New models for motivic K-theory spectra or Universality of algebraic K-theory

j.w. Marc, Joachim, Denis, Burt
+ Tom Bachmann   recent progress :)

Goal for today: understand a new universal property of alg. (and hermitian) K-theory and its applications.

Intro. k - perfect base field

$K : Sm_k^{op} \to Spc$ defined as $K = L_{Zar} Vect^{gp}$,
where stack Vect is the presheaf of $\mathcal{E}_\infty$-spaces
$Sm_k \ni X \mapsto$ (gpd of v.b. over X, $\oplus$).

Some **features of K**:

1. descent $Zar < \text{(Nis)} < \cancel{et}$
2. $\mathbb{A}^1$-htpy invariance $K(X) \xrightarrow{pr^*} K(X \times \mathbb{A}^1)$ $\mathbb{A}^1$-equiv.

] motivic equiv.

3. infinite $\mathbb{P}^1$-loop space, thanks to Bott periodicity $K \simeq \Omega_{\mathbb{P}^1} K$
4. finite flat transfers $f : X' \to X$ finite flat $\rightsquigarrow$ $f_* : K(X') \to K(X)$, induced by $f_* : Vect(X') \to Vect(X)$

Moral: K is universal among presheaves $Sm_k^{op} \to Spc$ with these features

①, ② – properties   ③, ④ – data

Towards precise formulation.

$SH(k) :=$ $\mathbb{P}^1$-spectra in $\mathbb{A}^1$-inv. Nis. shvs of spaces on Sm_k

$SH \simeq$ Spectra — htpy inv. — M-V property — SmMan

$SH^{fflat}(k) :=$ —||— on $Corr_k^{flat}$, where

$Corr^{fflat}(k)$ is a $(2,1)$-cat. with maps

$$\begin{array}{ccc} & Z \in Sch_k & \\ \text{fin. flat} \swarrow & & \searrow \\ X \in Sm_k & & \ni Y \end{array}$$

$kgl :=$ $\mathbb{P}^1$-connective spectrum representing K, so under $Re_{\mathbb{C}} : SH(\mathbb{C}) \to SH$ we have $kgl_{\mathbb{C}} \mapsto ku$

Main Thm (us + Tom): There's an equivalence of symm. mon. ∞-cats

$$Mod_{kgl} SH(k)[\tfrac{1}{e}] \simeq SH^{fflat}(k)[\tfrac{1}{e}]$$

$$\searrow SH(k)[\tfrac{1}{e}] \swarrow$$

$$e = \begin{cases} 1, & \text{char } k = 0 \\ p, & \text{char } k = p \end{cases}$$

In this sense, kgl represents the universal generalized motivic cohomology theory with finite flat transfers

The proof consists of two parts:

① $\Omega^\infty_{\mathbb{P}^1} kgl \overset{Zar}{\underset{def.}{\simeq}} Vect^{gp} \rightleftarrows FFlat^{gp}$, (as shvs of spaces with finite flat transfers) — $\mathbb{A}^1$-equivalence (on affines)

so $\Omega^\infty_{\mathbb{P}^1} kgl$ is the universal grouplike $\mathbb{A}^1$-inv. sheaf of E_∞-spaces with finite flat transfers.

② deduce main thm, using a lot of general motivic theory: motivic recognition principle (EHKSY), cancellation thm for fflat transfers (Tom), etc.

Since ② is technical & involved, I'll talk about ①.
In fact, the base assumption is only used in ②, ① works over ∀ base. The main ingredient is the following **Thm** ($d \geq 1$): $\mathrm{Vect}_{d-1} \to \mathrm{FFlat}_d$ is an $\mathbb{A}^1$-equivalence

square zero extension $\mathcal{E} \mapsto \mathcal{O} \oplus \mathcal{E}$

Proof: The $\mathbb{A}^1$-inverse $\mathrm{FFlat}_d \to \mathrm{Vect}_{d-1}$ is $\mathcal{A} \mapsto \mathcal{A}/\mathcal{O}$.
Need to show: $\mathrm{FFlat}_d \to \mathrm{Vect}_{d-1} \to \mathrm{FFlat}_d$ is $\mathbb{A}^1$-equiv. to $\mathrm{id}|_{\mathrm{FFlat}_d}$

$\mathcal{A} \longmapsto \mathcal{A}/\mathcal{O} \oplus \mathcal{O}$

We'll define an explicit homotopy along $\mathbb{A}^1$.

Rees algebra. S - ring, A - S-algebra with increasing filtration $0 \subset A_0 \subset A_1 \subset \dots$
$1 \in A_0$, $A_i \cdot A_j \subset A_{i+j}$, $A = \bigcup A_i$. Then
$R(A) := \bigoplus \underbrace{A_i \cdot t^i}_{\deg i} \subset A[t]$ is graded $S[t]$-algebra with (grading comes from $A[t]$, A deg 0, t deg 1)

$R(A)/(t) \simeq gr(A)$ and $R(A)/(t-1) \simeq A$.

If all $gr_i A := A_i / A_{i-1}$ are flat over S, so is $R(A)$ over $S[t]$.

We apply this to the canonical filtration
$\underset{A_0}{S \cdot 1} \subset \underset{A_1}{A}$ of any $A \in \mathrm{FFlat}_d(S)$, $d \geq 1$.

We get $R(A)/(t) \simeq S \oplus \underbrace{A/S}_{\text{zero product here}}$; $R(A)/(t-1) = A$.

So the map $A \mapsto R(A)$ is natural and extends to
$\mathbb{A}^1 \times \mathrm{FFlat}_d \to \mathrm{FFlat}_d$ (aka $\mathrm{FFlat}_d(S) \to \mathrm{FFlat}_d(\mathbb{A}^1_S)$)
which provides the $\mathbb{A}^1$-htpy between $\mathcal{A} \mapsto \mathcal{A}/\mathcal{O} \oplus \mathcal{O}$ and $\mathrm{id}_{\mathcal{A}}$. ∎

$\mathrm{Maps}(\mathbb{A}^1, \mathrm{FFlat}_d(S))$

With this thm, we prove that

$FFlat^{gp} \to Vect^{gp}$ is an $\mathbb{A}^1$-equiv. (on affines), using that

$$\begin{array}{ccc} FFlat^{gp} & \underset{\mathbb{A}^1}{\simeq} & \mathbb{Z} \times FFlat_\infty \\ \downarrow & & \downarrow \\ Vect^{gp} & \underset{\mathbb{A}^1}{\simeq} & \mathbb{Z} \times Vect_\infty \end{array}$$

$FFlat_\infty = \mathrm{colim}(FFlat_0 \xrightarrow{+1} FFlat_1 \dots)$

and that $Vect_{d-1} \underset{\text{square 0}}{\overset{\mathbb{A}^1}{\simeq}} FFlat_d \xrightarrow{\text{forget}} Vect_d$, with composite $+1$,

where $+1 \overset{\mathbb{A}^1}{\simeq} id$ on $Vect_\infty$. (It's in fact more subtle because square zero ext. map doesn't commute with +1 maps)

Applications

① **Hilbert scheme model** for K

As we saw,

$$\Omega^\infty_{\mathbb{P}^1} kgl = K \underset{\mathbb{A}^1}{\simeq} \mathbb{Z} \times Vect_\infty \underset{\mathbb{A}^1}{\simeq} \mathbb{Z} \times FFlat_\infty.$$

Prop: The forgetful map

$Hilb_d(\mathbb{A}^\infty) \to FFlat_d$ is an $\mathbb{A}^1$-eq. (on affines).

$Hilb_d(\mathbb{A}^\infty) = \mathrm{colim}_n Hilb_d(\mathbb{A}^n)$

"space of embeddings into $\mathbb{A}^\infty$ is $\mathbb{A}^1$-contr."

$$\left(Z \hookrightarrow \mathbb{A}^N_T,\ Z \xrightarrow{\text{fin. flat}} T \right) \mapsto Z \to T$$

So, $K \underset{\mathbb{A}^1}{\simeq} Hilb(\mathbb{A}^\infty)^{gp} \underset{\mathbb{A}^1}{\simeq} \mathbb{Z} \times Hilb_\infty(\mathbb{A}^\infty)$

its $\mathbb{A}^1$-htpy type is an E_∞-space, because in $\mathbb{A}^\infty$ you can make any closed subschemes disjoint

②. Higher spaces of kgl

More generally, we can show that for $X \in Sm_k$

$$\Omega^\infty_{\mathbb{P}^1}(kgl \otimes \Sigma^\infty_{\mathbb{P}^1} X_+)[\tfrac{1}{e}] \simeq_{mot} Corr^{flat}(-,X)^{gp}[\tfrac{1}{e}] \qquad \text{(Thm)}$$

and $Corr^{flat}(-,X) \underset{\mathbb{A}^1}{\simeq} Hilb(\mathbb{A}^\infty_X)$. ($E_\infty$ via $\sqcup$)

This can be thought of as analogue of Segal: $\Omega^\infty(ku \otimes \Sigma^\infty X_+)$ (↑ space) is weakly equivalent to gp of the "labelled configuration space" of X, where points in a tuple are labelled by pairwise orthogonal complex v.sp. embedded into $\mathbb{C}^\infty$, and E_∞-str. is defined so that you sum v.sp. when pts collide.

In our case, we have a forgetful map

$$Hilb(\mathbb{A}^\infty_X) \xrightarrow{\text{Hilb-Chow}} Sym(\mathbb{A}^\infty_X) \xrightarrow{(pr_X)_*} Sym(X)$$

(support + degree; pushf. on cycles)

$$\left[\begin{array}{c} Z \overset{i}{\hookrightarrow} \mathbb{A}^\infty_X \times T \xrightarrow{p} X \times T \to T \\ Z \to T \text{ finite flat} \end{array}\right] \longmapsto \sum_{x \in X \times T} \deg\,(p \circ i)^{-1}(x) \cdot x$$

We can show by similar methods that the fiber of this map over, say, $\sum_{i=1}^{r} a_i p_i$, $p_i \in X(k)$ distinct, is mot. equivalent to $\prod_{i=1}^{r} Vect_{r-1}$.

Upshot: We get Hilbert scheme models for $\Omega^\infty_{\mathbb{P}^1} \Sigma^n_{\mathbb{P}^1} kgl$, $n \geq 0$, over any perfect field (don't need $[\tfrac{1}{e}]$!)

Dream: Use this and $kgl/\beta \simeq H\mathbb{Z}$ to compute Steenrod alg. in char p!