Kaufman and Falconer Estimates for Radial Projections

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Outline

Background on Projection Theory

Proof of the Falconer-type
radial projection theorem

Proof of the Kaufman-type
radial projection theorem

Applications + Continuum Results

Background on Projection Theory

Marstrand's Projection Theorem

We define $\pi_V : \mathbb{R}^n \to \mathbb{R}^m$ to be the orthogonal projection onto the subspace $V \in G(n, m)$.

Given $Y \subset \mathbb{R}^n$ Borel, what is the (Hausdorff) dimensional relationship between Y and $\pi_V(Y)$?

Clearly, $\dim \pi_V(Y) \leq \min\{m, \dim Y\}$. In fact,

Theorem (Marstrand's Projection Theorem)

For almost every $V \in G(n, m)$,

$$\dim \pi_V(Y) = \min\{m, \dim Y\}.$$

How often is the size of the projection *smaller*?

Exceptional Set Estimates

Let $0 \leq s < \min\{m, \dim Y\}$, and define

$$E_s(Y) := \{V \in G(n,m) : \dim \pi_V(Y) < s\}.$$

Then, we have

- (Falconer and Peres–Schlag):

$$\dim E_s(Y) \leq \max\{m(n-m) + s - \dim Y, 0\}$$

- (Kaufman):

$$\dim E_s(Y) \leq m(n-m-1) + s.$$

Can we get similar looking estimates for *radial* projections?

Radial Projection Estimates

We define $\pi_x : \mathbb{R}^n \setminus \{x\} \to \mathbb{S}^{n-1}$ to be the radial projection onto the sphere centered at x:

$$\pi_x(y) := \frac{y-x}{|y-x|}.$$

By B.–Gan (2022), we have

- (Conjectured by Lund–Pham–Thu): For $0 < s < \lfloor \dim Y \rfloor$,

  $$\dim(\{x \in \mathbb{R}^n \setminus Y : \dim \pi_x(Y) < s\}) \leq \min\{\lfloor \dim Y \rfloor + s - \dim Y, 0\}$$

- (Conjectured by Bochen Liu):

  $$\dim(\{x \in \mathbb{R}^n \setminus Y : \dim \pi_x(Y) \leq \dim Y\}) \leq \lceil \dim Y \rceil.$$

Orponen–Shmerkin–Wang (2022)

Shortly after B.–Gan's work, Orponen, Shmerkin, and Wang (2022) proved stronger versions of these results (and much more) in

"Kaufman and Falconer estimates for radial projections and a continuum version of Beck's theorem".

Main Results

Their main results (in the plane) are as follows.

Theorem

Let $X \subset \mathbb{R}^2$ be a (non-empty) Borel set which is not contained on any line. Then, for every Borel set $Y \subset \mathbb{R}^2$,

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X, \dim Y, 1\}.$$

Theorem

Let $X, Y \subset \mathbb{R}^2$ be Borel sets with $X \neq \emptyset$ and $\dim Y > 1$. Then,

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X + \dim Y - 1, 1\}.$$

Notes

1. OSW's main theorems are stronger versions of Kaufman's and Falconer's dimension estimates.
2. The first result was generalized to higher dimensions in OSW, and there is ongoing work of B.–Fu–Ren generalizing the second result.
3. These results have a number of applications to
   - exceptional set estimates (OSW),
   - a continuum version of Beck's Theorem (OSW),
   - (sticky) Kakeya sets (Wang–Zahl),
   - the ABC sum-product conjecture (Orponen–Shmerkin),
   - and more!

Proof of the Falconer-type radial projection theorem

Warm up dimension lower bound

Proposition

If A supports a probability measure ν with $\nu(B_r) \leq r^t$, then $\dim A \geq t$.

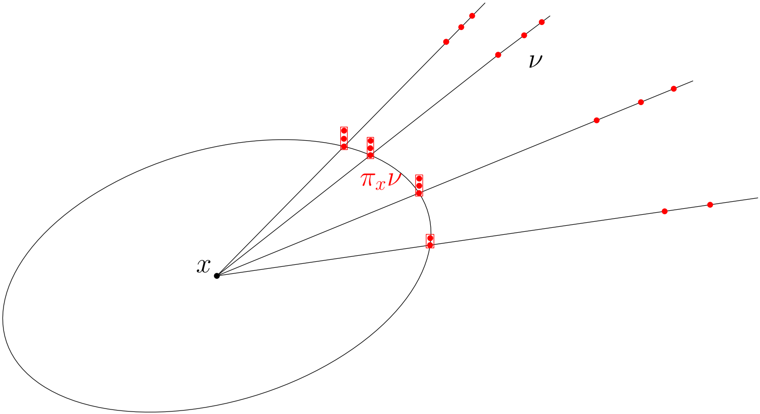
Proof.

$\sum r_i^t \geq \sum \nu(B_i) \geq \nu(A) > 0$, so $\mathcal{H}^t(A) > 0$. □

What condition guarantees lower bounds for radial projections?

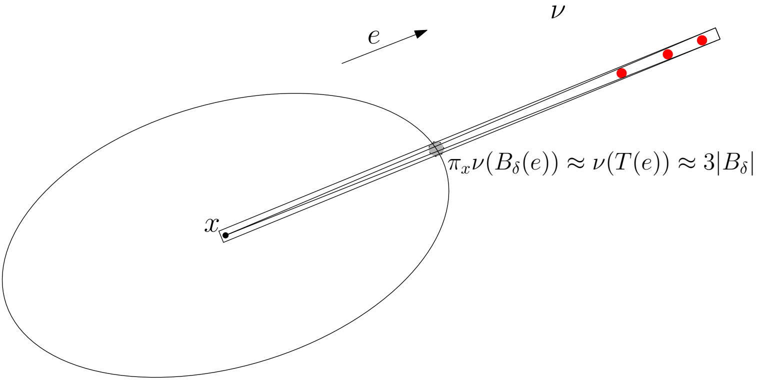
If $\mathcal{H}^t(\pi_x(Y \setminus \{x\})) > 0$, then $\dim \pi_x(Y \setminus \{x\}) \geq t$. We can use the idea of a Frostman measure.

If ν is a measure supported in $Y \setminus \{x\}$ and $\pi_x \nu$ is t-Frostman, then $\mathcal{H}^t(\pi_x(Y \setminus \{x\})) > 0$.



Tubes and fans

$\pi_x \nu(B_\delta(e)) \approx \nu(T(e))$ if $dist(x, \operatorname{supp} \nu) \approx 1$



If $G \subset X \times Y$, we have x-sections $G|_x = \{y : (x, y) \in G\}$.

Definition ((t, K, c)-thin tubes)

We say probability measures (μ, ν) has (t, K, c)-thin tubes if for some $G \subset X \times Y$ with $(\mu \times \nu)(G) \geq c$, and each $x \in X$,

$$\nu(T \cap G|_x) \leq K \cdot r^t, \qquad r > 0$$

for each r-tube T containing x.

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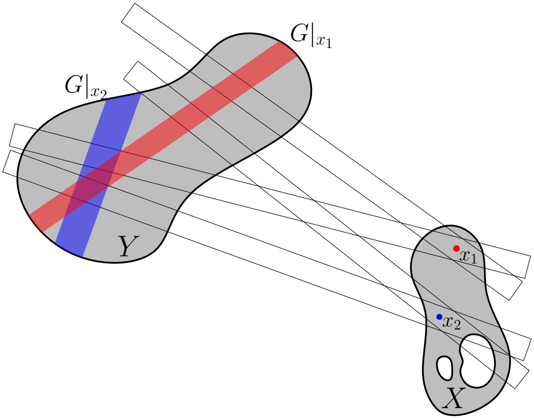


Proposition

If $t > 0$, and (μ, ν) has t-thin tubes, then

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq t.$$

Proof.

$t > 0$ implies $\exists x$ such that $\pi_x \nu$ satisfies a t-Frostman condition. $\square$

Warm up with thin tubes

Proposition

If $\nu(B_r) \leq Kr^{1+\delta}$, and μ is arbitrary, then (μ, ν) has $(\delta, K, 1)$-thin tubes.

Proof.

Any r-tube T can be covered by r^{-1} many r-balls.

$$\nu(T) \leq r^{-1} \cdot Kr^{1+\delta} = K \cdot r^{\delta}.$$

We can take $G = \operatorname{supp} \mu \times \operatorname{supp} \nu$ with $(\mu \times \nu)(G) = 1$. □

Falconer-type projection theorem

Theorem (Falconer-type estimate)
If $X, Y \subset \mathbb{R}^2$, *and* $\dim Y > 1$, *then*

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X + \dim Y - 1, 1\}$$

The strategy is to prove that "X, Y" has σ-thin tubes for any $\sigma < \min\{\dim X + \dim Y - 1, 1\}$.

From sets to thin tubes

Lemma (Frostman measure)

If $\dim A > s$*, then* A *supports a measure* μ *with* $\mu(B_r) \leq r^s$*.*

From sets to thin tubes

Sets X, Y	$\dim X > s$	$\dim Y > t$
$\downarrow$	$\downarrow$	$\downarrow$
Measures μ, ν	$\mu(B_r) \leq r^s$	$\nu(B_r) \leq r^t$

Base case and inductive step

If $t > 1$, then (μ, ν) has $(t-1)$-thin tubes.

Lemma (Key Lemma)

If $\sigma < \min\{s+t-1, 1\}$ and μ, ν has $(\sigma, K, 1-\epsilon)$-thin tubes, then there exists $\eta > 0$ (uniform) and $K' = K'(\sigma, s, t, K)$ such that μ, ν has $(\sigma + \eta, K', 1-4\epsilon)$-thin tubes.

Idea of proof of Key Lemma

If μ, ν have σ-thin tubes, but not $\sigma + \eta$-thin tubes, we discretize measures μ and ν to r-separated sets of points P_X, P_Y for some very small scale r.

Because P_X, P_Y do *not* have $\sigma + \eta$-thin tubes, for each $x \in P_X$, we can find a (r, σ)-set of tubes $\mathcal{T}_x$ with the property

$$|P_Y \cap T| \geq r^{\sigma+\eta} |P_Y|.$$

Theorem (Fu–Ren incidence theorem)

For all $t \in (1,2]$, $\sigma < 1$, and $\zeta > 0$, there exists $\eta > 0$ such that for all $s \in [0,2]$, if P_X is a (r,s)-set, P_Y is a (r,t)-set, and for each $x \in P_X$, $\mathcal{T}_x$ is a (r,σ)-set of tubes such that

$$|P_Y \cap T| \geq r^{\sigma+\eta}|P_Y|, \qquad T \in \mathcal{T}_x,$$

then $\sigma \geq s + t - 1 - \zeta$.

Because $\sigma < s + t - 1$, we can fit in a $\zeta > 0$ so $\sigma < s + t - 1 - \zeta$.

Fu–Ren's theorem implies $\sigma \geq s + t - 1 - \zeta$, so we reached a contradiction.

Theorem (Falconer-type estimate)

If $X, Y \subset \mathbb{R}^2$, *and* $\dim Y > 1$, *then*

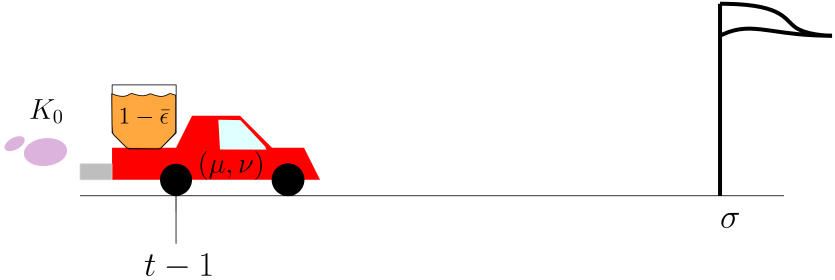
$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X + \dim Y - 1, 1\}.$$

Theorem (Bootstrapping theorem)

Let $s \in [0,2]$, $t > 1$, $\sigma < \min\{s + t - 1, 1\}$, *and* $\epsilon > 0$. *There exists* $K > 0$ *so that if* $\mu(B_r) \leq r^s$ *and* $\nu(B_r) \leq r^t$, *then* (μ, ν) *has* $(\sigma, K, 1 - \epsilon)$*-thin tubes.*

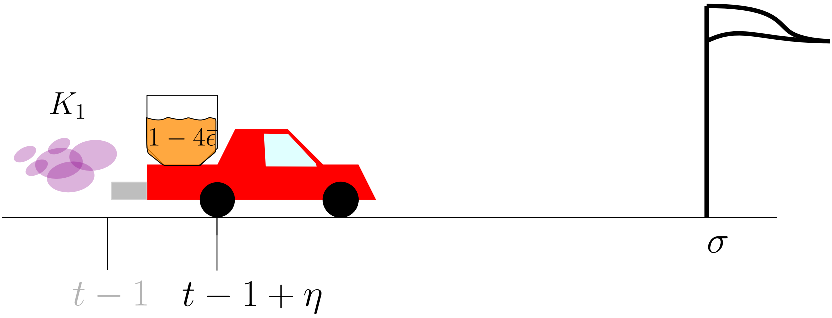
Proof of bootstrapping theorem

As we noted, μ, ν start out with $(t-1, K_0, 1)$-thin tubes.



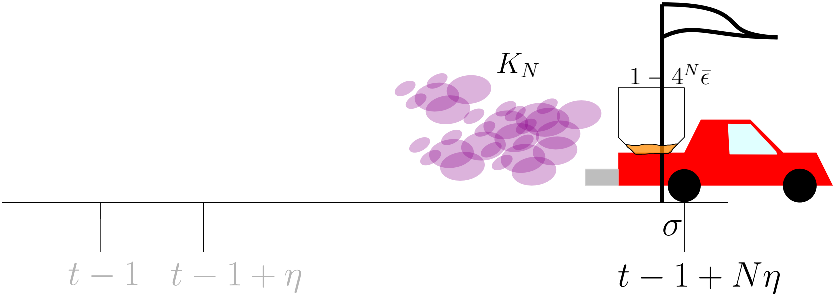
Proof of bootstrapping theorem

One application of the Key Lemma improves the thin-tubes information at the cost of a worse "K"



Proof of bootstrapping theorem

We keep going until we pass σ



The conclusion is (μ, ν) has $(\sigma, K_N, 1-\epsilon)$-thin tubes, where $\epsilon = 4^N\bar{\epsilon}$.

Proof of the Kaufman-type radial projection theorem

Kaufman-type projection theorem

Theorem (Kaufman-type estimate)

Let $X \subset \mathbb{R}^2$ be a (non-empty) Borel set which is not contained on any line. Then, for every Borel set $Y \subset \mathbb{R}^2$,

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X, \dim Y, 1\}.$$

Two main ingredients go into the proof:

1. A weak version of the above (Shmerkin 2021)
2. The ϵ-improved (s,t)-Furstenberg set dimension bound (Orponen–Shmerkin 2021)

Proof scheme

The proof of Theorem 1.1 is by bootstrapping.

0. The "weak version" due to Shmerkin tells us that, if μ, ν are s-Frostman, then (μ, ν) has β-thin tubes.
1. If μ, ν are s-Frostman and (μ, ν) has σ-thin tubes ($\sigma \geq \beta$), find an (s, s)-Furstenberg subset of $\operatorname{spt} \nu$ (at some scale r) and conclude from the ϵ-improvement that (μ, ν) has $(\sigma + \eta)$-thin tubes.
2. Bootstrap to conclude that (μ, ν) has σ-thin tubes for all $\sigma < s$.
3. Applying the result to s-Frostman measures μ on X and ν on Y, conclude from the thin tubes that the radial projection of Y about some point of X has large dimension.

Step 0. The base case

Proposition (Weak version of Theorem 1.1)

For all $C, \delta, \epsilon, s > 0$*, there exist*

$$\beta \in (0, s), \quad \tau > 0, \quad \textit{and} \quad K > 0$$

such that the following holds. If $\mu, \nu \in \mathcal{P}(B^2)$ *are* s*-Frostman with constant* C*, if* $\operatorname{dist}(\operatorname{spt}\mu, \operatorname{spt}\nu) \geq C^{-1}$*, and if* $\mu(T) \leq \tau$ *for all* δ*-tubes* T*, then* (μ, ν) *has* $(\beta, K, 1-\epsilon)$*-thin tubes.*

Loosely, if $\dim X, \dim Y \geq s$ aren't concentrated on each other and if X isn't too concentrated on lines, then $\sup_{x \in X} \pi_x(Y \setminus \{x\}) \geq \beta$.

Step 1. The bootstrapping scheme

Theorem (Bootstrapping scheme)

Let $0 < s \leq 1$ and $\epsilon \in (0, \frac{1}{10})$, let $\mu, \nu \in \mathcal{P}(B^2)$ be s-Frostman with constant C and $\mathrm{dist}(\mathrm{spt}\,\mu, \mathrm{spt}\,\nu) \geq C^{-1}$, and suppose that both (μ, ν) and (ν, μ) have $(\sigma, K, 1-\epsilon)$-thin tubes for some $\beta \leq \sigma < s$. Then there exists $\eta > 0$ such that (μ, ν) and (ν, μ) have $(\sigma + \eta, K', 1 - 5\epsilon)$-thin tubes.

Roughly, if $\dim X, \dim Y \geq s$ and if $\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \sigma$, then in fact $\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \sigma + \eta$.

Step 2. A basic criterion for thin tubes

Corollary (Thin tube criterion 1)

Let $0 < \sigma < s \leq 1$ and let $C, \epsilon, \delta > 0$. Then there exist $\tau, K > 0$ such that the following hold. If $\mu, \nu \in \mathcal{P}(B^2)$ are s-Frostman with constant C, if $\operatorname{dist}(\operatorname{spt}\mu, \operatorname{spt}\nu) \geq C^{-1}$, and if

$$\max\{\mu(T), \nu(T)\} \leq \tau \qquad \forall\, \delta\text{-tubes } T,$$

then (μ, ν) and (ν, μ) have $(\sigma, K, 1-\epsilon)$-thin tubes.

Proof. If $\beta < \sigma$, then Step 0 gives $\eta > 0$ such that (μ, ν) and (ν, μ) have $(\beta + \eta, K, 1 - 5\epsilon)$-thin tubes. After $N \sim_{s,\sigma} \eta^{-1}$ steps, one gets $(\sigma', K_N, 1 - 5^N\epsilon)$-thin tubes for some $\sigma \leq \sigma' < s$. Conclude the desired result by replacing ϵ with $\epsilon/5^N$. □

Step 2'. A more basicer criterion for thin tubes

Corollary (Thin tube criterion 2)

If $\mu, \nu \in \mathcal{P}(B^2)$ are s-Frostman and if $\mu(\ell)\nu(\ell) < 1$ for all lines $\ell \subset \mathbb{R}^2$, then (μ, ν) and (ν, μ) have σ-thin tubes for all $0 \leq \sigma < s$.

Proof. Treat in case $\nu(\ell) > 0$ for some ℓ separately using Marstrand's projection theorem (Kaufman's version). For the case $\mu(\ell), \nu(\ell) = 0$ for all ℓ, restrict and renormalize μ and ν to positively-separated sets and apply the previous corollary. □

Step 3. From thin tubes to radial projections

Theorem (Kaufman-type estimate)

Let $X \subset \mathbb{R}^2$ be a (non-empty) Borel set which is not contained on any line. Then, for every Borel set $Y \subset \mathbb{R}^2$,

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X, \dim Y, 1\}.$$

Proof. If Y is concentrated on a line ℓ, then radially project Y onto any point $x \in X$ a positive distance from ℓ. We get that $\dim \pi_x(Y) \geq \dim \pi_x(Y \cap \ell)$ is as large as possible in this case.

If instead Y is not concentrated on any line, then we can find s-Frostman measures μ on X and ν on Y such that $\mu(\ell)\nu(\ell) < 1$ for all lines ℓ. Apply the more basicer criterion for thin tubes to obtain σ-thin tubes for all $\sigma < s$. By the relation between thin tubes and radial projections, it follows that, for all $\sigma < s$, there exists $x \in X$ such that $\dim \pi_x(Y \setminus \{x\}) \geq \sigma$. $\square$

Step 1. Back to my boots(traps)

Theorem (Bootstrapping scheme)

Let $0 < s \leq 1$ and $\epsilon \in (0, \frac{1}{10})$, let $\mu, \nu \in \mathcal{P}(B^2)$ be s-Frostman with constant C and $\operatorname{dist}(\mathsf{spt}\,\mu, \mathsf{spt}\,\nu) \geq C^{-1}$, and suppose that both (μ, ν) and (ν, μ) have $(\sigma, K, 1-\epsilon)$-thin tubes for some $\beta \leq \sigma < s$. Then there exists $\eta > 0$ such that (μ, ν) and (ν, μ) have $(\sigma + \eta, K', 1 - 5\epsilon)$-thin tubes.

The general idea is to show by contradiction that, if (μ, ν) does *not* have $(\sigma + \eta)$-thin tubes, then we can find a (σ, σ)-Furstenberg subset of $Y := \mathsf{spt}\,\nu$ (at some scale $\bar{r}$), whose dimension is necessarily at least $2\sigma + \epsilon$. The geometry and dimension of this set can be used to contradict our hypothesis against the existence of thin tubes.

ϵ-Improvement to the (s, t)-Furstenberg set estimate

Theorem (ϵ-improved Furstenberg set estimate)

Given $s \in (0,1)$ *and* $t \in (s,2]$, *there exist* $\delta_0, \epsilon > 0$ *such that the following holds* $\forall\, \delta \in (0, \delta_0]$*: if* $X \subseteq B^2$ *is a* $(\delta, t, \delta^{-\epsilon})$*-set and if for each* $x \in X$ *there is a* $(\delta, s, \delta^{-\epsilon})$*-set* $\mathcal{T}_x$ *of* δ*-tubes through* x, *then*

$$\left| \bigcup_{x \in X} \mathcal{T}_x \right|_\delta \geq \delta^{-2s-\epsilon}.$$

Applications + Continuum Results

Points and Lines: Euclid's First Postulate Revisited

Figure: What is the most axiomatic property of Euclidean space?

Points and Lines: Euclid's First Postulate Revisited

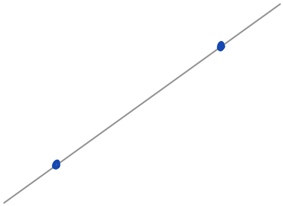


Figure: That any two non-equal points must determine a unique line.

Points and Lines: Beck's Theorem

In general, sets of $N \geq 3$ points in Euclidean space have two behaviours.

1. Either $\approx N$ many points lie on some line; or,
2. The points determine $\gtrsim N^2$-many (unique) lines.

This result is known as **Beck's Theorem**.

P.S. This is a consequence of a more nuanced result, the **Erdős-Beck Theorem**.

Theorem

Suppose that S is a set of N points in the plane, with at most $N - k$ many points collinear (for some $0 \leq k \leq N - 2$). Then, the set S determines $\gtrsim k \cdot N$ unique lines.

Points and Lines: Large Collinear Subsets

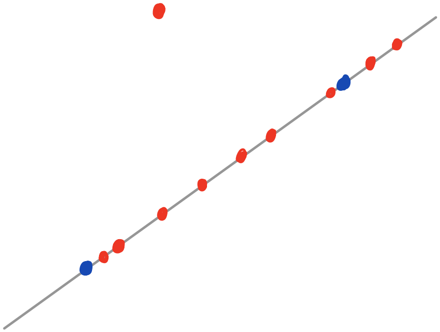


Figure: A point-set of cardinality $N = 12$ with a large subset ($k = 11$) of collinear points.

Points and Lines: Large Collinear Subsets

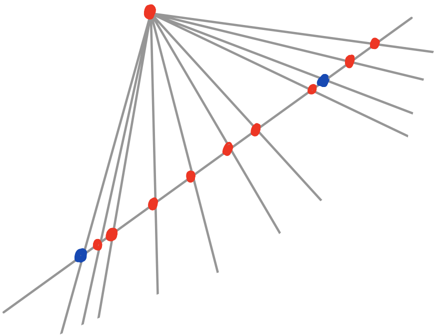


Figure: In this situation, the non-collinear point produces $N - 1 = 11$ unique lines; the remaining pairs-of-points produces only one single line, for a total of 12 lines.

Points and Lines: Large Line Sets

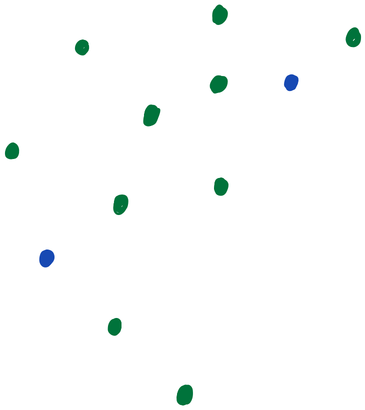


Figure: A point-set of cardinality $N = 12$ whose largest collinear subset has size $k = 4$.

Points and Lines: Large Line Sets

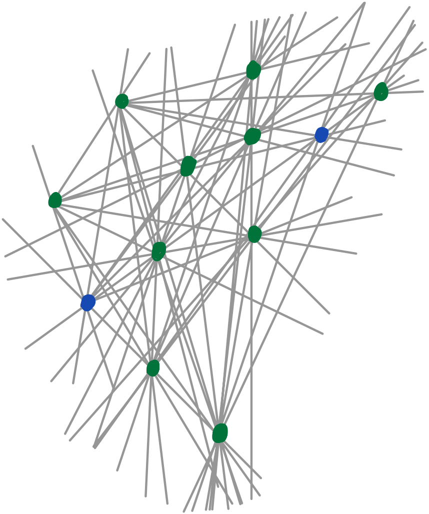


Figure: A bit of a mess; however, in this case, the number of lines is proportional to the number of pairs of unique points (around 55 lines).

A Continuum Version of Beck's Theorem

As a consequence of OSW's Main Theorem, one has the following estimate for line sets.

Corollary

Let $X \subset \mathbb{R}^2$ be a Borel set such that $\dim(X/\ell) = \dim X$ for all lines $\ell \subset \mathbb{R}^2$. Then, the line set $\mathcal{L}(X)$ spanned by (distinct) pairs of points in X satisfies,

$$\dim \mathcal{L}(X) \geq \min\{2 \dim X, 2\}.$$

OSW refer to this Corollary as a "continuum version of Beck's Theorem". Let's draw a picture to see why.

“Wiggly Fractals” & Their Line Sets

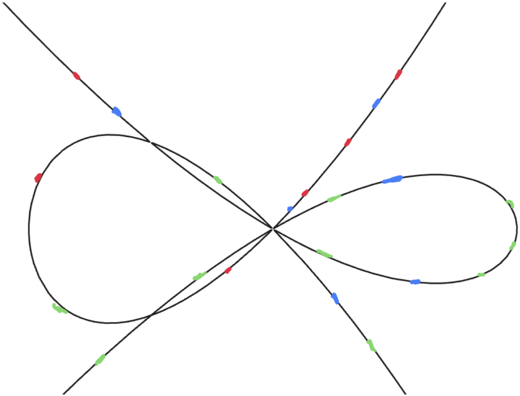


Figure: The black lines are the graph of some algebraic curve C. The coloured blobs $B, G, R \subset C$ represent three fractal subsets of C. What is a lower bound for $\dim \mathcal{L}(B \cup G \cup R)$?

A Short Remark on Incidence Theorems

Critical to the proof of Beck's Theorem is an **ϵ-improvement** over the Cauchy-Schwarz incidence estimate.

Proposition (Cauchy–Schwarz, ϵ-improvement)

If $\mathcal{S}$ is a set of points and $\mathcal{L}$ is a set of lines in the plane, let

$$I(\mathcal{S}, \mathcal{L}) = \{(p, l) \in \mathcal{S} \times \mathcal{L} : p \in l\}.$$

Then $|I(\mathcal{S}, \mathcal{L})| \leq |\mathcal{L}|^{1-\epsilon} + |\mathcal{L}|^{1/2-\epsilon}|\mathcal{S}|^{1-\epsilon}$.

Of course... this is goofy, because we know that a much stronger incidence estimate exists (the Szemerédi-Trotter theorem).

However, in the continuum, more delicate incidence estimates have to suffice.

Furstenberg sets and a continuous Szemerédi-Trotter estimate

To wrap-up our comparison of Theorem 1.1 with Beck's Theorem, we briefly sketch the necessary ϵ-improvement argument.

Theorem (T. Orponen, P. Shmerkin, 2021)

Given $s \in (0,1)$ and $t \in (s,2)$, there exists $\epsilon(s,t) > 0$ such that the following holds for all $0 \leq \delta \leq \delta_0(s,t)$. Suppose that $X \subset \mathbb{B}^2$ satisfies,

$$|X \cap B(x,r)|_\delta \leq \delta^{-\epsilon} r^t |X|_\delta,$$

and for each $x \in X$, there exists a family of $(\delta, s, \delta^{-\epsilon})$ tubes $\mathcal{T}_x$. Then,

$$\left| \bigcup_{x \in X} \mathcal{T}_x \right|_\delta \geq \delta^{-2s-\epsilon}.$$

Sum-Product Problems

Dimensional estimates for radial projections have some remarkable consequences for arithmetically-structured sets.

Corollary

Let $A, B \subset \mathbb{R}$ be Borel sets. Then

$$\dim\left(\frac{A-B}{A-B}\right) \geq \min\{\dim A + \dim B, 1\}.$$

Proof.

Included in the study guide! However, this gives a hint as to how one may utilize radial projection theorems to "lift" arithmetic problems in $\mathbb{R}$ to product sets in $\mathbb{R}^2$. □

ABC Sum-Product Theorem

A much more complex—and enticing—application of the OSW radial projection estimates is the following.

Theorem (T. Orponen, P. Shmerkin, 2023)

Let $0 < \beta \leq \alpha < 1$ *and* $\kappa > 0$. *Then there exists an* η *(depending continuously upon* α, β, κ*) such that whenever* $A, B, C \subset \mathbb{R}$ *are Borel and satisfy* $\dim A = \alpha$ *and* $\dim B = \beta$*, there exists some* $c \in C$ *satisfying,*

$$\dim(A + cB) \geq \dim A + \eta,$$

so long as $\dim C \geq \alpha - \beta + \kappa$.

The situation $\dim A = \dim B$ and $\dim C > 0$ follows from the earlier (2010) work of J. Bourgain.

Exceptional Set Estimates: Kaufman's Bound Revisited

Recall Kaufman's exceptional set estimate for $\mathbb{R}^2$.

Theorem (R. Kaufman, 1968)

Let $Y \subset \mathbb{R}^2$ be a Borel set. Then, one has,

$$\dim\{\theta \in \mathbb{S}^1 : \dim \pi_\theta(Y) < s\} \le s,$$

for $0 \le s \le \min\{\dim Y, 1\}$.

Again, as a consequence of the radial projection result of OSW, one has the following improvement.

Theorem (T. Orponen, P. Shmerkin, 2023)

Suppose that $Y \subset \mathbb{R}^2$ has equal packing and Hausdorff dimension. Then, one has:

$$\dim\{\theta \in \mathbb{S}^1 : \dim \pi_\theta(Y) < s\} \le \max\{2s - \dim Y, 0\}.$$

Sticky Kakeya Sets

Lastly, we mention one final recent advancement which incorporates the bootstrapping thin-tubes argument of OSW.

Definition

A compact $K \subset \mathbb{R}^3$ is called a *sticky Kakeya set* if there exists a family of lines $\mathcal{L}$ of packing dimension 2 such that $\ell \cap K$ contains a unit line segment for each $\ell \in \mathcal{L}$.

Theorem (H. Wang, J. Zahl, 2023)

The Hausdorff dimension of any sticky Kakeya set $K \subset \mathbb{R}^3$ is 3.