Topological cyclic homology of local fields

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Topological Hochschild Homology

Fixed a prime p. We will work in the p-completed world throughout.
Let A be an E_∞-ring spectrum. We have

$$THH(A) = A^{\otimes \mathbb{T}}$$

to be the free $\mathbb{T}$-E_∞-ring spectrum generated by A.
We will follow Nikolaus-Scholze's definition of a cyclotomic structure.
There is a cyclotomic structure on $THH(A)$, i.e. an E_∞-homomorphism

$$\varphi : THH(A) \to THH(A)^{tC_p}$$

equivariant with respect to the group isomorphism $\mathbb{T} \cong \mathbb{T}/C_p$.

Topological Cyclic Homology

Topological periodic homology

$$TP(A) = THH(A)^{t\mathbb{T}}$$

Topological negetive cyclic homology

$$TC^{-}(A) = THH(A)^{h\mathbb{T}}$$

Topological cyclic homology

$TC(A)$ is the equalizer of the canonical map

$$can : TC^{-}(A) \to TP(A)$$

and the Frobenius

$$\varphi : TC^{-}(A) \to TP(A)$$

Relative THH

Let

$$S_n = \mathbb{S}[z_0, \ldots, z_n]$$

be the E_∞-ring spectrum $\mathbb{S} \wedge \mathbb{N}_+^{n+1}$. Then the ∞-category of S_n-modules is symmetric monoidal.

For any E_∞-S_n-algebra A, define

$$THH(A/S_n) = A^{\otimes_{S_n} \mathbb{T}}$$

as the free $\mathbb{T}$-E_∞-S_n-algebra generated by A.

Cyclotomic Structure

By construction of Bhatt-Morrow-Scholze, there is a cyclotomic structure on S_n with trivial $\mathbb{T}$-action such that the Frobenius

$$\phi : S_n \to S_n^{t\mathbb{T}}$$

is defined by sending z_i to z_i^p.

For any E_∞-S_n-algebra A, $THH(A/S_n) \cong THH(A) \otimes_{THH(S_n)} S_n$ has a structure as a cyclotomic E_∞-spectrum over S_n.

Relative TP

$$TP(A/S_n) = THH(A/S_n)^{t\mathbb{T}}$$

Relative TC^-

$$TC^-(A/S_n) = THH(A/S_n)^{h\mathbb{T}}$$

Locally Complete Intersections

Let K to be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$. Let K_0 be the maximal unramified subextension in K.
Let

$$P = \mathcal{O}_K[z_1, \dots, z_n]$$

Let I be an ideal of P which is a locally complete intersection, i.e. Zariski locally generated by a regular sequence.
Let $R = P/I$. Then $L = I/I^2$ is a projective R-module.

The above data amounts to a locally complete intersection algebra R together with a set of generators z_i. For fixed R, the choices of set of generators form a filtered system.

We make P an S_n-algebra by sending z_0 to a fixed uniformizer ϖ of K.
We further assume that ϖ is not a zero divizor in R.

Relative TP for P

We will call the filtration defined by the Tate spectral sequence the Nygaard filtration.

$$THH(P/S_n) \cong P[u]$$

with $|u| = 2$ being the Bökstedt element.

The Tate spectral sequence for $TP(P/S_n)$ collapses for degree reasons, but by Bhatt-Morrow-Scholze there is a non-trivial extension:

Let E be the minimal equation for ϖ over $\mathcal{O}_{K_0}$ with constant term p.

$$TP_0(P/S_n) = \mathcal{O}_{K_0}[x_0, \ldots, x_n]^\wedge$$

with Nygaard filtration defined by powers of E.

$$TP_*(P/S_n) = TP_0(P/S)[\sigma^{\pm}]$$

Relative TC^- and the Frobenius

Relatice TC^-

$$TC^-_*(P/S_n) = TP_0(P/S)[u, v]/(uv - E)$$

Formula for Frobenius

$$can(v) = \sigma^{-1}$$

$$\varphi(u) = \sigma$$

It is essential that the constant term of E to be p to have the above formula without an unspecified unit.

Relative THH for R

- $THH(R/S_n)$ is concentrated in even degrees.
- The Tate and homotopy fixed point spectral sequences for $THH(R/S_n)$ collapses.
- There is a filtration on $THH(R/S_n)$ such that the graded pieces is isomorphic to
$$R[u] \otimes \Gamma(L)$$
with $L = I/I^2$ lying in degree 2.

Relative TP for R

Divisibility property

If $\alpha \in TP_0(R/S_n)$ has Nygaard filtration i, then $\varphi(\alpha)$ is divisible by $\varphi(E)^i$. In fact $\varphi(\alpha\sigma^i) = \frac{\varphi(\alpha)}{\varphi(E)^i}\sigma^i$.

For any $f \in I$, define $h_f = \frac{\varphi(f)}{\varphi(E)}$.

Theorem

$TP_0(R/S_n)$ is the completion under the Nygaard filtration of the δ-ring over $\mathcal{O}_{K_0}[x_0, \ldots, x_n]$ generated by h_f for $f \in I$, modulo the relations

$$\varphi(E)h_f = \varphi(f)$$

$$h_{af} = \varphi(a)h_f$$

We can call this to be the h-envelope of I, which is a deformation of the divided polynomial ring.

The Structure of $TP_0(R/S_n)$

For $f \in I$, define

$$f^{(1)} = \frac{f^p - h_f E^p}{p}$$

Inductively, we define $f_i^{(k)}$ and $h_f^{(k)}$ by:

- Define
$$f^{(k)} = \frac{(f^{(k-1)})^p - h_f^{(k-1)} E^{p^k}}{p}$$
- $f^{(k)}$ lies in $\mathbb{Z}_p[x_0, \ldots, x_n][h_f, \ldots, h_f^{(k-1)}]$.
- $f_i^{(k)}$ lies in Nygaard filtration p^k.
- $\varphi(f^{(k)})$ is divisible by $\varphi(E)^{p^k}$.
- Define $h_f^{(k)}$ by the equation

$$h_f^{(k)} \varphi(E)^{p^k} = \varphi(f^{(k)})$$

(Continued)

- By construction

$$\varphi(f^{(k)}) = \frac{\varphi(f^{(k-1)})^p - \varphi(h_f^{(k-1)})\varphi(E)^{p^k}}{p}$$

- Since $\varphi(E)$ is not a zero divisor,

$$h_f^{(k)} = \frac{(h_f^{(k-1)})^p - \varphi(h_f^{(k-1)})}{p}$$

- $(f^{(k)})^p - \varphi(f^{(k)})$ is divisible by p.

Resolution of the base

We have an Adams resolution for $\mathbb{S}$:

$$\mathbb{S} \to S_n \to S_n \otimes_{\mathbb{S}} S_n \to S_n^{\otimes 3} \to \dots$$

R is a $S_n^{\otimes m}$-algebra via the map

$$S_n^{\otimes m} \to S_n \to R$$

We have the augmented cosimplicial cyclotomic E_∞ spectrum

$$THH(R) \to THH(R/S_n) \to THH(R/S_n^{\otimes 2}) \to THH(R/S_n^{\otimes 3}) \to \dots$$

Convergence

- $THH(R) \xrightarrow{\cong} Tot(THH(R/S_n^{\otimes \bullet}))$.
- $TP(R) \xrightarrow{\cong} Tot(TP(R/S_n^{\otimes \bullet}))$.
- $TC^-(R) \xrightarrow{\cong} Tot(TC^-(R/S_n^{\otimes \bullet}))$.

The descent spectral sequence

- $TP_0(R/S_n^{\otimes 2})$ is flat over $TP_0(R/S_n)$.
- $(TP_0(R/S_n), TP_0(R/S_n^{\otimes 2}))$ forms a Hopf algebroid.
- $TP_*(R/S_n)$ is a $TP_0(R/S_n^{\otimes 2})$-comodule.
- $TP_*(R/S_n^{\otimes \bullet})$ is isomorphic to the cobar complex:

$$C^\bullet(TP_*(R/S_n), TP_0(R/S_n^{\otimes 2}), TP_0(R/S_n))$$

- We have a spectral sequence

$$Ext^j_{TP_0(R/S_n^{\otimes 2})}(TP_0(R/S_n), TP_i(R/S_n)) \Rightarrow TP_{i-j}(R)$$

Descent spectral sequences for TC^- and TC

TC^-

The coskeleton filtration on $Tot(TC^-(R/S_n^{\otimes \bullet}))$ gives the descent spectral sequence for TC^-:

$$TC_*^-(R/S_n^{\otimes *}) \Rightarrow TC_*^-(R)$$

TC

Define the filtration on $TC(R)$ such that $TC(R)_{(n)}$ to be the fiber of

$$can - \varphi : Tot_n(TC^-(R/S_n^{\otimes \bullet})) \to Tot_{n-1}(TP(R/S_n^{\otimes \bullet}))$$

Then $E_1^{*,*}(TC(R))$ is the mapping cone of

$$can - \varphi : E_1^{*,*}(TC^-(R)) \to E_1^{*,*}(TP(R))$$

Relationship with Bhatt-Morrow-Scholze theory

Let

$$S_n^{\infty} = \mathbb{S}[z_0^{\frac{1}{p^{\infty}}}, \ldots, z_n^{\frac{1}{p^{\infty}}}]$$

There are maps

$$THH(R/S_n) \to THH(R \otimes_{S_n} S_n^{\infty}/S_n^{\infty}) \xleftarrow{\cong} THH(R \otimes_{S_n} S_n^{\infty})$$

Note that the p-completion of $R \otimes_{S_n} S_n^{\infty}$ is semi-perfectoid.
The above map induces a morphism from the descent resolution to the Čech resolution in the quasi-syntomic site.

Conjecture

The descent spectral sequence is isomorphic to the BMS spectral sequence.

Structure of $TP_0(R/S_n^{\otimes 2})$

For simplicity, suppose R is a complete intersection defined by $f_1(z), \dots, f_k(z)$. We also have:

$R = \mathcal{O}_{K_0}[z_1, \dots, z_n, z'_0, z'_1, \dots, z'_n]/(f_1(z), \dots, f_k(z), z'_0 - \varpi, z'_1 - z_1, \dots, z'_n - z_n)$

$TP_0(R/S_n^{\otimes 2})$ is the completion under the Nygaard filtration of the δ-ring generated over $TP_0(R/S_n)[z'_0, \dots, z'_n]$ by $h_{z'_0 - z_0}, h_{z'_1 - z_1}, \dots, h_{z'_n - z_n}$, modulo the relations

$$h_{z'_0 - z_0}\varphi(E(z_0)) = z_0'^p - z_0^p$$

$$h_{z'_i - z_i}\varphi(E(z_0)) = z_i'^p - z_i^p$$

Hopf Algebroid Structures

units

$$\eta_L(z_i) = z_i$$
$$\eta_R(z_i) = z_i'$$

comultiplication

$$\psi(z_i) = z_i \otimes 1$$
$$\psi(z_i') = 1 \otimes z_i'$$

comodule structure on $TP_*(R/S_n)$

$$TP_*(R/S_n) = TP_0(R/S_n)[\sigma^{\pm}]$$
$$\psi(\sigma) = \epsilon^{-1}\sigma$$
$$\epsilon^{-1}\varphi(\epsilon) = \frac{\varphi(E(z_0'))}{\varphi(E(z_0))}$$

Naturality

Let

$$\tilde{R} = \mathcal{O}_K[y_1, \dots, y_l]/(h_1(y), \dots, h_m(y))$$

with $p, h_1(y), \dots, h_m(y)$ a regular sequence.
Let $g_1(z), \dots, g_l(z) \in \mathcal{O}_K[z_1, \dots, z_n]$ be polynomials such that

$$h_i(y) \in (f_1(z), \dots, f_k(z), y_1 - g_1(z), \dots, y_l - g_l(z))$$

Then $g_i(z)$ defines a ring homomorphism $g : \tilde{R} \to R$

$$\begin{tikzcd}
S_l \arrow[r] \arrow[d] & S_{l+n} \arrow[d] & S_n \arrow[l] \arrow[d] \\
\mathcal{O}_K[y_i] \arrow[r] \arrow[d] & \mathcal{O}_K[z_j, y_i] \arrow[d] & \mathcal{O}_K[z_j] \arrow[l] \arrow[d] \\
\tilde{R} \arrow[r] & R & R \arrow[l]
\end{tikzcd}$$

Naturality

- We have a morphism of Hopf algebroids

$$(TP_0(\tilde{R}/S_l), TP_0(\tilde{R}/S_l^{\otimes 2})) \to (TP_0(R/S_{l+n}), TP_0(R/S_{l+n}^{\otimes 2}))$$

- There is a Morita equivalence

$$(TP_0(R/S_n), TP_0(R/S_n^{\otimes 2})) \to (TP_0(R/S_{l+n}), TP_0(R/S_{l+n}^{\otimes 2}))$$

- We have a morphism of spectral sequences

$$\begin{tikzcd} Ext_{TP_0(\tilde{R}/S_l^{\otimes 2})}(TP_*(\tilde{R}/S_l)) \arrow[r] \arrow[d, Rightarrow] & Ext_{TP_0(R/S_n^{\otimes 2})}(TP_*(R/S_n)) \arrow[d, Rightarrow] \\ TP_*(\tilde{R}) \arrow[r] & TP_*(R) \end{tikzcd}$$

Localization

Let $h(x) \in \mathcal{O}_K[z_1, \ldots, z_n]$ be a polynomials. We have:

$$R[h^{-1}] = \mathcal{O}_K[z_1, \ldots, z_n, y]/(f_1(z), \ldots, f_k(z), yh(z) - 1)$$

Theorem

The Hopf algebroid

$$(TP_0(R[h^{-1}]/S_{n+1}), TP_0(R[h^{-1}]/S_{n+1}^{\otimes 2}))$$

is Morita equivalent to

$$(TP_0(R/S_n)[h(x)^{-1}], TP_0(R/S_n^{\otimes 2})[h(x)^{-1}])$$

Breuil-Kisin Twists

Let T be the free comodule of rank 1 generated by σ, such that

$$\psi(\sigma) = \epsilon^{-1}\sigma$$

$$\epsilon^{-1}\phi(\epsilon) = \frac{\varphi(E(z_0))}{\varphi(E(z_0'))}$$

For any comodule A, we have its Breuil-Kisin twist by

$$A\{i\} = A \otimes T^{\otimes i}$$

Algebraic Tate/homotopy fixed points spectral sequences

The E_1 terms of the descent spectral sequences has the Nygaard filtration, which induces the algebraic Tate spectral sequence:

$$E_2(THH(R))[\sigma^{\pm}] \Rightarrow E_2(TP(R))$$

and the algebraic homotopy fixed points spectral sequence:

$$E_2(THH(R))[v] \Rightarrow E_2(TC^{-}(R))$$

Square of four spectral sequences

$$\begin{tikzcd}
 & TP_*(R) & \\
THH(R)[\sigma^{\pm}] \arrow[ur, Rightarrow, "\textbf{Tate SS}"] & & E_2(TP(R)) \arrow[ul, Rightarrow, "\textbf{descent SS}"'] \\
 & E_2(THH(R))[\sigma^{\pm}] \arrow[ul, Rightarrow, "\textbf{descent SS}"] \arrow[ur, Rightarrow, "\textbf{algebraic Tate SS}"'] &
\end{tikzcd}$$

The Hopf algebroid for $\mathcal{O}_K$

$$TP_0(\mathcal{O}_K/\mathbb{S}[z]) = \mathcal{O}_{K_0}[z]^\wedge$$

$TP_0(\mathcal{O}_K/\mathbb{S}[z, z'])$ is the completion of the δ-ring over $\mathcal{O}_{K_0}[z, z']$ generated by h, modulo the relation

$$h\varphi(E(z)) = \varphi(E(z'))$$

The structure maps are

$$\eta_L(z) = z$$

$$\eta_R(z) = z'$$

Decent SS for $THH(\mathcal{O}_K; \mathbb{F}_p)$

We have the following description of the E_2 term of the descent spectral sequence for $THH(\mathcal{O}_K; \mathbb{F}_p)$:

Theorem

- $E_2^{0,*}(THH(\mathcal{O}_K; \mathbb{F}_p))$ is generated by $z^l u^n$ for $1 \leq l \leq e-1$ or $p|en$ if $e > 1$, and by u^n for $p|n$ if $e = 1$.
- $E_2^{1,*}(THH(\mathcal{O}_K; \mathbb{F}_p))$ is generated by $z^l u^{n-1} dz$ for $0 \leq l \leq e-2$ or $p|en$ if $e > 1$, and by $u^{n-1} dz$ for $p|n$ if $e = 1$.
- $E_2^{i,*}(THH(\mathcal{O}_K; \mathbb{F}_p)) = 0$ for $i \geq 2$.

It follows that in this case the descent spectral sequence collapses.

The mod p algebraic Tate differentials

For $n \geq 0$, $j \in \mathbb{Z}$, $l = v_p(n - \frac{pej}{p-1})$, we have algebraic Tate differnetials:

$$d(z^n \sigma^j) \doteq z^{pe\frac{p^l-1}{p-1}+n-1} \sigma^j dz$$

This is in agreement with results of Bökstedt, Hesselholt, Madsen, Rognes, Tsalidis.

The descent spectral sequence for $TC(\mathcal{O}_K)$

Let $d = [K(\zeta_p) : K]$. There is a class $\beta \in E_2^{0,2d}(TC(\mathcal{O}_K); \mathbb{F}_p)$ detecting the Bott element. As an $\mathbb{F}_p[\beta]$-module,

$$E_2^{0,*}(TC(\mathcal{O}_K); \mathbb{F}_p \cong \mathbb{F}_p[\beta]$$

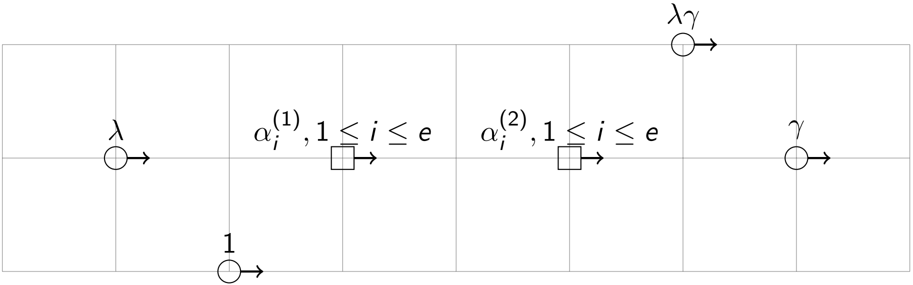
$$E_2^{1,*}(TC(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbb{F}_p[\beta]\{\lambda, \gamma\} \oplus k[\beta]\{\alpha_i^{(j)} | 1 \leq i \leq e, 1 \leq j \leq d\}$$

$$E_2^{2,*}(TC(\mathcal{O}_K); \mathbb{F}_p) \cong \mathbb{F}_p[\beta][\lambda\gamma]$$

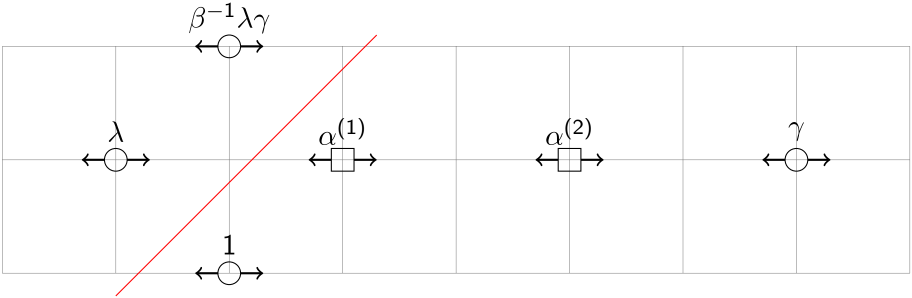
$$E_2^{i,*}(TC(\mathcal{O}_K); \mathbb{F}_p) = 0 \text{ for } i \geq 3$$

It follows that the descent spectral sequence for $TC(\mathcal{O}_K)$ collapses.

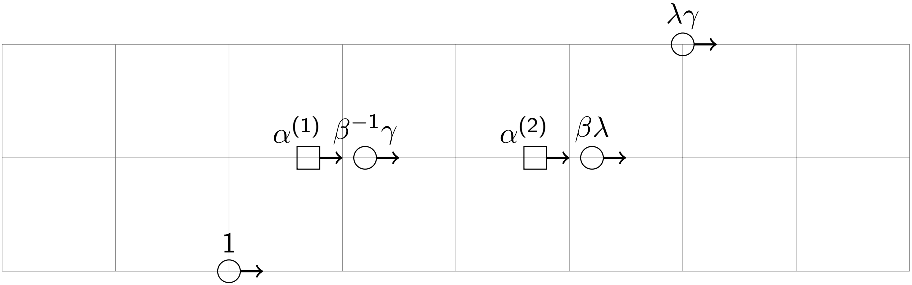
The descent spectral sequence for $TC(\mathcal{O}_K; \mathbb{F}_p)$



The étale spectral sequence for $\mathbb{K}^{\text{ét}}(K;\mathbb{F}_p)$



The motivic spectral sequence for $\mathbb{K}(K;\mathbb{F}_p)$



Computations for $K = \mathbb{Q}_p$ with p odd

Theorem

Let $e = \frac{p-y^p}{p-x^p}$ The element

$$log(e) - \frac{1}{p} log(\phi(e)) = \frac{x^p - y^p}{p} + \frac{x^{2p} - y^{2p}}{2p^2} + \dots$$

lies in $TP_0(\mathbb{Z}_p/\mathbb{S}[x, y])$.

So we have

$$x^p - y^p \doteq \frac{x^{2p} - y^{2p}}{p} + \dots \mod p$$

$$d_p(z^p) \doteq z^{2p-1} dz$$

Computations for $K = \mathbb{Q}_p$ with p odd

Applying Frobenius, we get

$$x^{p^2} - y^{p^2} \doteq \frac{x^{2p^2} - y^{2p^2}}{p} + \dots \mod p$$

Expanding

$$p^{2p}(log(e) - \frac{1}{p}log(\phi(e)))$$

we get

$$\frac{x^{2p^2} - y^{2p^2}}{p} \doteq \frac{x^{2p^2+p} - y^{2p^2+p}}{p} + \dots \mod p$$

These imply:

$$d_{p^2+p}(z^{p^2}) \doteq z^{2p^2+p-1}dz$$

Higher Tate differentials can be obtained by induction.

Thanks!