\|S_0u[0]\|_{Y_T}$. Then setting $j = -1$ in (31), we get

$$\|u_{k+1} - u_0\| \leq C\|u_k\|_{Y_T}^5 \leq C\|S_0u[0]\|_{Y_T}^5$$

And, since $u_0 = S_0u[0]$, the above implies

$$\|u_{k+1}\|_{Y_T} \leq \|S_0u[0]\|_{Y_T} + C\|S_0u[0]\|_{Y_T}^5 \leq 2C\|S_0u[0]\|_{Y_T}.$$

Now, since we have chosen T so that $2C\|S_0u[0]\|_{Y_T} \leq \frac{1}{4}$ we can set $j + 1 = k$ in (31) to get

$$\|u_{k+1} - u_{j+1}\|_{Y_T} \leq \frac{1}{2}\|u_k - u_{k-1}\|_{Y_T}$$

And this implies that the sequence $\{u_k\}$ is Cauchy in Y_T . Hence there exists a $u \in Y_T$ such that $u_k \rightarrow u$ in Y_T satisfying $\|u\|_{Y_T} \leq 2C\|S_0u[0]\|_{Y_T}$. We show that u is our desired solution. To see this observe that for every j we have

$$\begin{aligned}(32) \quad u_j(t, \cdot) &= \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g + \cos(t\sqrt{-\Delta})f + \int_0^t \frac{\sin((t-r)\sqrt{-\Delta})}{\sqrt{-\Delta}}u_{j-1}^5(s, \cdot) dr. \\ &= K_t * g + K'_t * f + \int_0^t K_{t-r} * u_{j-1}^5\end{aligned}$$

where $K_t := \mathcal{F}^{-1}(\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}})$. By our above work, the left hand side converges in Y_T to u . We would like to show that the left hand side converges to $K_t * g + K'_t * f + \int_0^t K_{t-r} * u^5$ in Y_T . By the estimate for the inhomogeneous term in the Strichartz estimates, ((29)), and the estimates in (31), we have

$$\begin{aligned}\| \int_0^t K_{t-r} * (u_k^5 - u^5) \|_{Y_T} &\leq \| (u_k^5 - u^5) \|_{L^1([0,T];L_x^2)} \\ &\leq \| u_k - u \|_{Y_T} (\| u_k \|_{Y_T}^4 + \| u \|_{Y_T}^4) \\ &\leq C \| u_k - u \|_{Y_T} \rightarrow 0\end{aligned}$$

Hence the right hand side also converges in Y_T proving that $u \in Y_T$ is a solution. To see that u is also in X_T , we can first assume that $(f, g) \in C_0^\infty \times C_0^\infty$. Then the sequence $\{(u_k, \partial_t u_k)\}$ is a Cauchy sequence in $C^0([0, T]; \dot{H}^1) \cap C^1([0, T]; L^2)$ converging to $(u, \partial_t u)$. Then recall that we can combine the Strichartz and energy estimates to obtain estimates of the form

$$(33) \quad \|u_k\|_{X_T} := \|Du_k\|_{L^\infty([0, T]; L_x^2)} + \|u_k\|_{Y_T} \leq C(\|u_k[0]\|_{\dot{H}^1 \times L^2} + \|u_{k-1}^5\|_{L^1([0, T]; L_x^2)})$$

And we can use these estimates along with an approximation argument to remove the smoothness assumption on the initial data. This proves existence.

To prove uniqueness suppose u and v are two solutions. Then again we can combine Strichartz and energy estimates as in (33) to obtain

$$\begin{aligned} \|u - v\|_{X_T} &\leq \|u^5 - v^5\|_{L^1([0, T]; L_x^2)} \\ &\leq \|u - v\|_{L^5([0, T]; L_x^{10})} (\|u\|_{L^5([0, T]; L_x^{10})}^4 + \|v\|_{L^5([0, T]; L_x^{10})}^4) \\ &\leq \|u - v\|_{X_T} 2(2C\|S_0 u[0]\|_{Y_T})^4 \\ &\leq \frac{1}{2} \|u - v\|_{X_T} \end{aligned}$$

which implies that $u = v$. Continuous dependence on the initial data follows easily as well adapting the method we used in the subcritical case to our present situation. Persistence of regularity also follows from adapting the proof used in the subcritical case. \square

Remark 5.4. There are also global existence and uniqueness results for the critical equation. As in the subcritical case, we have a global theory for the defocusing equation with large data due to the fact that energy can not concentrate at a point. The proof, which will not be addressed here, involves the Morawetz identities and can be found, for example, in [3]. We can, however, adapt the proof of Theorem 5.3 to prove a global result if we assume that our initial data is small.

Theorem 5.5. *Let $u[0] := (f, g) \in \dot{H}^1 \times L^2$. Then, there exists an $\epsilon > 0$ so that if $\|u[0]\|_{\dot{H}^1 \times L^2} < \epsilon$, the Cauchy problem*

$$(34) \quad \begin{aligned} \square u &= u^5 \\ u[0] &= (f, g) \end{aligned}$$

is globally well-posed in $X := C^0([0, \infty); \dot{H}^1) \cap C^1([0, \infty); L^2) \cap L^5([0, \infty); L^{10})$

Proof of Theorem 5.5. By the Strichartz estimates for solutions to the inhomogeneous wave equation we have

$$(35) \quad \|S_0 u[0]\|_{L_t^5 L_x^{10}} \leq C \|u[0]\|_{\dot{H}^1 \times L^2}$$

Then if we choose ϵ small enough so that $2C\|u[0]\|_{\dot{H}^1 \times L^2} < 2C\epsilon < \frac{1}{4}$, we have, by (35), that $2C\|S_0 u[0]\|_{L_t^5 L_x^{10}} < \frac{1}{4}$. (Here C is the constant arising in the Strichartz estimates.) Then we can carry out the same proof from Theorem 5.3 to prove Theorem 5.5, with the smallness of $\|S_0 u[0]\|_{L_t^5 L_x^{10}}$ achieved without having to restrict the size of T . \square

Remark 5.6. There is a similar global result for the subcritical problem with small data if we assume that the initial data, $(f, g) \in \dot{H}^\gamma \times \dot{H}^\gamma$ where γ is chosen so that $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$ is the scale invariant space for the given subcritical equation.

6. SCATTERING

Roughly speaking, a nonlinear problem for which there exists a global existence theory exhibits scattering if, as time becomes large, the effects of the nonlinearity become negligible and solutions begin to resemble solutions to a homogeneous problem. To be more precise, scattering theory consists of the following questions: (I) Given “nonlinear data” $(f, g) \in Y$ and a corresponding solution $u \in Z$ to the nonlinear problem

$$(36) \quad \begin{aligned} \square u &= F(u) \\ u[0] &= (f, g) \end{aligned}$$

can we find data $(f_0, g_0) \in Y$ such that the solution $u_0 \in Z$ to the corresponding homogeneous problem

$$(37) \quad \begin{aligned} \square u_0 &= 0 \\ u_0[0] &= (f_0, g_0) \end{aligned}$$

is such that $\|u(t) - u_0(t)\|_Z \rightarrow 0$ as $t \rightarrow 0$; and (II) Given “free data” $(f_0, g_0) \in Y$ and the corresponding solution $u_0 \in Z$ to (37), can we find nonlinear data $(f, g) \in Y$ so that the solution $u \in Z$ to (36) satisfies $\|u(t) - u_0(t)\|_Z \rightarrow 0$ as $t \rightarrow 0$. (I) is called completeness of wave operators and (II) is called existence of wave operators. We will prove scattering for the critical semilinear problem with small data (see Theorem 5.5). To do this we first formulate the wave equation as a Hamiltonian system, $\dot{U} = JE'(U)$ where J is a skew symmetric matrix and $E'(U)$ is the Fréchet derivative of the conserved quantity. Solutions u to (36) satisfy

$$(38) \quad -\partial_t \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}.$$

Setting $\begin{pmatrix} u \\ u_t \end{pmatrix} =: U$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: J$, $\begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} =: H$, and $\begin{pmatrix} 0 \\ F(u) \end{pmatrix} =: F(U)$, we can rewrite (37) as

$$(39) \quad \begin{aligned} -\dot{U}_0 + JHU_0 &= 0 \\ U_0[0] &= \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \end{aligned}$$

and (36) as

$$(40) \quad \begin{aligned} -\dot{U} + JHU &= F(U) \\ U[0] &= \begin{pmatrix} f \\ g \end{pmatrix}. \end{aligned}$$

Then the solution to (37) is given by $U_0(t) = e^{tJH}U_0[0]$ and by Duhamel’s formula, the solution to (36) is given by

$$U(t) = e^{tJH}U[0] + \int_0^t e^{(t-s)JH}F(U(s)) ds$$

where $\int_0^t e^{(t-s)JH}F(U(s)) ds$ means to integrate each component of the vector $e^{(t-s)JH}F(U(s))$ from 0 to t .

We would like this matrix formulation to be in Hamiltonian form. To see this recall that the energy functional $E(u)$ is given by

$$E(u) := \frac{1}{2} \int -u\Delta u + (u_t)^2 dx$$

Then the Fréchet derivative $E'(u)$ satisfies

$$\langle E'(u), h \rangle = \int -h\Delta u + u_t h_t dx$$

In matrix notation we have then that $E'(U) = \begin{pmatrix} -\Delta u \\ u_t \end{pmatrix} = HU$. Also observe that J is symplectic. Hence we have indeed formulated the homogeneous wave equation in Hamiltonian form, $\dot{U} = JE'(U)$.

Remark 6.1. The Hamiltonian formulation provides us with another way of deriving the conservation of energy. To see this observe that we have

$$\begin{aligned} \partial_t E(u) &= \langle E'(U), \dot{U} \rangle \\ &= \langle E'(U), JE'(U) \rangle \\ &= 0 \end{aligned}$$

where the second line follows since U satisfies $\dot{U} = JE'(U)$ and the last line follows from the fact that J is symplectic.

With this matrix formulation we can address the scattering theory for the critical semilinear problem with small data. The global well-posedness theory for such problems was addressed in Theorem 5.5.

Theorem 6.2 (Completeness of Wave Operators). *Let $U[0] := (f, g)^t \in \dot{H}^1 \times L^2$ satisfy $\|U[0]\|_{\dot{H}^1 \times L^2} < \epsilon$ where $\epsilon > 0$ is chosen as in Theorem 5.5. Let $U = (u, u_t)^t \in Z := C^0([0, \infty); \dot{H}^1) \times C^0([0, \infty); L^2)$ be the solution to the critical problem, (34), with initial data $U[0]$, given by Theorem 5.5. Then, there exists free data $U_0[0] := (f_0, g_0)^t \in \dot{H}^1 \times L^2$ such that*

$$\|U(t) - e^{tJH}U_0[0]\|_Z \longrightarrow 0$$

as $t \longrightarrow \infty$.

The idea for the proof is to fix a time T , evolve $U(T)$ backwards in time via the free evolution to obtain an approximation of the desired free data, and then send $T \rightarrow \infty$ to obtain the free data.

Proof of Theorem 6.2. By Duhamel's formula we have that for each T , $U(T)$ satisfies

$$U(T) = e^{TJH}U[0] + \int_0^T e^{(T-s)JH}F(U(s)) ds.$$

Applying the backwards, free evolution operator, e^{-TJH} to both sides we get

$$e^{-TJH}U(T) = U[0] + \int_0^T e^{-sJH}F(U(s)) ds.$$

Letting $T \rightarrow \infty$ above we define our free data $U_0[0]$ by

$$U_0[0] := U[0] + \int_0^\infty e^{-sJH}F(U(s)) ds.$$

Applying the free evolution operator to $U_0[0]$ we get

$$U_0(t) := e^{tJH}U_0[0] = e^{tJH}U[0] + \int_0^\infty e^{(t-s)JH}F(U(s)) ds$$

Then

$$U_0(t) - U(t) = \int_t^\infty e^{(t-s)JH}F(U(s)) ds.$$

We would like to show that

$$\left\| \int_t^\infty e^{(t-s)JH}F(U(s)) ds \right\|_Z \longrightarrow 0$$

as $t \longrightarrow \infty$. We show this by proving that we have a global bound

$$\left\| \int_0^\infty e^{(t-s)JH}F(U(s)) ds \right\|_Z \leq C$$

By expanding out the $e^{(t-s)JH}$ we see that the first component of the vector $e^{(t-s)JH}F(U(s))$ is given by $\frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(u(s))$ and the second component is $\cos(t\sqrt{-\Delta})F(u(s))$. Then

$$\begin{aligned} \left\| \int_0^\infty \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(u(s)) ds \right\|_{\dot{H}^1} &\leq \int_0^\infty \|K_{(t-s)} * F(u)\|_{\dot{H}^1} ds \\ &\leq C\|F(u)\|_{L_t^1 L_x^2} \\ &= C\|u^5\|_{L_t^1 L_x^2} \\ &= C\|u\|_{L_t^5 L_x^{10}}^5 \leq C' \end{aligned}$$

where the first inequality is Minkowski and the second is by Theorem 2.3. Also, we have

$$\begin{aligned} \left\| \int_0^\infty \cos(t\sqrt{-\Delta})F(u(s)) ds \right\|_{L^2} &\leq C\|F(u)\|_{L_t^1 L_x^2} \\ &\leq C\|u\|_{L_t^5 L_x^{10}}^5 \\ &\leq C' \end{aligned}$$

These last two estimates imply that

$$\left\| \int_0^\infty e^{(t-s)JH}F(U(s)) ds \right\|_Z \leq C.$$

Hence

$$\|U(t) - U_0(t)\|_Z \longrightarrow 0$$

as $t \longrightarrow \infty$ proving Theorem 6.2. □

Theorem 6.3 (Existence of Wave Operators). *Let $U_0[0] := (f_0, g_0)^t \in \dot{H}^1 \times L^2$ satisfy $\|U_0[0]\|_{\dot{H}^1 \times L^2} < \frac{\epsilon}{4C}$ where ϵ is chosen as in Theorem 5.5 and C is the constant arising in the Strichartz estimates. Let $U_0(t) := e^{tJH}U_0[0]$ be the free evolution. Then there exists data $U[0] := (f, g)^t \in \dot{H}^1 \times L^2$ such that $\|U[0]\|_{\dot{H}^1 \times L^2} < \epsilon$ and a corresponding $u(t) \in X := C^0([0, \infty); \dot{H}^1) \cap C^1([0, \infty); L^2) \cap L^5([0, \infty); L^{10})$ solving the nonlinear equation, (34), such that*

$$\|U_0(t) - U(t)\|_Z \longrightarrow 0$$

as $t \longrightarrow \infty$. Here, as before, $U(t) := (u, u_t)^t$ and $Z := C^0([0, \infty); \dot{H}^1) \times C^0([0, \infty); L^2)$.

Proof of Theorem 6.3. We begin by deriving the Yang-Feldman equation. Solutions $U(t)$ to the nonlinear problem (40) satisfy

$$U(t) = e^{tJH}U[0] + \int_0^t e^{(t-s)JH}F(U(s)) ds.$$

Applying the operator e^{-tJH} to both sides gives

$$e^{-tJH}U(t) = U[0] + \int_0^t e^{-sJH}F(U(s)) ds.$$

Differentiating with respect to t we get,

$$\partial_t(e^{-tJH}U(t)) = e^{-tJH}F(U(t)).$$

Now integrate this expression from T to t to obtain

$$e^{-tJH}U(t) = e^{-TJH}U(T) + \int_T^t e^{-sJH}F(U(s)) ds.$$

which implies the Yang-Feldman equation

$$(41) \quad U(t) = e^{(t-T)JH}U(T) + \int_T^t e^{(t-s)JH}F(U(s)) ds.$$

Now, if we would like $U(t)$ to agree with the free evolution $U_0(t) = e^{tJH}U_0[0]$ at time T , we need $U(T) = e^{TJH}U_0[0]$. Plugging this into (41) we get that $U(t)$ must satisfy

$$U(t) = e^{tJH}U_0[0] - \int_t^T e^{(t-s)JH}F(U(s)) ds$$

In our case, we would like $U(t)$ to asymptotically agree with the free evolution. Hence we let $T \rightarrow \infty$ above to get

$$(42) \quad U(t) = e^{tJH}U_0[0] - \int_t^\infty e^{(t-s)JH}F(U(s)) ds$$

To find a $U \in Z$ that satisfies (42), we use a contraction argument in the space $X_R := \{v \in X : \|v\|_X \leq R := 2C\|U_0[0]\|_{\dot{H}^1 \times L^2}\}$ where C is the constant arising in the Strichartz estimates. Define an operator L such that for each $v \in X_R$ we have

$$L(v)(t) = e^{tJH}U_0[0] - \int_t^\infty e^{(t-s)JH}F(V(s)) ds$$

where again $V := (v, v_t)^t$. We first show that $L : X_R \rightarrow X_R$.

$$\|L(v)\|_X \leq \|e^{tJH}U_0[0]\|_X + \left\| \int_t^\infty e^{(t-s)JH}F(V(s)) ds \right\|_X$$

Combining the Strichartz, (Theorem 4.2), and energy estimates, (Lemma 2.4), for the homogeneous wave equation, we have

$$\|e^{tJH}U_0[0]\|_X = \|Du_0\|_{L_t^\infty L_x^2} + \|u_0\|_{L_t^5 L_x^{10}} \leq C\|U_0[0]\|_{\dot{H}^1 \times L^2}$$

where $u_0(t) = \cos(t\sqrt{-\Delta})f_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g_0$. The first component of $e^{(t-s)JH}F(V(s)) ds$ is $\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(v(s))$. Using the interval (t, ∞) instead of $(0, t)$ when applying the Christ-Kiselev lemma, (Lemma 4.11),

in the the proof of the Strichartz estimates we get as in (29) that

$$\begin{aligned}
\left\| \int_t^\infty \frac{e^{(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(v(\cdot, s)) ds \right\|_{L_t^5 L_x^{10}} &\leq C \|F(v)\|_{L_t^1 L_x^2} \\
&= C \|v\|_{L_t^5 L_x^{10}}^5 \\
&\leq C \|v\|_X^5 \\
&\leq C \|U_0[0]\|_{\dot{H}^1 \times L^2}^5
\end{aligned}$$

And as in the proof of Theorem 6.2 we have

$$\begin{aligned}
\left\| \int_t^\infty \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} F(v(s)) ds \right\|_{\dot{H}^1} &\leq \int_t^\infty \|K_{(t-s)} * F(v)\|_{\dot{H}^1} ds \\
&\leq C \|F(v)\|_{L_t^1 L_x^2} \\
&= C \|v\|_{L_t^5 L_x^{10}}^5 \\
&\leq C \|v\|_X^5 \\
&\leq C \|U_0[0]\|_{\dot{H}^1 \times L^2}^5
\end{aligned}$$

Taking the supremum over t on both sides gives

$$\left\| \int_t^\infty \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} F(v(s)) ds \right\|_{L_t^\infty \dot{H}_x^1} \leq C \|U_0[0]\|_{\dot{H}^1 \times L^2}^5$$

Similarly we have

$$\left\| \int_0^\infty \cos(t\sqrt{-\Delta}) F(v(s)) ds \right\|_{L_t^\infty L_x^2} \leq C \|U_0[0]\|_{\dot{H}^1 \times L^2}^5$$

Putting these together we get

$$\left\| \int_t^\infty e^{(t-s)JH} F(V(s)) ds \right\|_X \leq C \|U_0[0]\|_{\dot{H}^1 \times L^2}^5$$

And hence $\|L(v)\|_X \leq R$. We now show that L is a contraction on X_R . Let $v_1, v_2 \in X_R$. Then, using the same techniques as above we have

$$\begin{aligned}
\|L(v_1) - L(v_2)\|_X &= \left\| \int_t^\infty e^{(t-s)JH} (F(V_1) - F(V_2)) ds \right\|_X \leq C \|F(v_1) - F(v_2)\|_{L_t^1 L_x^2} \\
&\leq C \|(v_1 - v_2)(|v_1|^4 + |v_2|^4)\|_{L_t^1 L_x^2} \\
&\leq C \|(v_1 - v_2)\|_{L_t^5 L_x^{10}} \|(|v_1|^4 + |v_2|^4)\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}} \\
&\leq C \|(v_1 - v_2)\|_X (\|v_1\|_{L_t^5 L_x^{10}}^4 + \|v_2\|_{L_t^5 L_x^{10}}^4) \\
&\leq 2C \|U_0[0]\|_{\dot{H}^1 \times L^2}^4 \|(v_1 - v_2)\|_X
\end{aligned}$$

And since we have chosen ϵ so that $2C \|U_0[0]\|_{\dot{H}^1 \times L^2}^4 \ll 1$ we have proven that L is a contraction. Hence L has a unique fixed point $u \in X_R$. This means that $U := (u, u_t)^t$ satisfies

$$U(t) = e^{tJH} U_0[0] - \int_t^\infty e^{(t-s)JH} F(U(s)) ds$$

We would like that u solves the nonlinear wave equation, (34). To see this set

$$U[0] := U(0) = U_0[0] - \int_0^\infty e^{-sJH} F(U(s)) ds$$

We would like $\|U[0]\|_{\dot{H}^1 \times L^2} < \epsilon$. And this is indeed the case since $\|U_0[0]\|_{\dot{H}^1 \times L^2} < \frac{\epsilon}{2}$ and by the same methods as above

$$\left\| \int_0^\infty e^{-sJH} F(U(s)) ds \right\|_{\dot{H}^1 \times L^2} \leq C \|U_0[0]\|_{\dot{H}^1 \times L^2} < \frac{\epsilon}{2}.$$

Now, let N denote the nonlinear evolution operator. Evolving this initial data with N we get

$$\begin{aligned} N(U[0]) &= e^{tJH} \left(U_0[0] - \int_0^\infty e^{-sJH} F(U(s)) ds \right) + \int_0^t e^{(t-s)JH} F(U(s)) ds \\ &= e^{tJH} U_0[0] - \int_t^\infty e^{(t-s)JH} F(U(s)) ds \\ &= U(t) \end{aligned}$$

Hence U is the unique solution to (34) given by Theorem 5.5. Finally,

$$\|U_0(t) - U(t)\|_Z \longrightarrow 0$$

as $t \longrightarrow \infty$ by the same argument as in the proof of Theorem 6.2. □

Remark 6.4.

- We can prove scattering for the subcritical problem with small data if we assume, as in Remark 5.6, that the data $(f, g) \in \dot{H}^\gamma \times \dot{H}^\gamma$, where $\dot{H}^\gamma \times \dot{H}^\gamma$ is the scale invariant space for the given subcritical problem.
- To address the scattering theory for problems with large data we would need some sort of Morawetz identity or other forms of global control that are beyond the scope of these notes.

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