Bökstedt's theorem on $\mathrm{THH}_*(\mathbb{F}_p)$

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In this talk, we present a proof of Bökstedt's theorem $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$ using a result of Steinberger on Dyer-Lashof operations and Hopkins-Mahowald's theorem that there is an equivalence of $\mathbb{E}_2$-ring spectra $H\mathbb{F}_p \simeq M(\Omega^2 S^3 \to BGL_1(\mathbb{S}_p^\wedge))$.

Note that Bökstedt's theorem reduces to the following fact:

Claim 0.1. $\mathrm{THH}(\mathbb{F}_p) = H\mathbb{F}_p \otimes \Sigma_+^\infty \Omega S^3$.

Then $\mathrm{THH}_*(\mathbb{F}_p) = \pi_*(\mathrm{THH}(\mathbb{F}_p))$ is the homology of ΩS^3 with $\mathbb{F}_p$ coefficients, which has a single generator in degree 2. Bökstedt's theorem thus follows.

Convention: We denote by $\mathcal{S}$ and Sp the infinity categories of spaces and spectra. Unless specifically noted, the tensor product $\otimes$ is taken over $\mathbb{S}$.

1 Thom spectrum

We begin by recalling the construction of the generalized Thom spectrum following [1].

Let R be a ring spectrum and $X \in \mathcal{S}$. We want to obtain an analogous construction of a line bundle with fiber R over a topological space.

First off, we can put a free rank 1 R-module $L_p \simeq R$ at each point $p \in X$. Then we need to specify homotopy coherence conditions: for every path γ connecting points p, q, there should be an equivalence L_γ of R-modules between the fibers L_p, L_q; for every homotopy h between γ and γ', there should be a homotopy $L_\gamma \to L_{\gamma'}$ in the space of R-module equivalences $L_p \to L_q$; etc.

To sum up, we define a *bundle of R-line bundle over X* to be a functor of infinity groupoids

$$f : X \simeq X^{op} \to BGL_1(R)$$

sending each point of X to the unique point of $BGL_1(R)$. Here $GL_1(R) \simeq \mathrm{Aut}_R(R)$ is the grouplike space of units of the ring R. Explicitly, it is the pullback

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Omega^\infty(R) \arrow[d] \\ \pi_0(R)^\times \arrow[r] & \pi_0(R) \end{tikzcd}$$

in $\mathcal{S}$.

Definition 1.1. The (generalized) *Thom spectrum* Mf of a map $f : X \to BGL_1(R)$ in $\mathcal{S}$ is the spectrum

$$Mf := \mathrm{colim}(X \xrightarrow{f} BGL_1(R) \to R\text{-mod}).$$

For fixed R, we obtain a Thom spectrum functor

$$M : \mathcal{S}_{/BGL_1(R)} \to R\text{-mod}.$$

Note that if f is an $\mathbb{E}_n$ map, then Mf inherits an $\mathbb{E}_n$-ring structure.

Remark 1.2. There is a "change of fiber" formula. Given a morphism $r : R \to R'$, there is a commutative diagram connecting two line bundles
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="] & BGL_1(R) \arrow[r] \arrow[d, "\hat{r}"] & R\text{-mod} \arrow[d, "-\otimes_R R'"] \\ X \arrow[r] & BGL_1(R') \arrow[r] & R'\text{-mod} \end{tikzcd}.$$

The Thom spectrum of the bottom row is given by

$$\begin{aligned} M(\hat{r}\circ f) =& \operatorname{colim}(X \xrightarrow{f} BGL_1(R) \xrightarrow{\hat{r}} BGL_1(R') \to R\text{-mod}) \\ \simeq& R' \underset{R}{\otimes} \operatorname{colim}(X \xrightarrow{f} BGL_1(R) \to R'\text{-mod}) \\ =& R' \underset{R}{\otimes} Mf \end{aligned}$$

Example 1.3. The "trivial line bundle with fiber R" is the constant functor $\underline{R}$, and

$$M\underline{R} = R \otimes \Sigma^\infty_+ X$$

is the generalized homology with coefficient in R.

Example 1.4. If $X = BG$ is the classifying space of an $\mathbb{E}_1$ group, then $f : BG \to BGL_1(R)$ induces an action of G on R via $\Omega f : G \to GL_1(R)$. Hence Mf is by definition the homotopy orbit space

$$R_{hG} := \operatorname{colim}(BG \to R\text{-mod} \to Sp),$$

where the map in the colimit sends the unqiue point of BG to R.

One may compare this with the following unstable construction: given a fibration $F \to E \to BG$ of topological spaces, there is a $G \simeq \Omega BG$-action on the fiber F, whose homotopy orbit space is

$$E \simeq F_{hG} \simeq F \times_G EG.$$

Example 1.5. Taking $R = \mathbb{S}$ recovers the classical construction of stable spherical fibrations. A map $BG \to BGL_1(\mathbb{S})$ produces an action of G on the fiber $\mathbb{S}$, which is equivalent to a $\Sigma^\infty_+ G$-module structure on S. Hence

$$(\mathbb{S})_{hG} \simeq \mathbb{S} \underset{\Sigma^\infty_+ G}{\otimes} \mathbb{S}$$

with the two augmentations given respectively by the module structure and the trivial map $G \to *$.

Example 1.6. The universal real Thom spectra MO can be obtained using the J-homomorphism

$$BJ : BO \to BGL_1(\mathbb{S}).$$

Example 1.7. Let $\mathrm{Conf}_k(M)$

2 Hopkins-Mahowald's Theorem

Recall that a two-fold loop map $\Omega^2 S^3 \to BGL_1(\mathbb{S}^\wedge_p)$ is induced from a map $S^1 \to BGL_1(\mathbb{S}^\wedge_p)$. Equivalently, such a map is the adjoint to an element of

$$\pi_1(BGL_1(\mathbb{S}^\wedge_p)) = \pi_0(GL_1(\mathbb{S}^\wedge_p)) = \mathbb{Z}_p^\times.$$

Let f_p be the two-fold loop map corresponding to $1 + u \cdot p \in \mathbb{Z}_p^\times$, where u is a unit.

Theorem 2.1 (Hopkins-Mahowald)**.** *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$Mf_p \simeq H\mathbb{F}_p.$$

Proof. We follow the proof in [8, A.1]. By definition $Mf_p = (\mathbb{S}_p^\wedge)_{h\Omega^3 S^3}$. We want to compute $\pi_0(Mf_p)$. Note that $\pi_0 : \tau_{\geq 0}Sp \to \mathbb{A}$ has right adjoint the Eilenberg-MacLane spectrum functor, so it commutes colimits for connective spectra. Furthermore, since Ab is a 1-category, the 1-truncation of any colimit diagram D in Ab are cofinal, i.e. taking colimit indexed by D is equivalent to taking colimit indexed by the 1-truncation of D. Hence we deduce that [1]

$$\begin{aligned}
\pi_0(Mf_p) = \pi_0((\mathbb{S}_p^\wedge)_{h\Omega^3 S^3}) &= \pi_0(\mathrm{colim}(\Omega^2 S^3 \to BGL_1(\mathbb{S}_p^\wedge))) \\
&\simeq \underset{B\Omega^3 S^3}{\mathrm{colim}} \pi_0(BGL_1(\mathbb{S}_p^\wedge)) \\
&\simeq \underset{B\pi_0(\Omega^3 S^3)}{\mathrm{colim}} \pi_0(BGL_1(\mathbb{S}_p^\wedge)) \\
&\simeq (\pi_0(BGL_1(\mathbb{S}_p^\wedge))_{h\pi_0(\Omega^3 S^3)} = (\mathbb{Z}_p)_{\mathbb{Z}}.
\end{aligned}$$

Since $1 \in \mathbb{Z}$ acts on $\mathbb{Z}_p$ by $1-p$, we have $(\mathbb{Z}_p)_{\mathbb{Z}} \cong \mathbb{Z}_p/(1-(1+u\cdot p)) \cong \mathbb{F}_p$. Thus we obtain an $\mathbb{E}_2$-map

$$\phi : Mf_p \to H\mathbb{F}_p$$

to the 0th stage of the Postnikov tower of Mf with all stages and maps $\mathbb{E}_2$.[2]

We claim that ϕ is an equivalence. Note that Mf_p and $H\mathbb{F}_p$ are p-torsion[3] and connective, so it suffices to show that

$$\phi_* : H_*(Mf_p;\mathbb{F}_p) \to H_*(H\mathbb{F}_p;\mathbb{F}_p)$$

is an isomorphism on homology. To understand $H_*(Mf_p;\mathbb{F}_p)$, we compare two ways of computing the Thom spectrum of the $\mathbb{E}_2$-map

$$\Omega^2 S^3 \xrightarrow{f_p} BGL_1(\mathbb{S}_p^\wedge) \xrightarrow{r} BGL_1(H\mathbb{F}_p) = B\mathbb{F}_p^\times,$$

where r is induced by the reduction mod p. The change of fiber formula gives an equivalences of $\mathbb{E}_2$-ring spectrum

$$M(r\circ f_p) = \mathrm{colim}_{\Omega^2 S^3}(r\circ f_p) \simeq H\mathbb{F}_p \underset{\mathbb{S}_p^\wedge}{\otimes} \mathrm{colim}_{\Omega^2 S^3} f_p \simeq H\mathbb{F}_p \otimes Mf_p.$$

On the other hand, the 2-fold loop map $r\circ f_p$ is a lift of

$$S^1 \xrightarrow{1+u\cdot p} BGL_1(\mathbb{S}_p^\wedge) \xrightarrow{r} BGL_1(H\mathbb{F}_p),$$

which has to be null-homotopic since $1+u\cdot p = 1 \bmod p$. Hence there is an equivalence of $\mathbb{E}_2$-ring spectra

$$H\mathbb{F}_p \otimes \Sigma_+^\infty \Omega^2 S^3 \simeq M(r\circ f_p) \simeq H\mathbb{F}_p \otimes Mf_p,$$

both sides of which are the $\mathbb{F}_p$-homology.

As a result, we only need to check that

$$\phi_* : H_*(Mf_p;\mathbb{F}_p) \cong H_*(\Omega^2 S^3;\mathbb{F}_p) \to H_*(H\mathbb{F}_p;\mathbb{F}_p)$$

is an isomorhism in degree 0 and 1. Then the following classical results [black-box] ensure that ϕ_* extends to an isomorphism, since both sides are generated by $\mathbb{E}_2$-Dyer-Lashof operations from degree 1 as $\mathbb{E}_2$-rings.

[1] Alternatively, one can see this using a spectral sequence for the homotopy orbit space E_{hG} with

$$E^2_{*,*} = H_*(BG, \pi_*(E)) \Rightarrow \pi_*(E_{hG}),$$

which is first-quadrant for connective G-spectrum E. [6, 2.5] Here we have an action of $\Omega^3 S^3$ on $\mathbb{S}_p^\wedge$. The $(0,0)$ term survives and is given by the 0th homology group with local coefficients

$$\pi_0(Mf_p) = H_0(\Omega^2 S^3; \pi_0(\mathbb{S}_p^\wedge)) = \mathrm{Tor}^{\pi_1(\Omega^2 S^3)}(\mathbb{Z}; H_0(\mathbb{S}_p^\wedge)) = \mathrm{Tor}^{\mathbb{Z}}(\mathbb{Z};\mathbb{Z}_p) = (\mathbb{Z}_p)_{\mathbb{Z}}.$$

[2] Such a tower is constructed by using exclusively $\mathbb{E}_n$ cells to kill off higher homotopy groups in the usual Postnikov tower construction. See [2, Section 4] for details.

[3] This is because the unit $1 \in \pi_0(Mf_p)$ of the associative graded ring $\pi_*(Mf_p)$ is p-torsion.

Lemma 2.2. *1). $H_*(\Omega^2S^3;\mathbb{F}_p)=\mathbb{F}_p[y_0,y_1,\dots;z_1,z_2,\dots]/(y_i^2)$, where $|y_i|=2p^i-1$ and $|z^i|=2p^i-2$. The elements y_i and z_i are generated from the degree 1 element y_0 via Dyer-Lashof operations.*

2). There is an ($\mathbb{E}_2$-) ring isomorphism $H_*(\Omega^2S^3;\mathbb{F}_p)\cong H_*(H\mathbb{F}_p;\mathbb{F}_p)$ of Pontryagin rings, i.e. this isomorphism is compatible with the Dyer-Lashof operations.

Consider the $\mathbb{E}_1$ map $\mathbb{Z}\to\Omega^3S^3$ induced from the canonical map $g:S^1\to\Omega^2S^3$. Then we get a map

$$\mathbb{S}/p=(\mathbb{S}_p^\wedge)_{h\mathbb{Z}}\to(\mathbb{S}_p^\wedge)_{h\Omega^3S^3}=Mf_p.$$

This is an isomorphism on homology in degree 0 and 1 since g is 1-connected. The composition

$$\mathbb{S}/p\to Mf_p\to H\mathbb{F}_p$$

is the map to the 0th section of the postnikov tower of $\mathbb{S}/p$, which is an isomorphism in degree 0 and 1 $\mathbb{F}_p$-homology. This concludes the proof. $\square$

3 An algebraic proof

Now we prove that $\mathrm{THH}(\mathbb{F}_p)=H\mathbb{F}_p\otimes\Sigma_+^\infty\Omega S^3$. This proof was sketched in [7, 1.2] and explained in [9] using an argument in [5, 5.7].

Proof of Claim 0.1. There is a homotopy fiber sequence

$$\Omega X\times\Omega X\to\Omega X\xrightarrow{ev_{1/2}}X,$$

where $ev_{1/2}$ sends a loop $\gamma:I\to X$ to $\gamma(1/2)$. Then $\Omega X=(\Omega X\times\Omega X)_{h\Omega X}$ is the homotopy orbit space of the fibration

$$\Omega X\to\Omega X\times\Omega X\to\Omega X.$$

Take $\Omega X=\Omega^2S^3\in\mathcal{S}_{/BGL_1(\mathbb{S}_p^\wedge)}$ with the augmentation f_p and

$$\Omega^2S^3\times\Omega^2S^3\xrightarrow{f_p\times f_p}BGL_1(\mathbb{S}_p^\wedge)\times BGL_1(\mathbb{S}_p^\wedge)\xrightarrow{\mu}BGL_1(\mathbb{S}_p^\wedge).$$

Then we have $\Omega^2S^3=(\Omega^2S^3\times\Omega^2S^3)_{h\Omega^2S^3}$ in $\mathcal{S}_{/BGL_1(\mathbb{S}_p^\wedge)}$. Apply the Thom spectrum funtor on both sides. Then the Hopkins-Mahowald result implies that the $H\mathbb{F}_p\otimes H\mathbb{F}_p$-module structure of $H\mathbb{F}_p$ is given by

$$H\mathbb{F}_p=(H\mathbb{F}_p\otimes H\mathbb{F}_p)_{h\Omega^2S^3}=(H\mathbb{F}_p\otimes H\mathbb{F}_p)\underset{\Sigma_+^\infty\Omega^2S^3}{\otimes}\mathbb{S}.$$

Thus we have

$$\begin{aligned}
\mathrm{THH}(\mathbb{F}_p)&=H\mathbb{F}_p\underset{H\mathbb{F}_p\otimes H\mathbb{F}_p}{\otimes}H\mathbb{F}_p &(1)\\
&=H\mathbb{F}_p\underset{H\mathbb{F}_p\otimes H\mathbb{F}_p}{\otimes}(H\mathbb{F}_p\otimes H\mathbb{F}_p)\underset{\Sigma_+^\infty\Omega^2S^3}{\otimes}\mathbb{S} &(2)\\
&=H\mathbb{F}_p\underset{\Sigma_+^\infty\Omega^2S^3}{\otimes}\mathbb{S} &(3)\\
&=H\mathbb{F}_p\otimes\mathbb{S}\underset{\Sigma_+^\infty\Omega^2S^3}{\otimes}\mathbb{S} &(4)\\
&=H\mathbb{F}_p\otimes\Sigma_+^\infty(*\underset{\Omega^2S^3}{\otimes}*) &(5)\\
&=H\mathbb{F}_p\otimes\Sigma_+^\infty(B(*,\Omega^2S^3,*)) &(6)\\
&=H\mathbb{F}_p\otimes\Sigma_+^\infty\Omega S^3. &(7)
\end{aligned}$$

$\square$

4 A topological proof

Next we look briefly at a proof in [3] that is more hands-on. The main idea is that $\mathrm{THH}(Mf)$ can be expressed as a Thom spectrum. Under good conditions, a generalized version of Thom isomorphism allows one to factor out Mf.

Theorem 4.1. *Let $F : X \to BGL_1(R)$ be any R-line bundle over a connected $\mathbb{E}_1$-algebra X. Then*

$$\mathrm{THH}(MF) = M(\mathcal{L}BX \to BGL_1(R))$$

as R-modules, where $\mathcal{L}$ is the free loop space functor. If F is an $\mathbb{E}_2$-map, then they are equivalent as $\mathbb{E}_1$-R algebras.

Proof. Since the Thom spectrum functor preserves colimits and tensor product, and thus the cyclic bar construction, we have

$$\mathrm{THH}(MF) = \mathrm{HH}(MF/Sp) = M(\mathrm{HH}(X/\mathcal{S}_{/BGL_1(R)})).$$

We want to understand the map we are taking the Thom spectrum funtor over on the right hand side. Recall a classical theorem of Goodwillie [4] that for a topological loop space ΩZ, the ordinary Hochschild complex of the singular chain complex $\mathrm{HH}(C_*(\Omega Z))$ is isomorphic to the chain complex $C_*(\mathcal{L}Z)$ of the free loop space on Z.

Now consider the Hochshild complex of X in the over category $\mathcal{S}_{/BGL_1(R)}$. Then $\mathrm{HH}(X/\mathcal{S}_{/BGL_1(R)})$ consists of a map

$$\mathrm{HH}(X/\mathcal{S}) \to \mathrm{HH}(BGL_1(R)/\mathcal{S}) \to BGL_1(R).$$

Using the theorem above and the fact that the two constructions Hochschild complex of a DGA in the classical and the ∞-categorical settings agree [8, Proposition 3.6], this is equivalent to a map of spaces

$$h : \mathcal{L}BX \to \mathcal{L}B^2GL_1(R) \to BGL_1(R).$$

Thus we conclude that $\mathrm{THH}(MF) = Mh$.

If F is an $\mathbb{E}_2$ map, then BF is $\mathbb{E}_1$ and the structure is preserved by taking Hochshild complex. □

Recall that the fibration

$$X \simeq \Omega BX \to \mathcal{L}BX \xrightarrow{ev_0} BX$$

admits a section given by the constant loops. If X is an $\mathbb{E}_1$-space, then the composite

$$X \times BX \to \mathcal{L}BX \times \mathcal{L}BX \xrightarrow{\mu} \mathcal{L}BX$$

is an equivalence. Note that this is an $\mathbb{E}_1$-equivalence only after taking Thom spectrum. Under good conditions, we can use the following version of the Thom isomorphism to factor the Thom spectrum of $\mathcal{L}BX$.

Lemma 4.2. *Let Y be an $\mathbb{E}_n$-space and X an $\mathbb{E}_{n+1}$-grouplike space over $BGL_1(R)$ such that*

$$\begin{tikzcd} Y \arrow[d, "g"] \arrow[dr] & \\ X \arrow[r, "f"] & BGL_1(R) \end{tikzcd}$$

commutes. Suppose that $Mg : MY \to MX$ refines to a map of $\mathbb{E}_n$-ring spectra. Then there is an equivalence of $\mathbb{E}_n$-ring spectra $MX \otimes MY \xrightarrow{\simeq} MX \otimes Y$.

Then for an $\mathbb{E}_2$-map f with an $\mathbb{E}_3$ structure on Mf, we can deduce that there are equivalences of E_1-R-algebras

$$\mathrm{THH}(Mf) \simeq M(\mathcal{L}BX \to BGL_1(R)) \simeq M(X \times BX \to BGL_1(R)) \simeq Mf \otimes BX.$$

To conclude the proof of Bökstedt's theorem, we simply plug in the map $F = f_p : \Omega^2 S^3 \to BGL_1(\mathbb{S}_p^\wedge)$.

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