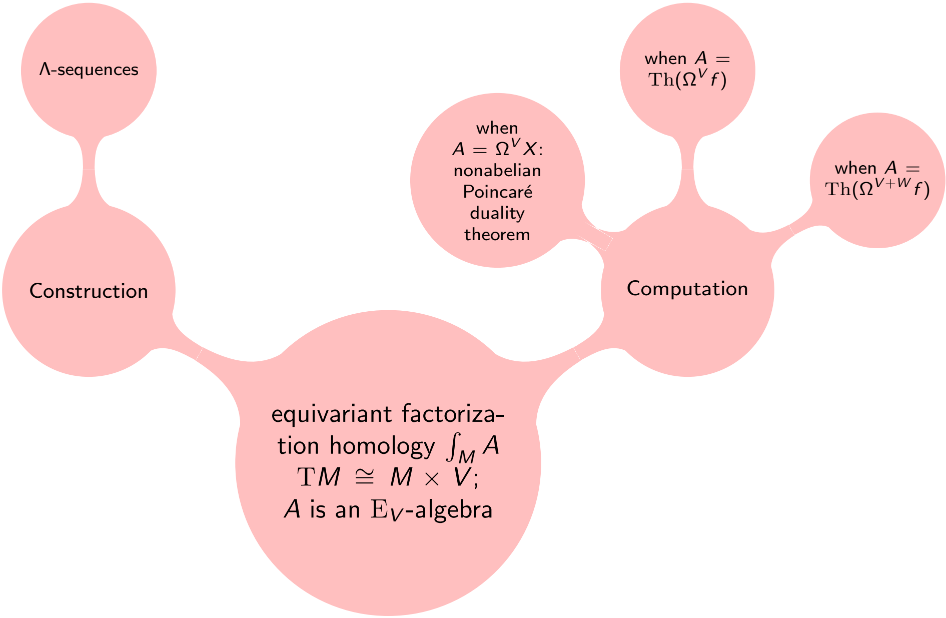
Nonabelian Poincaré duality theorem and equivariant factorization homology of Thom spectra

Foling Zou

University of Michigan

May 10, 2021

Mindmap


Λ-sequences
Construction
equivariant factorization homology $\int_M A$
$\mathrm{T}M \cong M \times V$;
A is an E_V-algebra
Computation
when $A = \Omega^V X$: nonabelian Poincaré duality theorem
when $A = \mathrm{Th}(\Omega^V f)$
when $A = \mathrm{Th}(\Omega^{V+W} f)$

Factorization homology, introduction

- Inputs: M and A
  M is a framed n-manifold (with given isomorphism $\mathrm{T}M \cong M \times \mathbb{R}^n$)
  A is an E_n-algebra in $(\mathrm{Top}, \times)$, $(\mathrm{Sp}, \wedge)$ or $(\mathcal{C}, \otimes)$
- Output: $\int_M A$, an object of Top, Sp or $\mathcal{C}$.
- Generalize ordinary homology theories on manifolds.
  - Special case: $\mathcal{C} = (\mathcal{D}(\mathbb{Z}), \oplus)$, where $\mathcal{D}(\mathbb{Z})$ consists of suitable chain complexes of $\mathbb{Z}$-modules. Let A be an abelian group, then

$$\int_M A \simeq C_*(M; A)$$

Properties

Eilenberg–Steenrod axioms	Ayala–Francis axioms
$\mathrm{H}_*(X, \mathbb{Z})$	$\int_M A$
$\mathrm{H}_*(\mathrm{pt}, \mathbb{Z}) \cong \mathbb{Z}$	$\int_{\mathbb{R}^n} A \simeq A$
$\mathrm{H}_*(X \sqcup Y, \mathbb{Z}) \cong \mathrm{H}_*(X, \mathbb{Z}) \oplus \mathrm{H}_*(Y, \mathbb{Z})$	$\int_{M \sqcup N} A \simeq \int_M A \otimes \int_N A$
excision / MV-sequence	tensor excision

Detour: Λ-sequences and reduced operads

Goal

Describe reduced operads and reduced monads categorically.

Notation

Let Σ be the category of finite sets and bijections; Λ be the category of based finite sets and based injections. Objects: $\mathbf{n} = \{0, 1, 2, \cdots, n\}$.

- We work with $(\mathrm{Top}, \times, *)$, $(G\mathrm{Top}, \times, *)$ or $(\mathcal{V}, \otimes, I)$.
- A symmetric sequence or Σ-sequence is a functor $\Sigma^{\mathrm{op}} \to \mathrm{Top}$;
  A Λ-sequence is a functor $\Lambda^{\mathrm{op}} \to \mathrm{Top}$.
- All such functors are denoted $\Sigma^{\mathrm{op}}[\mathrm{Top}]$ or $\Lambda^{\mathrm{op}}[\mathrm{Top}]$
- Examples. Suppose $\mathscr{C}$ is an operad. Then $\{\mathscr{C}(n)\}$ forms a Σ-sequence.
  If moreover $\mathscr{C}$ is reduced, meaning $\mathscr{C}(0) = *$, then $\{\mathscr{C}(n)\}$ forms a Λ-sequence. For $\mathbf{n} \hookrightarrow \mathbf{n+1}$, we have

$$\mathscr{C}(n+1) = \mathscr{C}(n+1) \times *^{n+1} \to \mathscr{C}(n+1) \times \mathcal{C}(1) \times \cdots \times \mathcal{C}(1) \times \mathcal{C}(0) \to \mathcal{C}(n).$$

First monoidal product: the Day convolution

$$\begin{array}{ccccc} \Lambda^{op} \times \Lambda^{op} & \xrightarrow{\mathscr{D} \times \mathscr{E}} & \mathrm{Top} \times \mathrm{Top} & \xrightarrow{\times} & \mathrm{Top} \\ \downarrow{\scriptstyle \vee} & & & \nearrow_{\mathscr{D} \boxtimes \mathscr{E}} & \\ \Lambda^{op} & & & & \end{array}$$

Unit: $\mathcal{I}_0 = \Lambda(-, \mathbf{0})$

Proposition (May-Zhang-Z.)

$i^* \mathscr{D} \boxtimes_\Sigma i^* \mathscr{E} \cong i^* (\mathscr{D} \boxtimes_\Lambda \mathscr{E})$

Explicitly,

$$(\mathscr{D} \boxtimes \mathscr{E})(\mathbf{n}) = \coprod_{n_1 + n_2 = n} (\mathscr{D}(\mathbf{n_1}) \times \mathscr{E}(\mathbf{n_2})) \times_{\Sigma_{n_1} \times \Sigma_{n_2}} \Sigma_n$$

Notation: Use $\Lambda^{\mathrm{op}}[\mathrm{Top}]_*$ to denote the category under the unit $\mathcal{I}_0$, i.e. Λ-sequences that are pointed at $\mathbf{0}$.
Remark: Symmetric envolope $\widetilde{\mathscr{C}}$ of an operad $\mathscr{C}$:
objects n; morphisms $\widetilde{\mathscr{C}}(m, n) = \mathscr{C}^{\boxtimes n}(m)$.

Second monoidal product: the Kelly product

For symmetric sequences, the Kelly product is

$$(\mathscr{C} \odot_\Sigma \mathscr{D})(\mathbf{n}) = \coprod_{k, n_1+\cdots+n_k=n} \mathscr{C}(\mathbf{k}) \times_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}} (\mathscr{D}(\mathbf{n_1}) \times \cdots \times \mathscr{D}(\mathbf{n_k}))$$

This formula can be reinterpreted in a way that works for Λ-sequences.
For $\mathscr{C} \in \Lambda^{\mathrm{op}}[\mathrm{Top}]$ and $\mathscr{D} \in \Lambda^{\mathrm{op}}[\mathrm{Top}]_*$, define

$$\mathscr{C} \odot_\Lambda \mathscr{D} = \mathscr{C} \otimes_\Lambda \mathscr{D}^{\boxtimes *}$$

- The unit of $\odot_\Lambda$ is $\mathcal{I}_{1,\Lambda} = \Lambda(-, \mathbf{1})$.
- There is an isomorphism

  $$(\mathscr{C} \boxtimes \mathscr{D}) \odot \mathscr{E} \cong (\mathscr{C} \odot \mathscr{E}) \boxtimes (\mathscr{D} \odot \mathscr{E}).$$

  Using this, we have associativity of $\odot$.
- Remark: $\mathcal{I}_1^{\boxtimes k} \cong \Lambda(-, \mathbf{k}) =: \mathcal{I}_k$, $\mathcal{I}_m \odot \mathcal{I}_n \cong \mathcal{I}_{mn}$.

Reduced operad and monad

Proposition

- *(Kelly) An operad is a monoid in* $(\Sigma^{\mathrm{op}}[\mathrm{Top}], \odot_\Sigma, \mathcal{I}_{1,\Sigma})$*.*
- *(May-Zhang-Z.) A reduced operad is a monoid in* $(\Lambda^{\mathrm{op}}[\mathrm{Top}], \odot_\Lambda, \mathcal{I}_{1,\Lambda})$*.*

Adjunction:

$$i_0 : \mathrm{Top}_* \rightleftharpoons \Lambda^{\mathrm{op}}[\mathrm{Top}]_* : p_0$$

$$i_0(X)(\mathbf{n}) = \begin{cases} X & n = 0 \\ \varnothing & n > 0 \end{cases} \qquad\qquad p_0(\mathscr{C}) = \mathscr{C}(\mathbf{0})$$

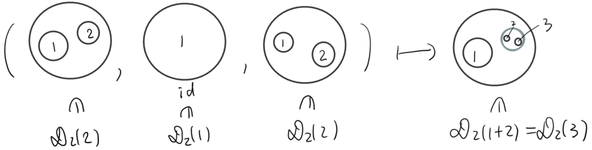
- Any $\mathscr{C} \in \Lambda^{\mathrm{op}}[\mathrm{Top}]$ yields a functor $\mathrm{C} : \mathrm{Top}_* \to \mathrm{Top}_*$ by

$$\mathrm{C}X = p_0(\mathscr{C} \odot i_0 X)$$

- The associated functor of $\mathscr{C} \odot \mathscr{D}$ is CD.
- When $\mathscr{C}$ is a reduced operad, C is a monad.
- Let X be a based space, then $i_0 X$ is a left $\mathscr{C}$-module for $\odot \Leftrightarrow X$ is a left C-module. Such X is called an algebra over the operad $\mathscr{C}$.

Two Λ-sequences: $\mathscr{D}_V$ and $\mathscr{D}_M$

- The little V-disk operad $\mathscr{D}_V$ (Guillou–May) has the following data:
  - $(G \times \Sigma_k)$-spaces $\mathscr{D}_V(k) = \{(e_1, \cdots, e_k) |$ conditions $\}$.
  - G acts on $\mathscr{D}_V(k)$ by conjugation.
  - G-equivariant structure maps $\gamma : \mathscr{D}_V(k) \times \mathscr{D}_V(j_1) \times \cdots \times \mathscr{D}_V(j_k) \to \mathscr{D}_V(j_1 + \cdots + j_k)$.



- $\mathscr{D}_M(k) = \{$ embeddings of V-framed G-manifolds $\coprod_k V \to M\}$
- $\mathscr{D}_V, \mathscr{D}_M \in \Lambda^{op}[G\mathrm{Top}]$.
  - $\mathscr{D}_M$ has the same homotopy type as the ordered configuration spaces of M;
  - We have a map $\mathscr{D}_M \odot \mathscr{D}_V \to \mathscr{D}_M$, which induces a map of functors $\mathrm{D}_M\mathrm{D}_V \to \mathrm{D}_M$.

Monadic bar construction of equivariant factorization homology

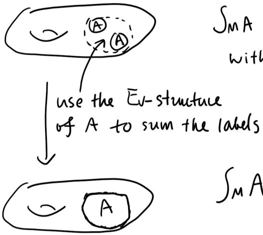
Let A be a $\mathscr{D}_V$-algebra in $G\mathrm{Top}$. Then we have:

- Associated functors

$$\mathrm{D}_V, \mathrm{D}_M : G\mathrm{Top}_* \to G\mathrm{Top}_*$$

- A simplicial G-space $\mathbf{B}_q(\mathrm{D}_M, \mathrm{D}_V, A) = \mathrm{D}_M(\mathrm{D}_V)^q A = p_0(\mathscr{D}_M \odot \mathscr{D}_V^{\odot q} \odot i_0 A)$.

$$\int_M A := \mathbf{B}(\mathrm{D}_M, \mathrm{D}_V, A) = p_0 \mathbf{B}(\mathscr{D}_M, \mathscr{D}_V, i_0 A).$$



$\int_M A$ is the "configuration space on M with summable labels in A"

$$\int_M A = \left| \cdots B_3 \substack{\to\\\to\\\to} B_2 \substack{\to\\\to} B_1 \right|$$

levels of iterated disks

Summary: Equivariant factorization homology of V-framed manifolds

G: finite group. V: orthogonal G-representation.

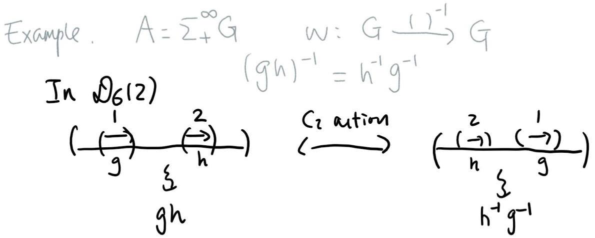
- Inputs: M and A
  M is a V-framed G-manifold
  A is an E_V-space
- Output: $\int_M A = \mathrm{B}(\mathrm{D}_M, \mathrm{D}_V, A)$, a G-space

(Andrade, Miller-Kupers, Z.)
Likewise, we can work with G-spectra A.

- Remark: If M is not framed, we require A to have more structure than an E_n-algebra.
- Related question: What is the E_2-page of the homology geometric realization spectral sequence? $\mathrm{H}_*(\mathrm{D}_n X, k)$ is an explicit functor of $\mathrm{H}_*(X, k)$. We need to know $\mathrm{H}_*(\mathrm{D}_M X, k)$ as a functor of $\mathrm{H}_*(X, k)$.
  Equivariantly, even the calculation of $\mathrm{H}_*^G(\mathrm{D}_V X)$ is unknown.

Examples of E_V-algebra

- $\Omega^V X = \mathrm{Map}_*(S^V, X)$ is an E_V-space.
- An E_n-G-space is an E_n-space such that the structure maps are G-equivariant.
- Let $G = C_2 = \{e, g\}$ and $V = \sigma$.
  An E_σ-C_2-space is an underlying E_1-space such that g acts by "anti-E_1-map".



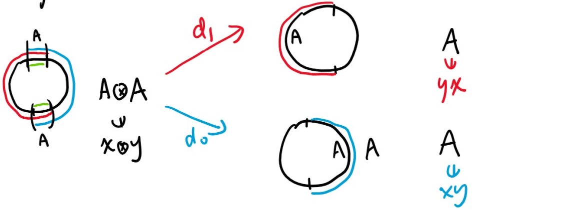
Examples of factorization homology

Examples

- $\int_{S^1} A \simeq \mathrm{THH}(A)$.
- $\int_{T^n} A$ *is iterated THH.*

Cardinality filtation of $\int_{S^1} A$

$\Rightarrow$ cyclic bar construction of $\mathrm{THH}(A)$.



Properties

- The extra degeneracy gives $\int_V A = \mathbf{B}(\mathrm{D}_V, \mathrm{D}_V, A) \simeq A$ for an E_V-algebra A.
- Recall
$$(\mathscr{C} \boxtimes \mathscr{D}) \odot \mathscr{E} \cong (\mathscr{C} \odot \mathscr{E}) \boxtimes (\mathscr{D} \odot \mathscr{E}).$$
This gives
$$\int_{M \sqcup N} A = p_0\mathbf{B}(\mathscr{D}_{M\sqcup N}, \mathscr{D}_V, i_0 A) \cong p_0\mathbf{B}(\mathscr{D}_M \boxtimes \mathscr{D}_N, \mathscr{D}_V, i_0 A) \cong \int_M A \otimes \int_N A$$

Examples

1. *$G = C_2$. σ = sign rep. Then S^σ is σ-framed.*
$\int_{S^\sigma} A \simeq \mathrm{THR}(A)$ for an E_σ-C_2-spectra A. (Horev)
(THR: Hessolholt–Madsen)

2. *$G = C_p$. Then S^1_{rot} is $\mathbb{R}$-framed.*
$\Phi^{C_p}(\int_{S^1_{\mathrm{rot}}} A) \simeq \mathrm{THH}(A; A^\tau) \simeq i_e^* \mathrm{THH}_{C_p} A$ for an E_1-C_p-spectra A. (Horev)
(twisted / relative THH: Angeltveit–Blumberg–Gerhardt–Hill–Lawson-Mandell)

Another construction: axiomatic approach (Horev)

$$\begin{array}{ccc} \mathrm{Disk}_n^{\mathrm{fr}} & \xrightarrow{A} & (\mathrm{Top}, \times) \\ \Big\downarrow & \nearrow_{\int_{\_} A} & \\ \mathrm{Mfld}_n^{\mathrm{fr}} & & \end{array}$$

- Uses G-∞-categories (Barwick–Dotto–Glasman–Nardin–Shah) and equivariant Morse theory (Wasserman).
- Coefficient: A is a package of H-spaces for each $H \subset G$.
- Input: H-manifold N. Output: H-space $\int_N A$.

$$\int_{\mathrm{Res}_K^H N} A \simeq \mathrm{Res}_K^H(\int_N A); \ \int_{G \times_H N} A \simeq N_H^G(\int_N A).$$

Comparison with my construction: (They are the same!)

- In the V-framed case, Horev's A is just an E_V-G-space. For a G-manifold M, Horev's $\int_M A$ is a derived operad left Kan extension.
- The derived operad left Kan extension agrees with the homotopy left Kan extension, which can be computed by the bar construction. (Berger-Moerdijk, Horel)

When the coefficient is a V-fold loop space.

Theorem (Equivariant nonabelian Poincaré duality theorem, Z.)

M: V-framed G-manifold, A: G-connected E_V-algebra in $G\mathrm{Top}$. Then there is a weak G-equivalence:

$$\int_M A \to \mathrm{Map}_*(M^+, \mathbf{B}^V A),$$

where $\mathbf{B}^V A$ is the V-fold deloop of A, i.e., $A \simeq \Omega^V \mathbf{B}^V A$ as G-spaces.

Non-equivariant statement: Salvatore, Lurie, Miller, Ayala-Francis, ...
Recall

$$\int_M A = |\mathrm{D}_M(\mathrm{D}_V)^\bullet(A)| \text{ and } \mathbf{B}^V A = |\Sigma^V(\mathrm{D}_V)^\bullet A|.$$

scanning map (McDuff, Segal, Bödigheimer, ...)

$$s : \mathrm{D}_M(X) \to \mathrm{Map}_*(M^+, \Sigma^V X).$$

Theorem (Rourke–Sanderson)

The scanning map $s : \mathrm{D}_M(X) \to \mathrm{Map}_*(M^+, \Sigma^V X)$ is a weak G-equivalence if X is G-connected.

Related question: scanning map when X is a spectrum?

(Equivariant) Thom spectra functor

(Ando-Blumberg-Gepner-Hopkins-Rezk)

- X: ∞-groupoid, R: ring spectra.
- R-line: the ∞-category of free rank one R-modules and equivalences. This is an ∞-groupoid that models BGL_1R.
- Definition. A bundle is $f : X \to R\text{-line}$.
- The Thom R-module spectrum of f is

$$Mf = \mathrm{colim}(X \xrightarrow{f} R\text{-line} \subset R\text{-mod})$$

(Horev-Klang-Z.)

- $\underline{\mathrm{Sp}}^G$: parametrized G-∞-category of G-spectra.
- $\underline{\mathrm{Pic}} := \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}^G)$: spanned by invertible objects.
- $\underline{\mathrm{Pic}}$ is a G-symmetric monoidal G-∞-groupoid (modeling a G-E_∞-space).
- Definition. The Thom spectra functor is the G-left Kan extension along Yoneda embedding

$$\begin{array}{ccc} \underline{\mathrm{Pic}} & \xrightarrow{\subset} & \underline{\mathrm{Sp}}^G \\ \downarrow{\scriptstyle j} & \nearrow{\scriptstyle Th} & \\ \underline{\mathrm{Top}}^G_{/\underline{\mathrm{Pic}}} & & \end{array}$$

- For $G = e$, this is equivalent to the ABGHR functor in the case of $R = \mathbb{S}$.

When the coefficient is the Thom spectra of a V-fold loop map

Lemma

M: a W-framed G-manifold, $A = \mathrm{Th}(\Omega^W X \to \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}^G))$. Then $\int_M A \simeq \mathrm{Th}(\mathrm{Map}_*(M^+, X) \to \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}^G))$.

Past results

- (Blumberg-Cohen-Schlichtkrull) n=1, $\mathrm{THH}(A) \simeq \mathrm{Th}(LX \to BF)$.
- (Klang) The nonequivariant statement.

Proof of the lemma

- Step 1: commute Thom spectra with factorization homology

$\Omega^W f$ is an E_W-algebra in Space^G/B. $B = \underline{\mathrm{Pic}}(\mathrm{Sp}^G)$.

Proposition $\int_M \mathrm{Th}(\Omega^W f) \simeq \mathrm{Th}(\int_M \Omega^W f)$.

Here, $\int_M \Omega^W f = (\int_M \Omega^W X \to \int_M B \to B)$.

- Step 2: use the nonabelian Poincare duality theorem.

$$\int_M \Omega^W X \simeq \mathrm{Map}_*(M^+, X).$$

When the coefficient is the Thom spectra of a $V+1$-fold loop map.

Theorem (Horev-Klang-Z.)

Let A be the Thom spectrum of an E_{V+1}-map $\Omega^{V+1}X \to \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}^G)$ such that X is suitably connected. Then

$$\int_{S^V \times \mathbb{R}} A \simeq A \wedge (\Omega X)_+.$$

$G = C_2$, σ: sign rep, ρ: regular rep

Theorem (Behrens–Wilson)

$\mathrm{H}\underline{\mathbb{F}_2}$ *is the Thom spectrum of a ρ-fold loop map $\Omega^\rho S^{\rho+1} \to B_{C_2}O$.*

$$\mathrm{THR}(\mathrm{H}\underline{\mathbb{F}_2}) \simeq \int_{S^\sigma \times \mathbb{R}} \mathrm{H}\underline{\mathbb{F}_2} \simeq \mathrm{H}\underline{\mathbb{F}_2} \wedge (\Omega S^{\rho+1})_+.$$

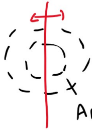
$$\mathrm{THR}_\star(\mathrm{H}\underline{\mathbb{F}}_2) \cong \pi_\star(\mathrm{H}\underline{\mathbb{F}}_2)[x_\rho]$$

This recovers Dotto–Moi–Patchkoria–Reeh.

$$H\underline{\mathbb{F}}_2 \simeq Th(\Omega^\rho S^{\rho+1} \to B) \quad \text{(Behrens-Wilson)}$$

By the lemma, we already have:

$$\int_{Ann} H\underline{\mathbb{F}}_2 \simeq Th(Map_*(Ann^+, S^{\rho+1}) \to B)$$



$Ann \cong S^\sigma \times \mathbb{R}$

$$Map_*(Ann^+, S^{\rho+1}) \simeq Map_*(S^\rho \vee S^1, S^{\rho+1})$$

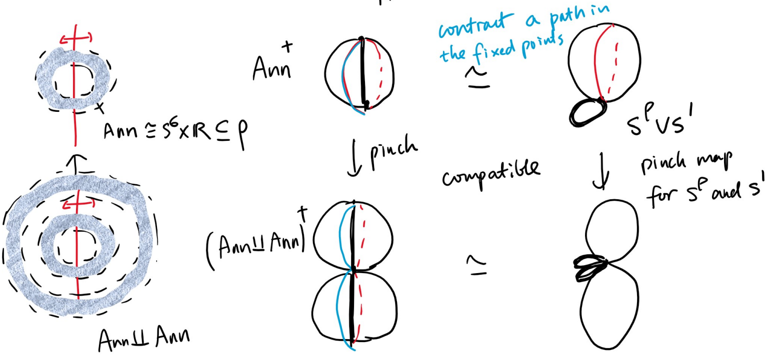
$$\simeq \Omega^\rho S^{\rho+1} \times \Omega S^{\rho+1} \longrightarrow B$$

$$\text{Th} \swarrow \qquad \downarrow \text{Th}$$

$$\Rightarrow \int_{Ann} H\underline{\mathbb{F}}_2 \simeq H\underline{\mathbb{F}}_2 \wedge \Sigma^\infty_{G,+} \Omega S^{\rho+1}.$$

A geometric proof that the splitting is E_1

$$\int_{Ann} H\underline{\mathbb{F}}_2 \simeq Th\Big(\underbrace{Map_*(Ann^+, S^{\rho+1})}_{\cong\, Map_*(S^\rho \vee S^1,\, S^{\rho+1})} \to B\Big)$$



Other applications

Using work of Hahn-Wilson, we also have ($G = C_2$, $\lambda = 2\sigma$)

Corollary (Horev-Klang-Z.)

1. $\mathrm{THR}(\mathrm{H}\underline{\mathbb{F}}_2) \simeq \mathrm{H}\underline{\mathbb{F}}_2 \wedge (\Omega S^{\rho+1})_+ \simeq \mathrm{H}\underline{\mathbb{F}}_2 \wedge (\Omega^\sigma S^{\lambda+1})_+$
2. $\mathrm{THR}(\mathrm{H}\underline{\mathbb{Z}}_{(2)}) \simeq \mathrm{H}\underline{\mathbb{Z}}_{(2)} \wedge (\Omega^\sigma (S^{\lambda+1}\langle \lambda+1\rangle))_+$
3. $\int_{S^\lambda} \mathrm{H}\underline{\mathbb{F}}_2 \simeq \mathrm{H}\underline{\mathbb{F}}_2 \wedge S^{\lambda+1}_+$
4. $\int_{S^\lambda} \mathrm{H}\underline{\mathbb{Z}}_{(2)} \simeq \mathrm{H}\underline{\mathbb{Z}}_{(2)} \wedge (S^{\lambda+1}\langle \lambda+1\rangle)_+$

Bökstedt proved

$$\mathrm{THH}(\mathrm{H}\mathbb{Z}) \simeq \mathrm{H}\mathbb{Z} \oplus (\oplus_{k\geq 2}\Sigma^{2k-1}\mathrm{H}\mathbb{Z}/k)$$

(2) is used to finish

Theorem (Dotto, Moi, Patchkoria, Reeh, Hahn, Wilson)

There is an equivalence of $\mathrm{H}\underline{\mathbb{Z}}$-module spectra

$$\mathrm{THR}(\mathrm{H}\underline{\mathbb{Z}}) \simeq \mathrm{H}\underline{\mathbb{Z}} \oplus (\oplus_{k\geq 2}\Sigma^{k\rho-1}\mathrm{H}\underline{\mathbb{Z}/k})$$

Other applications

Proposition (Horev-Klang-Z.)

$$i_e^* \mathrm{THH}_{C_2}(\mathrm{H}\underline{\mathbb{F}}_2) \simeq \mathrm{H}\mathbb{F}_2 \wedge (\Omega S^3)_+$$

PROOF:

$$\begin{aligned}\int_{S^1_{rot}} \mathrm{H}\underline{\mathbb{F}}_2 &\simeq \mathrm{Th}\left(\int_{S^1_{rot}} \Omega^\rho S^{\rho+1} \to \underline{\mathrm{Pic}}\right) \\ &\simeq \mathrm{Th}\left(\mathrm{Map}(S^1_{rot}, \Omega^\sigma S^{\rho+1}) \to \underline{\mathrm{Pic}}\right)\end{aligned}$$

Thom isomorphism

$$\mathrm{H}\underline{\mathbb{F}}_2 \otimes \int_{S^1_{rot}} \mathrm{H}\underline{\mathbb{F}}_2 \simeq \mathrm{H}\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+ \mathrm{Map}(S^1_{rot}, \Omega^\sigma S^{\rho+1})$$

Taking geometric fixed points

$$\mathrm{H}\mathbb{F}_2 \otimes \mathrm{THH}_{C_2}(\mathrm{H}\underline{\mathbb{F}}_2) \simeq \mathrm{H}\mathbb{F}_2 \otimes \Sigma^\infty_+ \mathrm{Map}_{C_2}(S^1_{rot}, \Omega^\sigma S^{\rho+1}) \simeq \mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2 \otimes \Sigma^\infty_+ \Omega S^3$$

□

Also proved by Adamyk-Gerhardt-Hess-Klang-Kong.

Thank you!