A sharp Mizohata–Takeuchi-type theorem for the cone in $\mathbb{R}^3$

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In $\mathbb{R}^3$, define

$$\mathbb{C}one^2 = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \tau = |\xi|, 1 < |\xi| < 2\}.$$

Given a smooth surface measure $d\sigma$ for $\mathbb{C}one^2$, define

$$Ef = (fd\sigma)^\vee.$$

The function $u(x, t) = Ef(x, t)$ solves the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u(x, t) = 0 \\ \widehat{u}(\xi, 0) = f(\xi). \end{cases}$$

For a set $X \subset \mathbb{R}^3$, let $U(X)$ be the smallest number such that

$$\int_X |Ef|^2 \leqslant U(X) \, \|f\|^2_{L^2(d\sigma)}$$

The number $U(X)$ should reflect the shape of X.

This is in contrast with

$$\int_{B_R} |Ef|^p \, dx \approx \alpha^p |\{x \in B_R : |Ef(x)| \approx \alpha\}|,$$

which is the left-hand side of a classic restriction or decoupling estimate.

Mizohata–Takeuchi

Let

$$P = \{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \tau = \xi^2, |\xi| < 1\}$$

and Ef be the Fourier extension of P.

Given $X \subset \mathbb{R}^2$, we let

$$\mathbf{T}(X) = \sup\{|X \cap T| : T \text{ is a } 1 \times R \text{ rectangle}\}$$

Conjecture (Mizohata–Takeuchi)

If $X \subset [0, R]^2$ is a union of lattice unit squares, then for each $\epsilon > 0$, there is $C_\epsilon > 0$ so that

$$\int_X |Ef|^2 \leqslant C_\epsilon R^\epsilon \, \mathbf{T}(X) \|f\|^2_{L^2(P)}.$$

Definition

A lightplank is a rectangular parallelepiped of dimensions $C \times AC \times A^2C$ whose long and short edges are in null directions.

For a set X, we let

$$\mathbf{P}(X) = \sup\{|X \cap P| : P \text{ is a } 1 \times R^{1/2} \times R\text{-lightplank}\}.$$

Theorem (O., 2023)

Suppose $X \subset [0, R]^3$ is a union of lattice unit cubes obeying the Frostman condition

$$|X \cap B_r| \lesssim r, \quad r > 1.$$

Then for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\int_X |Ef|^2 \leqslant C_\epsilon R^\epsilon \, \mathbf{P}(X)^{1/2} \|f\|^2_{L^2(d\sigma)}.$$

Theorem (O., 2023)

Suppose $X \subset [0,R]^3$ *is a union of lattice unit cubes obeying the Frostman condition*

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This theorem is equivalent to a Fourier average estimate.

Fourier averages

Given a measure ν in $\mathbb{R}^3$, $\widehat{\nu}(\xi)$ need not decay as $|\xi| \to \infty$.

Averages of $\widehat{\nu}$ tend to do better.

Question

Given a measure ν *with* $\operatorname{supp}\nu \subset [0,1]^3$ *and* $(\Gamma, d\sigma)$*, how does*

$$\int_{R\Gamma} |\widehat{\nu}|^2 \, d\sigma \quad \textit{decay as } R \to \infty?$$

Main theorem, Fourier average version

Theorem (O., 2023)

Suppose ν is a measure that agrees with Lebesgue measure on $X \subset [0, R]^3$, a union of lattice unit cubes obeying the Frostman condition

$$|X \cap B_r| \lesssim r, \quad r > 1.$$

Then for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\int_{\mathbb{C}one^2} |\widehat{\nu}|^2 \, d\sigma \leqslant C_\epsilon R^\epsilon \, \mathbf{P}(X)^{1/2} |X|.$$

History

For the circle in the plane, the following decay rates hold.

Theorem

In each case, assume $I_\alpha(\nu) = 1$ *with* $\operatorname{supp}\nu \subset [0,1]^2$. *The following sharp decay rates hold and are sharp:*

$$\int_0^{2\pi} |\widehat{\nu}(Re^{i\theta})|^2 \, d\theta \lessapprox \begin{cases} R^{-\alpha}, & \alpha \in (0, \frac{1}{2}] \quad \textit{(Mattila, 1987)} \\ R^{-1/2}, & \alpha \in (\frac{1}{2}, 1] \quad \textit{(Mattila, ——)} \\ R^{-\alpha/2}, & \alpha \in (1, 2) \quad \textit{(Wolff, 1999)} \end{cases}$$

History

In 2004, Erdoğan proved the following decay estimate for the cone segment in $\mathbb{R}^3$.

Theorem (Erdoğan)

Suppose $\alpha \in [1,2]$, *and* $\operatorname{supp} \nu \subset [0,1]^3$ *with* $I_\alpha(\nu) = 1$. *For every* $\epsilon > 0$, *there is* $C_\epsilon > 0$ *so that for each* $R > 1$,

$$\int_{\mathbb{C}one^2} |\widehat{\nu}(R\xi)|^2 \, d\sigma(\xi) \lesssim C_\epsilon R^\epsilon \, R^{-\alpha/2}.$$

The decay rate matches that of the circle, and is also sharp.

A feature of Wolff's and Erdoğan's work is that both authors used the connection with $\int_X |Ef|^2$ problems in the opposite direction.

In terms of the assumptions of our main theorem, we have the following slightly weaker corollary of Erdoğan's result, stated for $\alpha = 1$.

Corollary (Erdoğan)

Suppose ν is a measure that agrees with Lebesgue measure on $X \subset [0, R]^3$, a union of lattice unit cubes obeying the Frostman condition

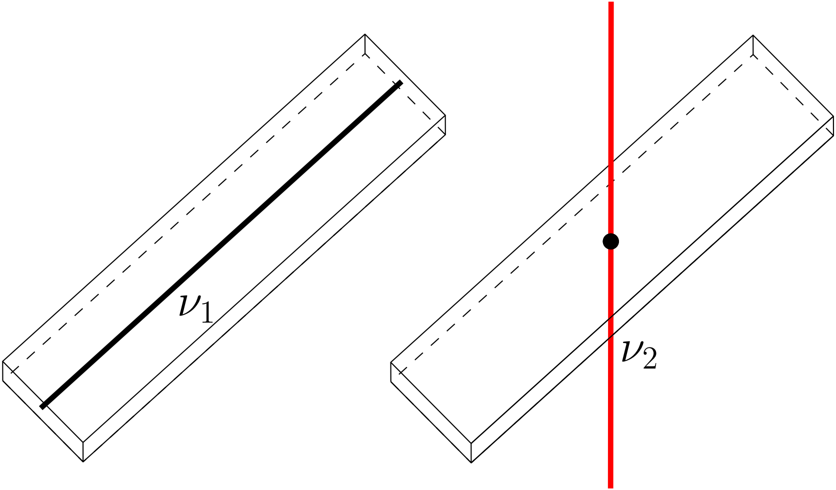
$$|X \cap B_r| \lesssim r, \quad r > 1.$$

Then for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\int_{\mathbb{C}one^2} |\widehat{\nu}|^2 \, d\sigma \leqslant C_\epsilon R^\epsilon \, R^{3/2}.$$

The main theorem strengthens this corollary by replacing the right-hand side with $C_\epsilon R^\epsilon \, \mathbf{P}(X)^{1/2} |X|$.

Examples



Here, $\mathbf{P}(\nu_1) \sim R$, while $\mathbf{P}(\nu_2) \sim 1$.

Proof of main theorem, Fourier average version

By Fourier transform properties,

$$\int_{\mathbb{C}one^2} |\widehat{\nu}|^2 \, d\sigma = \left| \iint \check{\sigma}(x-y) \, d\nu(x) \, d\nu(y) \right|$$
$$\leqslant \iint |\check{\sigma}(x-y)| \, d\nu(x) \, d\nu(y),$$

so it will suffice to estimate this double integral.

Lemma

For each $\epsilon, N > 0$, there exists $C(\epsilon, N)$ so that

$$|\check{\sigma}(x)| \leqslant C(\epsilon, N) \frac{1}{(1+|x|)^{\frac{1}{2}-\epsilon}} \frac{1}{(1+d(x,\Gamma_0))^N}.$$

Here, Γ_0 is the lightcone with vertex 0.

Let $\Delta(x,y) := d(x-y,\Gamma_0)$. By pigeonholing, for some $1 < \rho < R$,

$$\iint |\check{\sigma}(x-y)|\, d\nu(x)\, d\nu(y) \approx \rho^{-\frac{1}{2}} (\nu \times \nu)(\{(x,y) \in X \times X : |x-y| \approx \rho, \Delta(x,y) \leqslant R^{\epsilon}\}) + O(R^{-500}).$$

We let $\mathcal{L}_\rho(X) = \{(x,y) \in X \times X : |x-y| \approx \rho, \Delta(x,y) \leqslant R^{\epsilon}\}$.

Lemma (Key Lemma)

If X is a finite set of points in $[0,R]^2 \times [R,2R]^*$ such that $|X \cap B_r|_1 \lesssim r$, $1 < r < R$, then

$$|\mathcal{L}_\rho(X)| \lesssim R^{\epsilon} \rho^{\frac{1}{2}}\, \mathbf{P}(X)^{\frac{1}{2}} |X|.$$

Lemma (Key Lemma)

Fix $1 < \rho < R$. *If* X *is a finite set of points in* $[0, R]^2 \times [R, 2R]$ *such that* $|X \cap B_r|_1 \lesssim r$, $1 < r < R$, *then*

$$|\mathcal{L}_\rho(X)| \lesssim R^\epsilon \rho^{\frac{1}{2}} \, \mathbf{P}(X)^{\frac{1}{2}} |X|.$$

To prove the Key Lemma, we will transform the statement into one about thin annuli and their intersections by point-circle duality.

Point-circle duality

Given a point $x = (a, r) \in \mathbb{R}^2 \times (0, \infty)$, we can regard x as the circle

$$C_{a,r} = \{y \in \mathbb{R}^2 : |y - a| = r\}.$$

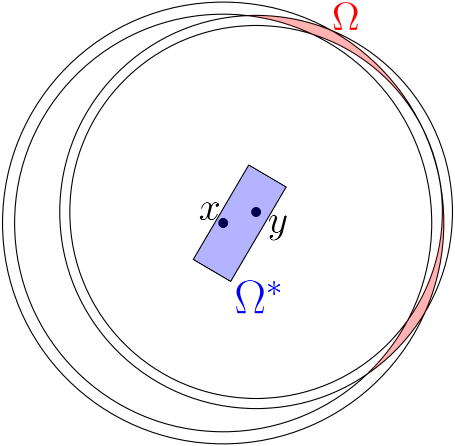
Likewise, $B(x, \delta) \subset \mathbb{R}^2 \times (0, \infty)$ can be identified with the δ-thin annulus

$$C_{\delta,a,r} = \{y \in \mathbb{R}^2 : r - \delta < |y - a| < r + \delta\}.$$

Point-circle duality

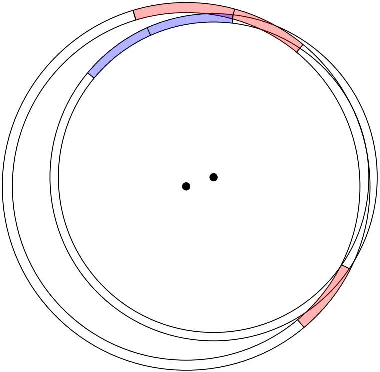
If $\delta < \rho < 1$ and $(x, y) \in \mathcal{L}_\rho(X)$, then x, y both belong to a lightplank Ω^* of dimensions $\delta \times \sqrt{\delta\rho} \times \rho$. (Here $X \subset [0,1]^2 \times [1,2]$.)

Equivalently, $C_{\delta,x} \cap C_{\delta,y}$ intersect in two $\delta, \sqrt{\frac{\delta}{\rho}}$-"rectangles."



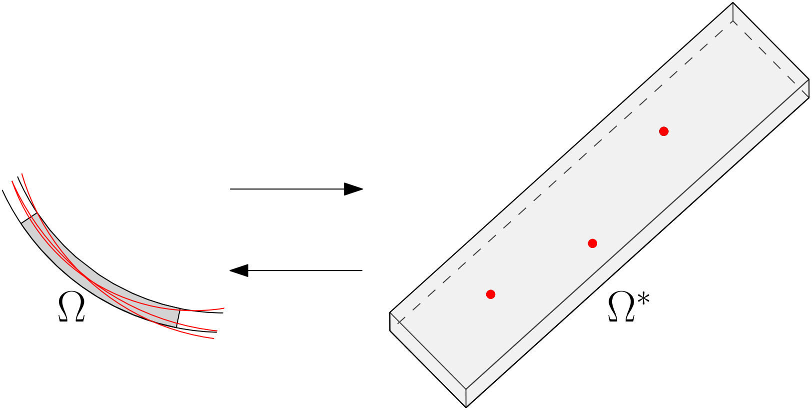
Point-circle duality

Cover $\bigcup_{x \in X} C_{\delta,x}$ by a maximal family $\mathcal{R}$ of pairwise incomparable $\delta, \sqrt{\frac{\delta}{\rho}}$-rectangles.



For each $\Omega \in \mathcal{R}$, we let

$$\nu(\Omega) = |\{x \in X : \Omega \subset C_{\delta,x}\}|.$$



Estimate of $|\mathcal{L}_\rho(X)|$

Therefore,

$$\mathcal{L}_\rho(X) \subset \bigcup_{\Omega \in \mathcal{R}} (\Omega^* \times \Omega^*) \cap (X \times X),$$

and by the union bound

$$\begin{aligned} |\mathcal{L}_\rho(X)| &\leqslant \sum_{\Omega \in \mathcal{R}} \nu(\Omega)^2 \\ &\leqslant \mathbf{P}(X)^{1/2} \sum_{\Omega \in \mathcal{R}} \nu(\Omega)^{3/2}. \end{aligned}$$

Wolff's circular maximal estimate

Let X be a set of $\leqslant R(=\delta^{-1})$ circles in $[0,1]^2 \times [1,2]$ with δ-separated radii.

Let $g_\delta(y)$ be the multiplicity function

$$g_\delta(y) = \sum_{x \in X} C_{\delta,x}(y).$$

Then

$$\|g_\delta\|_{L^{3/2}(\mathbb{R}^2)} \lesssim R^{\epsilon}(\delta|X|)^{2/3}$$

Proof of Key Lemma (scales < 1)

Let $\mathcal{R}$ be a maximal collection of pairwise incomparable δ, τ-rectangles (with $\tau = \sqrt{\frac{\delta}{\rho}}$).
Then,

$$g_\delta(y) \geqslant \sum_{\Omega \in \mathcal{R}} \nu(\Omega)\chi_\Omega(y).$$

By Wolff's maximal function estimate ($\alpha = 1!$),

$$\begin{aligned}
\delta|X| &\gtrapprox \int g_\delta^{3/2} \\
&\geqslant \int \Big(\sum_{\Omega \in \mathcal{R}} \nu(\Omega)\chi_\Omega(y)\Big)^{3/2} \\
&\geqslant \sum_{\Omega \in \mathcal{R}} \nu(\Omega)^{3/2}|\Omega|.
\end{aligned}$$

Proof of Key Lemma (scales < 1)

Recall

$$|\mathcal{L}_\rho(X)| \leqslant \sum_{\Omega \in \mathcal{R}} \nu(\Omega)^2 \leqslant \mathbf{P}(X)^{1/2} \sum_{\Omega \in \mathcal{R}} \nu(\Omega)^{3/2}.$$

By Wolff's maximal estimate, we have further

$$|\mathcal{L}_\rho(X)| \lessapprox \mathbf{P}(X)^{1/2} \cdot (\delta |X|) \cdot |\Omega|^{-1}.$$

Using $|\Omega| = \delta \cdot \sqrt{\frac{\delta}{\rho}}$ gives

$$|\mathcal{L}_\rho(X)| \lessapprox \sqrt{\frac{\rho}{\delta}} \cdot \mathbf{P}(X)^{1/2} |X|.$$

Proof of Key Lemma

Lemma (Key Lemma)

If X is a finite set of points in $[0,R]^2 \times [R,2R]$ such that $|X \cap B_r|_1 \lesssim r$, $1 < r < R$, then for each $1 < \rho < R$,

$$|\mathcal{L}_\rho(X)| \lesssim R^\epsilon \rho^{\frac{1}{2}}\, \mathbf{P}(X)^{\frac{1}{2}} |X|.$$

By the bound for scales < 1,

$$|\mathcal{L}_{\delta\rho}(\delta X)| \lessapprox \sqrt{\frac{\delta\rho}{\delta}} \cdot \mathbf{P}(X)^{1/2}|X| = \rho^{1/2}\mathbf{P}(X)^{1/2}|X|$$

By scale invariance,

$$|\mathcal{L}_\rho(X)| \lessapprox \rho^{1/2}\mathbf{P}(X)^{1/2}|X|.$$

Recap

We proved

$$\int_{\mathbb{C}one^2} |\widehat{\nu}|^2 \, d\sigma \lesssim R^{\epsilon} \, \mathbf{P}(X)^{1/2} |X|,$$

and it is sharp. Apart from R^{ϵ}, this is equivalent to

$$\int_X |Ef|^2 \lesssim R^{\epsilon} \, \mathbf{P}(X)^{1/2} \|f\|^2_{L^2(d\sigma)}.$$

The proof doesn't use any cancellation of $\check{\sigma}$ within lightplanks.
Let

$$\mathbf{L}(X) = \sup\{|X \cap T| : T \text{ is a } 1 \times 1 \times R \text{ tube in a null direction}\}$$

It seems natural to conjecture (at least for 1-dimensional X),

$$\int_X |Ef|^2 \lesssim R^{\epsilon} \, \mathbf{L}(X)^{1/2} \|f\|^2_{L^2(d\sigma)}$$

Thank you!